On the nuclear forces and the magnetic moments of the neutron and the proton

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INTRODUCTION

It was first suggested by Heisenberg that the forces between a proton and a neutron are connected with an exchange of charge between the two heavy particles. This exchange nature of the neutron-proton forces is now generally accepted. It would follow from this assumption that in suitable circumstances a proton (neutron) could emit a positively (negatively) charged particle transforming itself into a neutron (proton). At first sight it seemed that the emission of positive or negative electrons in the $\beta$-decay could in this way be made responsible for the nuclear forces. This was, in fact, suggested by Iwanenko (1934) and Tamm (1934). It has also been pointed out by Wick (1935) that the virtual emission of $\beta$-electrons might explain the values of the magnetic moments of the proton and the neutron. These theories, however, were not successful. The nuclear forces, for instance, turn out to be too small by a factor of more than $10^{10}$ and have far too small a range; this is due to the fact that the $\beta$-decay constant is extremely small. Since the $\beta$-decay is a process which, in nuclear dimensions, takes "geological ages", one might think that the ordinary properties of the heavy particles have no direct connexion with this process and that an approximate theory of the nuclear forces should be possible without the inclusion of the $\beta$-decay.

A new hope for such an "exchange theory" of the properties of nuclei is offered by the probable existence of a hitherto unknown type of particle constituting the hard component of cosmic radiation. Since these particles do not lose much energy by radiation, it has been suggested by Neddermeyer and Anderson (1937) that they are (positive and negative) "heavy electrons" with a mass between that of an electron and a proton. From cosmic-ray data the mass of these particles can hardly be determined yet,
but it can be limited to values between 3 and 300 electron masses. There are, however, some arguments favouring a mass nearer to the upper limit of 100–200 electron masses.†

The heavy electrons are certainly not stable. They seem to be absorbed strongly as soon as they reach an energy of less than $200 \times 10^6$ e-volts. It is probable that this absorption is due to some sort of nuclear processes. It seems, therefore, plausible to assume that a heavy electron can be absorbed directly by a proton or neutron. Thus we shall propose the existence of the following processes

\[
P \leftrightarrow N + Y_+,
\]

\[
N \leftrightarrow P + Y_-, \quad (1)
\]

where $Y_+$, $Y_-$ denotes a positive and negative heavy electron. There is no need for the introduction of any neutrino. It follows from (1) that the heavy electrons have zero (or an integral) spin and satisfy Bose statistics.

It has already been shown by Yukawa (1935) that processes of the type (1) can account for the nuclear forces in a reasonable way. The range of the forces $1/\lambda$ is directly connected with the mass $m_0$ of the heavy electron: $1/\lambda \sim \hbar/m_0c$. This leads to a value of $m_0 \sim 100$ electron masses. On the other hand, it will be shown in this paper that the processes (1) can also be made responsible for the magnetic moments of the neutron and proton. Independently of any detailed theory, it is possible to determine $m_0$ from the observed values of the magnetic moments (cf. Fröhlich and Heitler 1938). It turns out that $m_0$ is also about 100 electron masses.

The fact that $m_0$ can be determined in two independent ways both leading to the same value seems to us to give strong support for such an exchange theory of the nuclear properties based on assumption (1).

We think, therefore, that it might be a reasonable policy to try to link up the nuclear properties (forces and magnetic moments) with the cosmic-ray phenomena of the hard component rather than with the $\beta$-decay. As a first step in this direction, we shall try in this paper to build up a consistent scheme for the wave field of the heavy electrons and their inter-

† According to the theory the ionization of a fast charged particle passing through a gas has a minimum for energies of the order of the rest energy of the particle and increases logarithmically for higher energies. The cosmic-ray experiments, however, show that the ionization of particles of the order of $10^8$ e-volts is not appreciably higher than that of particles of $10^7$–$10^8$ e-volts. The particles with energy $> 3 \times 10^8$ e-volts are practically all heavy electrons. If the rest mass of the latter were only a few electron masses the ionization should be nearly the same as for electrons, which is contrary to the experiments. The above result suggests therefore a mass of the order of at least 100 electron masses.
action with the nuclear particles. It will be seen that such a scheme is possible and that it leads to a consistent theory of the nuclear forces and of the magnetic moments.

1. The wave equations for the heavy electron and its interaction with nuclear particles

Our first task is to find the wave equation for the heavy electron. The simplest relativistic wave equation for particles satisfying Bose statistics is the Klein-Gordon equation for a scalar wave function \( \psi \) in the quantized form of Pauli and Weisskopf (1934). This scheme has been used recently by Yukawa and Sakata (1937) to calculate the nuclear properties. It turns out, however, that such a theory cannot account for the magnetic moment of the proton and neutron and leads to a wrong spin dependence of the nuclear forces.

A scalar wave function is not the only possible one for Bose particles. Light quanta, for instance, are described by a vector field. It has been shown by Proca (1936) that such a vector field can also be used to describe Bose particles with a finite rest mass.

In this paper we shall assume a vector wave function for the heavy electron satisfying the equations given by Proca. In an accompanying paper Kemmer (1938) has studied a number of possible alternatives including both the case of a scalar wave function and the vector case. Only for the vector formalism the sign and spin dependence of the neutron-proton force is in agreement with empirical evidence. The other possibilities can therefore be disregarded here.

The quantization of Proca's field equations can be carried out by the method developed by Pauli and Weisskopf for a scalar wave function. The only difference is that a heavy electron can exist in three different states of polarization. For a particle with given momentum there are two waves with transverse polarization and—in contrast to the Maxwell case—one wave with longitudinal polarization.

An interesting feature of Proca's equations is their similarity to Maxwell's equations. Owing to this analogy the quantization may be performed in close correspondence to the well-known procedure in radiation theory. In the following we use a notation similar to that of Heitler's (1936a) book on radiation theory (cf. Chapter 1, §6). One slight formal difference must, however, be accepted from the start, due to the fact that the wave functions for the heavy electron are essentially complex quantities, whereas the radiation theory deals with real field variables. This results in a doubling
of the number of variables; to each quantity there exists the complex conjugate.

Let \( \phi_\alpha \) be the wave function of the heavy electron; \( \phi_\alpha \) corresponds to the 4-vector potential in radiation theory. We use Greek suffices (from 1 to 4) to denote a 4-vector and Latin suffices (from 1 to 3) for a 3-vector. Instead of \( \phi_4 \) we also introduce \( \phi_0 = -i\phi_4 \); \( \phi_0 \) corresponds to the scalar potential.

We now define an antisymmetrical tensor corresponding to the field strengths of electrodynamics:

\[
\lambda_{\alpha\beta} = \frac{\partial \phi_\beta}{\partial x_\alpha} - \frac{\partial \phi_\alpha}{\partial x_\beta}. \tag{2}
\]

For the complex conjugate quantities we have

\[
\lambda_{\alpha\beta}^* = \frac{\partial \phi_\beta^*}{\partial x_\alpha} - \frac{\partial \phi_\alpha^*}{\partial x_\beta}.
\]

The wave equations in the absence of an external field are then the following:

\[
\sum \frac{\partial \lambda_{\alpha\beta}}{\partial x_\alpha} = \lambda^2 \phi_\beta, \quad \lambda = \frac{m_0 c}{\hbar} \tag{3}
\]

A similar equation holds for \( \lambda_{\alpha\beta}^*, \phi_\beta^* \). \( m_0 \) is the rest mass of the heavy electron. For \( \lambda = 0 \) the equations (2) and (3) are identical with Maxwell's equations.

From (2) and (3) it follows that

\[
\sum \frac{\partial^2 \phi_\beta}{\partial x_\alpha^2} = \lambda^2 \phi_\beta \tag{4}
\]

and also

\[
\sum \frac{\partial \phi_\beta}{\partial x_\alpha} = 0. \tag{5}
\]

Thus, each wave function \( \phi_\beta \) satisfies the Klein-Gordon equation (4), Equation (5) corresponds to the "Lorentz condition" in Maxwell's theory. It is here a consequence of the field equations (2), (3). This is not the case in Maxwell's theory where (5) has to be assumed as an additional condition (it plays there only the role of a convenient normalization of the vector potential).

It can easily be shown that an adequate expression for the energy density of our system is given by

\[
W = \frac{1}{4\pi} \left( \sum_{i=1}^3 \chi_0^* \chi_0 + \frac{1}{4} \sum_{i,k=1}^3 \chi_i^* \chi_k + \lambda^2 \left( \phi_0^* \phi_0 + \sum_{i=1}^3 \phi_i^* \phi_i \right) \right). \tag{6}
\]
The space part $\phi_i$ of the wave function can be divided into a transverse (I) and a longitudinal (II) part:

$$\phi_i = \phi_i^I + \phi_i^{II} \quad (7)$$

with

$$\text{div} \phi_i^I = 0$$

$$\text{curl} \phi_i^{II} = 0.$$

We treat the two parts separately.

The transverse part is exactly analogous to the transverse waves in the radiation theory. We expand $\phi_i^I$ in a series of plane waves; let $A_\rho(r)$ be a complete set of orthogonal eigenfunctions with the condition of periodicity in a cube of volume 1:

$$A_\rho = \sqrt{(4\pi c^2)} j_\rho e^{i(k_\rho r)} \quad (8)$$

the $j_\rho$ being either of the two independent unit vectors perpendicular to the wave vector $k_\rho$. Putting now

$$\phi_i^I = \sum_\rho q_\rho(t) A_\rho(r),$$

$$\phi_i^{I*} = \sum_\rho q_\rho^*(t) A_\rho^*(r),$$

the $q_\rho$ and $q_\rho^*$ satisfy the equations

$$\ddot{q}_\rho + \nu^2 q_\rho = 0,$$

$$\ddot{q}_\rho^* + \nu^2 q_\rho^* = 0,$$

$$\frac{1}{\epsilon^2} \nu^2 = k_\rho^2 + \lambda^2. \quad (11)$$

The three components of the wave vector $k_\rho$ can assume all values which are integral multiples of $2\pi$. Using the definitions

$$q_\rho = p_\rho^*, \quad \dot{q}_\rho^* = p_\rho,$$

equations (10) can obviously be regarded as the canonical equations of the Hamiltonian

$$H^I = \sum_\rho H^I_\rho = \sum_\rho (p_\rho^* p_\rho + \nu^2 q_\rho^* q_\rho).$$

This expression can also be seen to be identical with the transverse part of the space integral of the energy density $W$ given by equation (6).
The longitudinal wave functions \( \phi_0^\Pi \) and \( \phi_0 \) may similarly be expanded in the form
\[
\phi_0 = \sum_{\sigma} a_{\sigma} \phi_{\sigma},
\phi_0^\Pi = \sum_{\sigma} q_{\sigma} \mathbf{B}_{\sigma}, \quad (\text{curl } \mathbf{B}_{\sigma} = 0)
\]
(13)
The \( \phi_{\sigma} \) are a complete set of scalar wave functions and \( \mathbf{B}_{\sigma} = (1/k_{\sigma}) \text{grad} \phi_{\sigma} \) (cf. Heitler 1936a, pp. 47–55). The quantities \( q_{\sigma}, a_{\sigma} \) and their complex conjugates satisfy the same differential equations (10) as the \( q_{\rho} \). If we put, therefore,
\[
\dot{q}_{\sigma} = p_{\sigma}^*, \quad \dot{p}_{\sigma}^* = p_{\sigma}; \quad \dot{a}_{\sigma} = -b_{\sigma}^*, \quad \dot{b}_{\sigma} = -b_{\sigma},
\]
(14)
we may describe the system by means of the Hamiltonian
\[
H_0^\Pi = \sum_{\sigma} \left[ (p_{\sigma} p_{\sigma} + q_{\sigma} q_{\sigma} q_{\sigma}) - (b_{\sigma}^* b_{\sigma} + q_{\sigma}^2 a_{\sigma}^* a_{\sigma}) \right]
\]
(15)
(the minus sign of the second term is explained below). The quantities \( a_{\sigma}, q_{\sigma}, b_{\sigma}, p_{\sigma} \) are not independent of each other. From (5) it follows that
\[
ck_{\sigma} q_{\sigma} = -b_{\sigma}^*,
\]
(16)
and therefore
\[
ck_{\sigma} p_{\sigma} = -b_{\sigma} = -q_{\sigma}^2 a_{\sigma}^*.
\]
(16')
Inserting these equations into the Hamiltonian (15), \( H_0^\Pi \) can be expressed by the \( q_{\sigma}, p_{\sigma} \) only:
\[
H_0^\Pi = \sum_{\sigma} \frac{c^2 \lambda^2}{v_{\sigma}^2} (p_{\sigma}^* p_{\sigma} + q_{\sigma}^2 q_{\sigma}^2).
\]
(17)
(15) or (17) is again identical with the space integral of the longitudinal part of the energy density defined by (6). For this purpose it was essential to choose the \( a_{\sigma}, b_{\sigma} \) part of the Hamiltonian (15) with the minus sign (cf. also Heitler 1936a, p. 55).

The fact that the longitudinal part \( H_0^\Pi \) does not vanish is a characteristic feature of our theory. In Maxwell’s case \( (\lambda = 0) \) \( H_0^\Pi \) vanishes except if charged particles are present and can then be reduced to a static Coulomb energy only. In our theory there are longitudinal waves even in vacuo.

The expression (17) has not exactly the form of an energy of an oscillator. But we can easily introduce new variables so as to re-establish this form. We consider the longitudinal part of the quantity \( \chi_0 \) and define a wave function
\[
\lambda \psi_i \equiv -\chi_0^\Pi = \frac{\partial \phi_0}{\partial x_i} + \frac{1}{c} \phi_i.
\]
(18)
Using the same $B_\sigma$ as above, we can expand
\[ \psi_i = \sum \sigma Q_\sigma B_\sigma, \] (19)
and putting
\[ \dot{Q}_\sigma = P^*_\sigma, \quad \dot{P}_\sigma = Q_\sigma \] (20)
we find
\[ P_\sigma = -c\lambda q^*_\sigma, \quad Q_\sigma = \frac{c\lambda}{\mu^2} P^*_\sigma. \] (21)
Hence
\[ H^{\Pi} = \sum \sigma (P^*_\sigma P_\sigma + \nu^2 Q^*_\sigma Q_\sigma). \] (22)
Thus the longitudinal part of the field $\phi_0$, $\phi^{\Pi}_i$ is equivalent to one set of oscillators with the Hamiltonian (22). Thus the total Hamiltonian of our system is the sum for the transverse and longitudinal oscillators
\[ H = \sum \rho (P^*_\rho P_\rho + \nu^2 q^*_\rho q_\rho) + \sum \sigma (P^*_\sigma P_\sigma + \nu^2 Q^*_\sigma Q_\sigma). \] (23)
The quantization of the field equations is now straightforward. We put
\[ P^*_\rho q^*_\rho - q^*_\rho P_\rho = -i\hbar, \quad P^*_\sigma Q^*_\sigma - Q^*_\sigma P_\sigma = -i\hbar \] (24)
and assume all other pairs of quantities to commute.
Following the procedure of Pauli and Weisskopf (1934) we further put
\[ p_\rho = \sqrt{\left(\frac{\epsilon_\rho}{2}\right)}(a^*_\rho + b_\rho), \quad p^*_\rho = \sqrt{\left(\frac{\epsilon_\rho}{2}\right)}(a_\rho + b^*_\rho), \]
\[ q_\rho = \frac{i\hbar}{\sqrt{(2\epsilon_\rho)}}(a_\rho - b^*_\rho), \quad q^*_\rho = \frac{-i\hbar}{\sqrt{(2\epsilon_\rho)}}(a^*_\rho - b_\rho), \]
\[ P_\sigma = \sqrt{\left(\frac{\epsilon_\sigma}{2}\right)}(A^*_\sigma + B_\sigma), \quad P^*_\sigma = \sqrt{\left(\frac{\epsilon_\sigma}{2}\right)}(A_\sigma + B^*_\sigma), \]
\[ Q_\sigma = \frac{i\hbar}{\sqrt{(2\epsilon_\sigma)}}(A_\sigma - B^*_\sigma), \quad Q^*_\sigma = \frac{-i\hbar}{\sqrt{(2\epsilon_\sigma)}}(A^*_\sigma - B_\sigma), \] (25)
where $\epsilon_\rho = \hbar \nu_\rho$ is the energy of the oscillator $\rho$.
Then the total energy becomes
\[ H = \sum \rho \epsilon_\rho(a^*_\rho a_\rho + b^*_\rho b_\rho + 1) + \sum \sigma \epsilon_\sigma(A^*_\sigma A_\sigma + B^*_\sigma B_\sigma + 1) \] (26)
and
\[ a^*_\rho a_\rho - a_\rho a^*_\rho = b^*_\rho b_\rho - b_\rho b^*_\rho = -1, \quad A^*_\sigma A_\sigma - A_\sigma A^*_\sigma = B^*_\sigma B_\sigma - B_\sigma B^*_\sigma = -1. \] (27)
From (27) it follows that the quantities

\[
\begin{align*}
\alpha^\ast \alpha &= N_\rho^+, \quad b^\ast b = N_\rho^-,
\alpha^\ast \alpha &= M_\sigma^+, \quad B^\ast B = M_\sigma^-.
\end{align*}
\]

(28)

all have the eigenvalues 0, 1, 2, ....

We can therefore interpret $N_\rho^\pm$ as the number of "transverse heavy electrons" with charge $\pm e$ in the state $\rho$ and $M_\sigma^\pm$ the number of "longitudinal heavy electrons" with charge $\pm e$ in the state $\sigma$.

The operators $a_\rho$, $a_\rho^\ast$, $b_\rho$, $b_\rho^\ast$, etc., have only matrix elements for transitions in which the number of positive (negative) electrons increases or decreases by one

\[
\begin{align*}
a_\rho^\ast (N_\rho^+ \to N_\rho^+ - 1) &= \sqrt{(N_\rho^+ + 1)}, \\
b_\rho^\ast (N_\rho^- \to N_\rho^- - 1) &= \sqrt{(N_\rho^- + 1)}, \\
a_\rho (N_\rho^+ \to N_\rho^+ + 1) &= \sqrt{N_\rho^+}, \\
b_\rho (N_\rho^- \to N_\rho^- + 1) &= \sqrt{N_\rho^-}.
\end{align*}
\]

(29)

Corresponding equations hold for the $A_\sigma$, ... and $M_\sigma^+$, $M_\sigma^-$. All other matrix elements vanish.

Our next task is to find an interaction between the heavy electrons and the nuclear particles proton and neutron leading to processes of the type (1). These processes are analogous to the emission of light by an electron or the emission of an electron and neutrino in the $\beta$-decay. To find the interaction in question we let ourselves be guided by the theory of light and Fermi's theory of the $\beta$-decay. In this paper we shall deal only with protons and neutrons moving very slowly compared with the velocity of light. We can therefore assume that the heavy particles are at rest at a given position $r_0$, say.

We then assume:

1. The interaction energy $H'$ shall depend only on the wave function of the heavy electron (and its derivative) at the position of the heavy particle.
2. It shall not depend on any variables of the heavy particles other than the spin vector $\sigma$.
3. It shall not contain higher order derivatives of the wave function of the heavy electron.

From relativistic arguments Kemmer (1938) has already deduced all possible interactions $H'$ satisfying the above conditions (cf. equation (63b) of his paper).† For our purpose we have simply to take the non-relativistic parts of this interaction.

† The field equations (2)-(5) and the interaction (30b) have also been obtained by Bhabha (1938).

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There are two independent interactions for the transverse and longitudinal waves:

\[ H'_I = \frac{\theta}{\lambda} \left[ \Pi (\sigma \text{curl} \phi_i) + \Pi^*(\sigma \text{curl} \phi_i^*) \right]_0, \] (30a)

\[ H'_\Pi = \frac{\gamma}{\lambda} \left[ \Pi \text{div} \psi_i + \Pi^* \text{div} \psi_i^* \right]_0, \] (30b)

\[ H' = H'_I + H'_\Pi. \] (30c)

Here \( \Pi \) is the operator transforming a neutron into a proton and is zero when applied to a proton. \( \Pi^* \) is the operator transforming a proton into a neutron and is zero when applied to a neutron. \( \phi_i, \psi_i \) are the wave functions of the heavy electron for the transverse and longitudinal waves respectively (cf. equations (9) and (18)). \( \theta \) and \( \gamma \) are two constants both with dimensions of a charge. We shall determine them later from experimental data. The index "0" in (30) indicates that the expressions in brackets have to be taken at the position of the heavy particle. \( \sigma \) is the Pauli spin vector:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (31)

The interaction (30) leads obviously to processes of the type (1). If we expand \( \phi_i \) and \( \psi_i \) in the form (9) (19) we see from (25) and (29) that \( \phi_i \) or \( \psi_i \) gives rise to an emission of a negative heavy electron or to an absorption of a positive heavy electron and this is, according to (30), connected with the transmutation of a neutron into a proton. The parts proportional to \( \Pi^* \) give rise to the inverse processes.

For later purposes, we finally give the matrix-elements of the interaction \( H' \) for the emission or absorption of a heavy electron represented by a plane wave \( j \exp i(kr) \) by a nuclear particle at the position \( r_0 \). Taking into account equations (8), (9), (19), (25), (29), we find for

Emission of a longitudinal \( Y^+ \) by proton: 

\[- \Pi^* \frac{\theta}{\lambda} \sqrt{\frac{2m^2c^2}{\varepsilon}} e^{-i(kr_0)} \cdot k, \]

Emission of a longitudinal \( Y^- \) by neutron:

\[ \Pi \frac{\theta}{\lambda} \sqrt{\frac{2m^2c^2}{\varepsilon}} e^{i(kr_0)} \cdot k, \]

Emission of a transverse \( Y^+ \) by proton:

\[ \Pi^* \frac{\gamma}{\lambda} \sqrt{\frac{2m^2c^2}{\varepsilon}} e^{-i(kr_0)} (\sigma \cdot [jk]), \]

Emission of a transverse \( Y^- \) by neutron:

\[ - \Pi \frac{\gamma}{\lambda} \sqrt{\frac{2m^2c^2}{\varepsilon}} e^{i(kr_0)} (\sigma \cdot [jk]). \]

The matrix elements for absorption are the complex conjugates to (32).
On the nuclear forces and the magnetic moments of the neutron

In the following sections we apply this theory to the physical properties of the proton and neutron.

2. The neutron-proton force

We first calculate the interaction energy of a proton and a neutron. The two particles are coupled with each other by the mutual emission and reabsorption of a heavy electron. This is accompanied by an exchange of charge between the proton and neutron and the force will, therefore, be an "exchange force". It follows from the perturbation theory that the operator of the interaction energy is given by

\[ V_{NP} = -\sum_{n} \frac{H'_{An}}{E_n - E_A} H'_{nF}. \]

Here \( A \) is the state where a proton is found at the position \( r_1 \) and a neutron at the position \( r_2 \). \( F \) is the state where the proton is at \( r_2 \) and the neutron at \( r_1 \). \( n \) are all possible intermediate states in which a positive or negative heavy electron is emitted by one of the nuclear particles. \( H'_{An} \), etc., are the matrix elements of \( H' \) given by equation (30). If we take for the intermediate states of the heavy electron plane waves with wave number \( k \), the matrix elements are given by (32). \( V_{NP} \) includes contributions from longitudinal and from transverse positive heavy electrons emitted by the proton (and absorbed by the neutron) and also from negative heavy electrons emitted by the neutron. The latter contribution is the same as that of the positive heavy electrons and results in a factor two.

The energy difference \( E_n - E_A \) is just equal to the energy of the heavy electron

\[ \epsilon = \hbar c \sqrt{(k^2 + \lambda^2)}. \]  

(33)

Denoting the operator \( II^*(1) II(2) \) which represents an exchange of charge between the two heavy particles by \( P \), we obtain for the longitudinal contribution of \( V_{NP} \)

\[ V_{II} = -2g^2 \frac{2\pi\hbar^2 c^2 P}{\lambda^2} \sum \frac{k^2 e^{ikr}}{e^2}, \]

(34)

where the sum \( \Sigma \) has to be extended over all values of \( k \). \( r \) is the distance of the two heavy particles \( r = r_1 - r_2 \). The sum can be replaced by an integral taking into account that the number of states with the direction...
of \( \mathbf{k} \) lying in the element of the solid angle \( d\Omega \) and \( k \) in the element \( dk \) is
\[
d\Omega k^2 dk/(2\pi)^3.
\]
Replacing \( k^2 \) by \( (k^2 + \lambda^2) - \lambda^2 \) we obtain, besides an integral
\[
\int \exp i(\mathbf{k}\cdot\mathbf{r}) \, dk = 0.
\]

For the contribution of the transverse waves we obtain in a similar way
\[
V_I = -2\int \frac{f^2}{\lambda^2} \frac{\hbar^2 e^2 P}{(2\pi)^2} \Sigma \frac{(\sigma|j\mathbf{k}|) (\rho|j\mathbf{k}|)}{e^2} e^{i(\mathbf{k}\cdot\mathbf{r})},
\]
where \( \sigma, \rho \) are the spin vectors of the particles 1 and 2 respectively. The \( \Sigma \)
has also to be extended over all directions of polarization \( j \). Carrying out the latter summation we obtain
\[
V_I = -2\int \frac{f^2}{\lambda^2} \frac{P}{(2\pi)^2} \frac{k^4 dk d\Omega}{k^2 + \lambda^2} e^{i(\mathbf{k}\cdot\mathbf{r})} \left[ (\sigma \rho) - \frac{(\sigma \mathbf{k}) (\rho \mathbf{k})}{k^2} \right].
\]

Working out the integral in the same way as above, we obtain finally
\[
V_I = +f^2 P e^{-\lambda r} \left[ (\sigma \rho) \left( 1 + \frac{1}{\lambda r} + \frac{1}{\lambda^2 r^2} \right) - \frac{(\sigma \mathbf{r}) (\rho \mathbf{r})}{r^2} \left( 1 + \frac{3}{\lambda r} + \frac{3}{\lambda^2 r^2} \right) \right].
\]

The total interaction operator is the sum of the two contributions
\[
V = V_I + V_{II}.
\]

In order to apply our result to the actual problem of the proton-neutron interaction we should have to insert \( V \) in the Schrödinger equation for the motion of these two particles. Equation (37) shows that there is a strong coupling between the orbital motion and the spins \( \sigma, \rho \). This makes an exact solution of the Schrödinger equation very difficult. The operator \( V_{NP} \) leads only to a coupling between \( S \) and \( D \)-terms but not between \( S \) and \( P \)-terms. Since the range of the forces is so small that the \( D \)-term can be neglected it will be sufficient, for a rough physical discussion, to separate the orbital motion and the spin and to take the average value of the spin operators occurring in (37).

For the wave function of the deuteron we assume then a product of three functions
\[
\Psi = \psi_1(1, 2) T(1, 2) U(1, 2).
\]

Here \( \psi_1(1, 2) \) depends only on the space coordinates of the two particles 1, 2, \( T(1, 2) \) is the spin function. \( U(1, 2) \) is the wave function describing which of the two particles, one and two, is a proton and which a neutron. Each
of these three functions can be symmetrical or antisymmetrical in 1 and 2, but \( \Psi \) must be antisymmetrical. Thus there are four possibilities:

- **Triplet:**
  - (1) \( T \text{ sym.}, \psi_1 \text{ sym.}, U \text{ antis.} \)
  - (2) \( T \text{ sym.}, \psi_1 \text{ antis.}, U \text{ sym.} \)

- **Singlet:**
  - (3) \( T \text{ antis.}, \psi_1 \text{ sym.}, U \text{ sym.} \)
  - (4) \( T \text{ antis.}, \psi_1 \text{ antis.}, U \text{ antis.} \)

We confine ourselves to the consideration of S terms. Then \( \psi_1 \) is symmetrical in the two particles, and we have only to consider the wave functions (1) and (3) of (40).

The operator \( P \) of equations (35) and (37) acts only on the wave function \( U \) and is

\[
\begin{align*}
PU\text{ sym.} &= + U\text{ sym.} \quad \text{(41)} \\
PU\text{ antis.} &= - U\text{ antis.}
\end{align*}
\]

The operators \( (\sigma \varphi) \) and \( (\sigma r)(\rho r)/r^2 \) act on \( T \). \( T \) has the following form:

- **Singlet:** \( \alpha(1)\beta(2) - \alpha(2)\beta(1) \),
  \[ (42a) \]
- **Triplet:** \( \alpha(1)\alpha(2) \),
  \[ \beta(1)\beta(2), \]
  \[ \alpha(1)\beta(2) + \alpha(2)\beta(1), \]  
  \[ (42b) \]

\( \alpha, \beta \) are the two spin functions for each particle. Since the S-term is spherically symmetrical the average of the spin operators over all directions results, for the \( ^3\text{S} \)-term, in taking the average over all three spin functions (42b). We then obtain

- **Singlet:** \( (\sigma \varphi)T = -3T, \)
  \( (\sigma r)(\rho r)/r^2T = -T, \)

- **Triplet:** \( (\sigma \varphi)T = +T, \)
  \( (\sigma r)(\rho r)/r^2T = \frac{1}{3}T. \)

Inserting now (41) and (43) into the expression for \( V_{NP} \) (35) and (37) taking into account the symmetry properties of the wave functions (40) we obtain for the mean value of \( V_{NP} \):

\[
\begin{align*}
^3\text{S} : V_{NP} &= -\frac{e^{-\lambda r}}{r} \left( g^2 + \frac{2}{3} f^2 \right), \\
^1\text{S} : V_{NP} &= -\frac{e^{-\lambda r}}{r} \left( 2f^2 - g^2 \right). \quad (44)
\end{align*}
\]
Thus we find that the potential for the $^3S$-term is always attractive. For the $^3S$ term $V$ is attractive if $2f^2 > g^2$. The experiments suggest that the proton-neutron potential in the lowest state $^3S$ can be represented by a hole of width $3 \times 10^{-13}$ cm. and a depth of $25 \times 10^6$ e-volts. For the $^1S$-state the potential is approximately half of that of the $^3S$-state. Hence we obtain at once

$$f \approx g.$$  \hspace{1cm} (45)

The range of the nuclear forces is $1/\lambda = m_0 c/\hbar$. The mass of the heavy electron will therefore be

$$m_0 = 100 \text{ electron masses.}$$ \hspace{1cm} (46)

As "depth" of the potential hole we can define the quantity $(g^2 + \frac{4}{3}f^2)\lambda$. Taking into account equation (45) we find for $g$ and $f$ (or for the dimensionless quantity $g^2/\hbar c$)

$$g = f = 5 \text{ electron charges, } \frac{g^2}{\hbar c} = \frac{1}{8}.$$ \hspace{1cm} (47)

All these figures are of course only very rough.

We have found that our scheme makes it possible to account in a reasonable way for the nuclear forces, including their right spin dependence. Although in our theory three constants are available, this success is by no means trivial. In the scalar theory (Yukawa and Sakata 1937) it turns out, for instance, that the $^3S$-state is always repulsive and the $^1S$-state attractive with the same absolute value, which is contrary to the experiments. Further the scalar theory does not allow for the magnetic moments of the proton and neutron. Only the vector scheme leads to a qualitative agreement with the experiments.

3. The Proton-Proton Interaction

From scattering experiments (Tuve, Heydenburg and Hafstadt 1936; Breit, Condon and Present 1936) it is known that there are also short range forces between two protons which seem to be of the same order of magnitude as the neutron-proton forces. Although the experiments do not allow any conclusions as to the exact form and magnitude of these forces it has been suggested by Breit, Condon and Present that the forces between all nuclear particles are exactly identical.

In our theory we have so far only assumed an interaction by which charged particles are emitted by a proton or neutron with the corresponding change of charge of the latter particles. It is obvious that this interaction
On the nuclear forces and the magnetic moments of the neutron

leads to a proton-proton force only in the fourth order of approximation. Although we shall see that the latter is not at all small, it is, of course, not to be expected that it is exactly the same as the neutron-proton force. It would, however, be possible to obtain the same forces irrespective of the charge of the nuclear particles if we would introduce a neutral particle of the same mass as that of the heavy electron, with a similar interaction with the nuclear particles to that given by equations (30).

It is quite possible that those particles exist. The cosmic-ray experiments, however, do not give any information yet about the existence of these particles, neither in the affirmative nor in the negative sense. Since the exact equality of the nuclear forces is by no means established experimentally, it seems in any case desirable to work out the consequences of our theory for the proton-proton forces in the fourth order of approximation without introducing neutral particles.

In the preceding section we have seen that the constant responsible for the nuclear forces, \( g^2 \) (or \( f^2 \)) is of the order of \( 25\alpha^2 \). The second, fourth ... etc., orders of approximation represent an expansion in a power series of \( g^2/\hbar c = 1/6 \) which is by no means small compared with unity. Thus the proton-proton forces will at least be comparable with the proton-neutron forces.

The perturbation energy is in the fourth approximation given by the formula

\[
V_{pp} = -\sum \frac{H'_{A1}H'_{II}H'_{III}H'_{IVA}}{E_I E_{II} E_{III}}. \tag{48}
\]

Here I, II, III are intermediate states in which one or two heavy electrons are emitted and \( H' \) is the corresponding matrix element of (30). The intermediate states are, for instance, the following:

(i) Proton 1 has emitted a positive heavy electron with wave number \( k_1 \).

\[
E_1^2 = \epsilon_1^2 = \hbar^2 c^2(k_1^2 + \lambda^2).
\]

(ii) Proton 2 has emitted a positive heavy electron with wave number \( k_2 \).

\[
E_{II} = \epsilon_1 + \epsilon_2. \quad \epsilon_2^2 = \hbar^2 c^2(k_1^2 + \lambda^2).
\]

(iii) The positive electron \( k_1 \) is absorbed by proton 2.

\[
E_{III} = \epsilon_2.
\]

The order of emission and absorption can be permuted for each \( k_1, k_2 \), and we obtain in this way four different sets of intermediate states. The matrix elements do not depend on this order.
There is another set of intermediate states in which a proton emits subsequently a positive and a negative heavy electron which are then absorbed by the second proton. The denominator in (48) is for this intermediate state
\[ \epsilon^2_2 (\epsilon_+ + \epsilon_-). \]

This intermediate state occurs twice since the roles of the two protons can be interchanged. The matrix elements are the same as for the first group of intermediate states. Writing \( \epsilon_1, \epsilon_2 \) (or \( \epsilon_2, \epsilon_1 \)) instead of \( \epsilon_+, \epsilon_- \) we obtain for the sum over all these intermediate states
\[
\frac{2}{\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2)} + \frac{1}{\epsilon_1^2 (\epsilon_1 + \epsilon_2)} + \frac{1}{\epsilon_2^2 (\epsilon_1 + \epsilon_2)} + \frac{2}{\epsilon_+^2 (\epsilon_+ + \epsilon_-)} = \frac{2 \epsilon_1 \epsilon_2}{\epsilon_1^2 - \epsilon_2^2} \left( \frac{1}{\epsilon_1^2} - \frac{1}{\epsilon_2^2} \right).
\]

Since the expression (48) is symmetrical in \( \epsilon_1, \epsilon_2 \) we obtain for \( V_{PP} \)
\[
V_{PP} = -4 \sum \epsilon_1 \epsilon_2 \frac{H'_{A1} H'_{IIIII} H'_{IIIIA}}{(\epsilon_1^2 - \epsilon_2^2) \epsilon_2^3},
\]
where the sum has now to be extended over all directions and values of the two wave vectors \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \), and all directions of polarizations. The matrix elements \( H'_{A1} \), etc., have to be inserted from equation (32). For \( H'_{A1} H'_{IIIII} \) and for \( H'_{IIII} H'_{IIIIA} \) we obtain very similar expressions as those obtained for the neutron-proton force in the preceding section. After carrying out the summation over all directions of polarization we obtain
\[
H'_{A1} H'_{IIIII} = \frac{2 \pi \lambda^2 c^2}{\epsilon_1 \lambda^2} \left( g^2 k_1^2 + f^2 [k_2^2 (\sigma \cdot \mathbf{k}_1) - (\sigma \cdot \mathbf{k}_1)] \right) e^{i(k_1 \cdot r)}. \tag{50}
\]
Here \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) is the radius vector of the distance between the two protons. \( H'_{IIII} H'_{IIIIA} \) is obtained by replacing \( \mathbf{k}_1 \) by \( \mathbf{k}_2 \) and \( \mathbf{r} \) by \( -\mathbf{r} \).

In all physical problems it is sufficient to consider the interaction of two protons only in the S-state because the range of the forces is very small. Our further calculations will be very much simplified if we confine ourselves to the evaluation of \( V_{PP} \) for the 1S-state. In this case the spin operators occurring in (50) are
\[
(\sigma \cdot \mathbf{k}_1) = -3, \quad (\sigma \cdot \mathbf{k}_1) (\sigma \cdot \mathbf{k}_1) = -k_1^2.
\]

For \( V_{PP} \) we thus obtain
\[
V_{PP} = -4 \left( \frac{2 \pi k^2 c^2}{\lambda^2} \right)^2 (g^2 - 2f^2)^2 \sum \frac{k_2^2 \epsilon^2_2 e^{i(k_1 \cdot r)} e^{i(k_3 \cdot r)}}{(\epsilon_1^2 - \epsilon_2^2) \epsilon_2^3}.
\]
Replacing the sum by an integral and carrying out the integration over the angles the sum becomes

$$
\Sigma = \frac{1}{4\pi^4(\hbar c)^5 r^2} \int_0^\infty \frac{k_1^2 k_2^2 \sin k_1 r \sin k_2 r \, dk_1 \, dk_2}{(k_1^2 - k_2^2) (k_2^2 + \lambda^2)^2}.
$$

If we put $k_1 r = x$, $k_2 r = y$ the integration over $x$ can easily be performed and yields for $V_{PP}$:

$$
V_{PP} = -\frac{2}{\pi \hbar c} \frac{(g^2 - 2f^2)^2}{\lambda^4 r^5} \int_0^\infty \frac{y^5 \sin y \cos y \, dy}{(y^2 + \lambda^2 r^2)^2}.
$$

The integral in (51) leads to Hankel functions. It can be reduced to the integral

$$
\int_0^\infty \frac{x \sin ax \, dx}{(x^2 + \beta^2)^{3/2}} = \frac{\pi}{2} \alpha H_\alpha^{(1)}(i \beta x)
$$

and its derivatives. Using $2 \sin y \cos y = \sin 2y$ and the well known relations between Hankel functions we obtain finally

$$
V_{PP} = -\frac{(g^2 - 2f^2)^2}{\hbar cr} \left[ \left(1 - \frac{1}{(2\lambda r)^2}\right) i H_{\lambda}^{(1)}(2i\lambda r) + \left(1 + \frac{1}{(2\lambda r)^2}\right) \frac{H_{\lambda}^{(1)}(2i\lambda r)}{\lambda r} \right].
$$

We compare this formula with the neutron-proton force (equation (44)) in the $^1S$-state and find, since $f \equiv g$

$$
V_{PP} = V_{NP} \frac{g^2}{\hbar c} e^{\lambda r} \times \text{[Hankel functions].}
$$

$g^2/\hbar c$ is of the order $1/6$. To discuss the factor containing the Hankel functions we give the values of this factor for a few values of $\lambda r$:

<table>
<thead>
<tr>
<th>$\lambda r$</th>
<th>$e^{\lambda r} \times \text{[Hankel functions]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>900</td>
</tr>
<tr>
<td>0.2</td>
<td>70</td>
</tr>
<tr>
<td>0.5</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Since $V_{NP}$ is attractive we find that the proton-proton force is always repulsive. For very small distances the absolute value of the force is very much larger than that of the proton-neutron force and decreases very rapidly with $r$. Thus the protons behave nearly like two rigid spheres with a diameter $d$ of about

$$
d = 1/2\lambda.
$$

The fact that the fourth order of approximation is greater than the second order for small distances means, of course, that the whole theory

† Cf. Jahnke-Emde (1933, pp. 286, 287).
diverges for small distances and our results can therefore only have a very qualitative significance. As a main result of our theory we can only say that the nuclear particles will have a finite radius of the order of equation (54). In this connexion we want to emphasize that the fourth order of the neutron-proton forces does not lead to any exchange force. The only way of exchanging two heavy electrons between a proton and a neutron is the successive emission of a positive and negative heavy electron by the proton or neutron and re-absorption by the other particle. It does obviously not lead to any exchange of charge. It would lead to a repulsion at very small distances of about half of the proton-proton repulsion.

The proton-proton force, as calculated in this section, leads to a strong scattering of protons by protons. It is not possible to say without detailed calculations whether the scattering due to these forces is of the order of magnitude observed experimentally. The authors mentioned above, however, concluded from the experimental data that the proton-proton force is mainly attractive. If this result is established it would inevitably lead to the conclusion, that we have also to introduce neutral particles ("Neutretto's") of the same mass as and similar properties to those of the heavy electrons. The theory of these particles can be developed on the same lines as for the charged particles. We shall not, however, discuss this question in detail here. We want to emphasize that the forces calculated in this section will exist in any case. The proton-proton force can therefore only be attractive for distances larger than $1/2 \lambda$ and will in any case be repulsive for smaller distances.

We may also mention that for a nucleus composed of three or more particles, forces also exist corresponding to a "triple exchange" of a heavy electron between three of the nuclear particles. These forces are of the same order as the fourth order proton-proton forces considered above.

4. The magnetic moment

In this section we show that the virtual emission of a heavy electron leads to an additional magnetic moment of the proton and the neutron. To calculate this additional moment we proceed in the following way: We consider a proton placed in a weak homogeneous magnetic field $H$ with spin momentum $+\frac{1}{2}$ in the direction of the magnetic field. The virtual emission and reabsorption of the heavy electron gives rise to a certain self-energy $W$ depending on the magnetic field $H$. Expanding $W$ in a power series of $H$

$$W = W_0 + \mu' H + \ldots,$$  (55)
the factor of $H$ gives the additional moment of the proton. Besides this surplus moment we shall assume the ordinary moment of one Bohr nuclear magneton $\mu_0$ for the proton.

It follows immediately from the interaction (30) that only the term $H'_1$ can give a contribution to the magnetic moment. $H'_{11}$ does not contain the spin $\sigma$ and cannot, therefore, give rise to a magnetic moment connected with the direction of the spin. In fact it would turn out that the term proportional to $H$ in (55) is precisely equal to zero.

The magnetic moment, therefore, is due to the emission and reabsorption of transverse waves.

Since we are interested in the magnetic energy of the emitted heavy electron, it will be convenient to introduce polar co-ordinates $r, \theta, \phi$, and to expand the transverse waves into spherical waves with the polar axis in the direction of the field (z-axis). These waves can be classified according to their angular momentum $m$ about the z-axis. Only waves with $m \neq 0$ will have a magnetic energy and therefore contribute to the magnetic moment. This expansion problem is almost identical with the expansion of electromagnetic waves into spherical waves in Maxwell’s theory. The only difference is the fact that the frequency is now connected with the wave number by equation (11) whereas in Maxwell’s theory $\nu = kc$. The latter problem has been treated already (Heitler 1936b). Our $\phi_i$ corresponds to the vector potential $A$ and curl $\phi_i$ to the magnetic field strength $\text{curl} A$. For our interaction $H'_1$ we need curl $\phi$ at the position of the proton ($r=0$) which therefore corresponds to the magnetic field strength for $r=0$.

The spherical electromagnetic waves can be divided into two groups representing an electrical and magnetic multipole radiation. Each of them is characterized by two integral “quantum numbers” $l$ and $m$, where

$$-l \leq m \leq l, \quad l = 0, 1, 2, \ldots$$

$m$ is proportional to the angular momentum about the z-axis. The expressions for the magnetic field strength for all these cases are explicitly given in the paper mentioned above. It turns out, however, that for $m \neq 0$ the magnetic field strength of the electrical multipole waves vanishes for $r=0$. For the magnetic multipole wave the magnetic field strength is given by the formulae (28) and (29) of the paper quoted above:

$$\text{curl}_x \phi \pm i \text{curl}_y \phi = \pm i e^{i\nu t} q \left[ \frac{l(l+m+2)(l+m+1)}{(l+1)(2l+1)(2l+3)} \right]^\frac{1}{2} Y^{m+1}_{l+1} f_{l+1}$$

$$- \left[ \frac{(l+1)(l+m-1)(l+m)}{l(2l+1)(2l-1)} \right]^\frac{1}{2} Y^{m-1}_{l-1} f_{l-1} \right], \quad (56)$$
where $Y_l^m$ are the normalized spherical harmonics, and

$$f_l = C J_{l+1}^2(kr)$$

$J_{l+1}$ are Bessel functions, $C$ is a constant which has to be taken in such a way that $f_l$ is a normalized wave function. If this function is normalized for a sphere with radius 1, then

$$C = k\sqrt{(8\pi c^2)}.$$

In formula (56) $q$ is the quantized amplitude of the oscillator representing the heavy electron (cf. equation (25)). We have written down only the $x$- and $y$-components. The $z$-components consist of terms proportional to $Y_{l+1}^m f_{l+1}$ and $Y_{l-1}^m f_{l-1}$ and will be seen to give no contribution to the magnetic moment.

For $r = 0$ all $f_l(0)$ vanish except $f_0(0)$. Thus it follows from (56) that only the wave with $l = 1$, $m = \pm 1$ gives a contribution. We then obtain for $r = 0$:

$$m = +1: \text{curl}_x \phi + i \text{curl}_y \phi = 0,
\text{curl}_x \phi - i \text{curl}_y \phi = i \frac{2}{\sqrt{3}} kq Y_0^0 f_0(0),$$

$$m = -1: \text{curl}_x \phi + i \text{curl}_y \phi = -i \frac{2}{\sqrt{3}} kq Y_0^0 f_0(0),
\text{curl}_x \phi - i \text{curl}_y \phi = 0.$$ (57)

The interaction $H'_I$ can be written in the following form:

$$(\sigma \text{curl} \phi) = \sigma_x \text{curl}_x \phi + \frac{1}{2}(\sigma_x + i\sigma_y) (\text{curl}_x - i \text{curl}_y) \phi + \frac{1}{2}(\sigma_x - i\sigma_y) (\text{curl}_x + i \text{curl}_y) \phi.$$ $$(\sigma \text{curl} \phi) = \sigma_x \text{curl}_x \phi + \frac{1}{2}(\sigma_x + i\sigma_y) (\text{curl}_x - i \text{curl}_y) \phi + \frac{1}{2}(\sigma_x - i\sigma_y) (\text{curl}_x + i \text{curl}_y) \phi.$$ (58)

According to (31) we have

$$\frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ (58)

Thus it follows, together with (25), (29), (57), that a proton with spin $\frac{1}{2}$ can emit only a positive electron with angular momentum $+1$, transforming itself into a neutron with spin $-\frac{1}{2}$ and vice versa. There are, of course, also transitions in which a heavy electron with angular momentum $m = 0$ is emitted but, as mentioned above, we are not interested in them here.

$\dagger$ As in equation (8) we normalize $Y_l^m f_l$ to $4\pi c^2$. 

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The self energy is given by the well-known formula
\[ W = -\sum^{\infty}_{n=1} \frac{\hbar^2 c^2}{2m} \left( k^2 + \lambda^2 \right) + \frac{e^2 m \hbar}{\hbar c}, \] (59)
where \( E_n \) is the energy of the emitted heavy electron in the magnetic field and \( E_0 \) is the energy in the initial state, i.e., the energy of the Dirac proton in the magnetic field. For the present we are interested in the contribution of the heavy electron to the magnetic energy and shall neglect \( E_0 \). We shall consider the effect of the Bohr magneton of the proton below.

The energy \( E_n \) is given by\( \dagger \)
\[ E_n^2 = \hbar^2 c^2 \left( k^2 + \lambda^2 \right) - \frac{e^2 m \hbar}{\hbar c}, \] (60)
neglecting terms in \( \hbar^2 \). We insert in (59) \( H_{0n} \) from equations (30a) and (57) and \( q \) from (25). In \( H_{n0} \) the matrices \( \sigma_x + i\sigma_y \), etc., have to be replaced by their hermitian conjugates. Furthermore the number of states in the interval \( dk \) of the wave number \( k \) is \( dk/\pi \). Replacing the sum by an integral and inserting \( Y_0^0 = 1/\sqrt{(4\pi)} \) and \( f_0(0) = k/(8\pi c^2) \) we obtain for the self-energy
\[ W = -\frac{4}{3\pi} \int \frac{k^4 dk}{\lambda^2} = W_{0} + \frac{4}{3\pi} \frac{e^2 \hbar}{\lambda^2 c} \int_{0}^{\infty} \frac{k^4 dk}{(k^2 + \lambda^2)^2}. \] (61)
The factor of \( H \) represents the surplus moment of the proton. The sign is the same as for a positive charge with positive angular momentum. The integral diverges for large values of the wave number \( k \). This shows that our theory cannot be valid for all wave-lengths. We have to introduce an upper limit \( k_0 \) for the wave number \( k \) marking the limit of validity of our theory. \( 1/k_0 \) can hardly be other than of nuclear dimensions. We have already seen

\( \dagger \) It may be considered doubtful whether the magnetic energy is really given by \( m\mu H \) (\( \mu = \) relativistic magneton) as it is the case for the relativistic particle with no spin, described, for instance, by the scalar Klein-Gordon equation. Strictly speaking, in order to calculate the magnetic energy, we should require the field equations of the heavy electron in an external field. Our field equations can easily be generalized for this case.

A consistent canonical scheme has been proposed by Proca (1936). We have calculated the magnetic moment also by a consequent application of the Proca equations with Born's approximation, and have obtained the same result as in the text, apart from a factor 2 which arises from heavy electron pairs created by the magnetic field (cf. Kemmer 1938). This factor 2 does not play any role since the absolute value of the magnetic moment depends in any case on the quantity \( k_0 \) (see equation (63)) of which only the order of magnitude is defined. We have preferred to deduce the magnetic moment by the method used in the text because it is far more simple than the rigorous application of the Proca equations.
in our calculations of the nuclear forces (§ 3) that the higher orders of approximation diverge for distances $> 1/(2\lambda)$. We had to conclude that our theory, with the form (30) for the Hamiltonian, is restricted to distances $> 1/(2\lambda)$. Thus we think it reasonable to assume that the limit of validity of the theory for the magnetic moment is the same as for the nuclear forces and we shall therefore put

$$k_0 \text{ of the order } \lambda.$$  

(62)

It will be seen that our theory leads then to the right order of magnitude of the surplus magnetic moment of the proton.

Writing for the integral simply $k_0$ we obtain for the surplus moment (dividing by the Bohr nuclear moment)

$$\frac{\mu'}{\mu_0} = \frac{8 f^2 M k_0}{3\pi \lambda^2 h^2} = \frac{8 f^2 M k_0}{3\pi h c m_0 \lambda}.$$  

(63)

If we insert for the value obtained from the nuclear forces (§ 2) $f^2 h c = 1/6$ and for $m_0$ 150 electron masses we obtain $\mu'/\mu_0 = 2k_0/\lambda$. This is of the right order of magnitude if $k_0 \sim \lambda$ as we have suggested above.

The total magnetic moment of the proton will be

$$\mu_p = \mu' + \mu_0.$$  

For the neutron the calculation is entirely analogous. The sign is the opposite because a negative electron is emitted. Therefore

$$\mu_n = -\mu'.$$

It would follow from these considerations that $\mu_p - |\mu_n| = \mu_0$. This is approximately the case since the magnetic moment of the deuteron is $\mu_d = \mu_p - |\mu_n| = 0.85 \mu_0$. The departure of this figure from $1 \times \mu_0$ can be explained by taking into account the magnetic energy of the Dirac proton $E_0 = \mu_0 H$ in equation (59) which we have so far neglected. For the neutron a similar term has to be taken into account since we have in the intermediate state a proton with spin direction $-\frac{1}{2}$ leading to a magnetic energy

$$-E_0 = -\mu_0 H.$$  

Similar terms would even occur in transitions to electronic states with $m = 0$ since also in this case† the proton with magnetic energy $+\mu_0 H$ is transformed into a neutron with no magnetic energy (correspondingly a

† Transitions of this kind will also be caused by the interaction with longitudinal waves, equation (30b).
neutron with no magnetic energy is transformed into a proton with magnetic energy \(+\mu_0 H\). We can take all this into account in the following simple way:

Let \(\alpha\) be the fraction of time which a proton (neutron) spends as a neutron (proton) with opposite spin direction and a positive (negative) heavy electron with angular momentum \(+1\). Let \(\beta\) be the fraction of time which a proton (neutron) spends as a neutron (proton) with the same spin direction and a positive (negative) electron with angular momentum 0. Then the magnetic moments of the proton and the neutron are given by the following formulae:

\[
\begin{align*}
\mu_P &= 1 - \alpha - \beta + \alpha \frac{M}{m^*} = 1 - \alpha + \alpha \left( \frac{M}{m^*} - \frac{\beta}{\alpha} \right), \\
\mu_N &= -\alpha + \beta - \alpha \frac{M}{m^*} = -\alpha - \alpha \left( \frac{M}{m^*} - \frac{\beta}{\alpha} \right).
\end{align*}
\]

(M is the mass of the proton, \(m^*\) is a certain mean value of the relativistic mass of the heavy electron. As can be seen from our preceding calculations \(m^*\) is given by

\[
\frac{k}{m^* c} = \int_0^{k_0} \frac{k^4 dk}{(k^2 + \lambda^2)^{\frac{3}{2}}}.
\]

Since \(k_0 \sim \lambda\), \(m_0\) differs from the rest mass only by a factor of the order of magnitude one. \(\beta\) is of the same order of magnitude as \(\alpha\), and the terms \(\beta/\alpha\) can therefore be neglected. In this form equations (64) have already been deduced by Fröhlich and Heitler (1938) and have been used to calculate the mass of the heavy electron. Inserting the observed values of the magnetic moments, we obtain \(m^* \approx 80\) electron masses, which is in good agreement with the mass obtained from the range of the nuclear forces (§ 2). It is very satisfactory that the masses deduced from two entirely different experimental sets of data agree with each other. We think that this agreement really suggests that the nuclear properties are closely connected with the existence of the heavy electron.

We want to emphasize that our last considerations concerning the determination of \(m^*\) from \(\mu_P\) and \(\mu_N\) are actually independent of the special form of our theory. This might be important in view of the fact that the value of the magnetic moment actually diverges. There are also other diverging quantities in this theory. The proton has, for instance, a self-energy given by \(W_0\) equation (61), which would diverge if we would integrate up to
infinite values of $k$ (cf. Kemmer 1938). If we cut off at wave numbers $k_0$ the relativistic invariance is destroyed. This shows that the whole scheme we have applied is very incomplete and unsatisfactory in the same way as the theory of radiation. On the other hand, the success with which we could apply this scheme to the nuclear properties suggests that there is a limited field of validity for this special form of our theory.

[Note added in proof.] Prof. L. Farkas has kindly communicated to us the results of his unpublished measurements of the ratio $\mu_D/\mu_F$ which is found to have the value 3.8. Using for $\mu_F$ the recent value found by Estermann, Simpson and Stern (Phys. Rev. 52, 535) $\mu_F = 2.46 \mu_0$ we obtain $\mu_N = -1.80$. This leads to a value of $m^* = 180$ electron masses which is in even better agreement with the value deduced from the range of the neutron-proton force.

Dr Nagendra Nath has further kindly drawn our attention to the fact that in equations (64) the recoil momentum of the nuclear particle due to the emission of the heavy electron should be taken into account. A simple calculation shows that equations (64) have then to be replaced by (neglecting $\beta$)

$$\mu_F = 1 - \alpha + \alpha \frac{M}{m^*} \frac{M}{M + m^*},$$

$$\mu_N = -\alpha \frac{M}{m^*}.$$

The value of $m^*$ is almost exactly the same as above.]

**SUMMARY**

An attempt is made to explain the properties of the nuclear particles proton and neutron by the hypothesis that these particles are capable of emitting a positive or negative "heavy electron" respectively with a rest mass $m_0$ between that of the proton and the electron. The existence of these particles has been made probable by cosmic ray observations.

The heavy electrons are assumed to have integral spin and satisfy Bose statistics. The wave functions of these particles are assumed to be of vectorial character, the components of which satisfy the Klein-Gordon equation. They are quantized according to the scheme given by Pauli and Weisskopf. Thus, a free heavy electron can exist in three different states of polarization; there are two transverse and one longitudinal wave with given

$$f^2 \int_0^{k_0} k^2 dk$$

which is small compared with $Mc^2$.

We should like to emphasize that the neutron-proton force is not appreciably influenced by this cutting-off at wave numbers $k_0$.\[† The self-energy is then of the order $\frac{f^2}{A^2} \int_0^{k_0} k^2 dk$ which is small compared with $Mc^2$.\]
momentum. The interaction with the nuclear particles is found by relativistic arguments and contains—apart from the mass $m_0$—two arbitrary constants $g$ and $f$, both with dimensions of an electric charge.

With this scheme we have calculated:

(1) The neutron-proton force. It is an exchange force and has a range $1/\lambda = \hbar/m_0 c$. In the $^3S$-state the force is always attractive. (This would not be the case for a scalar wave function.) $g$ and $f$ can be chosen so that the $^3S$ and $^1S$-states have the right position. The experiments suggest $g = f = 5$ electron charges.

(2) The magnetic moments $\mu_P$ and $\mu_N$ of the proton and neutron. They are found to be of nuclear dimensions and have the right sign.

(3) The mass $m_0$ can be determined independently from the range of the neutron-proton force and from the magnetic moments. In both ways we find $m_0 = 100$ electron masses.

(4) The proton-proton force is obtained only in the fourth order of approximation and leads to a strong repulsion for distances less than $1/(2\lambda)$. Attraction and equality with the neutron-proton force could be attained by introducing also neutral particles with mass $m_0$.

(5) The theory leads to a diverging self-energy of the proton and neutron.

REFERENCES


