ELECTRIC CONFINEMENT AND MAGNETIC SUPERCONDUCTORS

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Reinterpretation of the pioneering work of Nielsen and Olesen and others on magnetic confinement in (electric) superconductors leads to simple models which exhibit electric confinement of the type hoped for in quantum chromodynamics. "Electric" flux tubes joining "quarks" arise at a semi-classical level as a result of the existence of (pseudo) particles. The theory can be described in terms of a classical action principle only upon suitable modification of the action; this modification has profound topological significance as it is intimately related to the quantization condition for (pseudo) particles.

1. Introduction

The hypothesis of quark confinement is an attempt to reconcile the conspicuous absence of quark production in high-energy collisions with the plausible identification of partons as quarks.

A correct theory of confinement should provide a simple explanation of the linearly rising Regge trajectories ***. In fact, the rotational and vibrational modes of a confined tube with constant energy per unit length do exhibit the expected Regge behaviour in the semi-classical limit; this is illustrated by dual-string models * Therefore one is tempted to look for confinement schemes which, at least at high energy, can be depicted semi-classically by a flux-tube structure.

Such a linear structure emerges naturally in models of magnetic confinement based on the superconductor analogy [3]. In the Abelian model with colour gauge group U(1) [4], "quarks" are identified with Dirac monopoles [5] of magnetic charge $g$ satisfying the Dirac quantization condition $eg = 2\pi$; here $e$ is the charge of the condensed particles. Each quark carries with it a singly quantized flux tube $\Phi = g$ and hence has infinite energy except if the tube is absorbed by an antiquark

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*** Much of the material of the present paper has been reviewed by one of us (F.E.) in [1].
* For a discussion on dual strings, see for instance [2].
to produce a meson. A similar meson structure can be obtained from a non-Abelian
colour group SU(3)/Z3, in which case confined baryons also appear as a conse-
quence of the topological properties of monopoles and flux tubes in non-Abelian
gauge theories [6]. Magnetic schemes for confinement suffer however from two
defects. Firstly, Dirac monopoles are marred by theoretical ambiguities both
because of constraint conditions [5] and because of the present lack of a consis-
tent second-quantized field theory. Constructions using only regular 't Hooft-
Polyakov monopoles [7] can be devised at the expense of additional complications
but such constructions do not share the nice topological features of the models
based on Dirac monopoles [1]. Secondly, the monopole interpretation of quarks
does not shed any light on their quasi-free behaviour in the scaling region.

These shortcomings would be absent if confinement could be realized through
quantum chromodynamics (QCD). There, quark fields, minimally coupled to the
non-Abelian colour gauge group SU(3), are expected to bind permanently in colour
singlets; asymptotic freedom would then yield the required quasi-free small-distance
behaviour. Thus, such an “electric” confinement scheme might be favored despite
the fact that the emergence of flux tubes characterizing magnetic confinement is
not immediately apparent in QCD.

The physical analogy between magnetic confinement and electric confinement in
QCD is however suggested by renormalization-group arguments. The effective charge
felt by two “electric” quarks separated by a distance r is affected by the dielectric
properties of the vacuum. While, for distances r smaller than any quark mass, quark-
antiquark pairs tend to screen the charge, the Yang-Mills fields give rise to antiscreen-
ing; therefore, as long as not too many quarks are present, the net result is that the
effective charge increases with distance at least at small distances. In term of the
renormalized charge $e(\lambda)$ where $\lambda$ is a momentum of order $r^{-1}$, this means that
$e(\lambda) \to 0$ for $\lambda \to \infty$ so that the theory is asymptotically free [8]. If such a behaviour
can be extrapolated to the low energy momentum region without leading to a break-
down of the colour symmetry, it would imply an effective dielectric constant of the
vacuum $e(\lambda)$ which would tend to zero at large distances, that is for $\lambda \to 0$. This
property of $e(\lambda)$ suggests that the vacuum exhibits an “electric” Meissner effect [9]
because the displacement field $D$ due to an “electric” colour source cannot radiate
into the vacuum despite the fact that $\nabla \cdot D = 0$ outside the sources. Such a situation
is analogous to that of an Abelian superconductor; there the magnetic permeability
$\mu(\lambda)$ tends rapidly to zero when $\lambda^{-1}$ exceeds $m_{\gamma}^{-1}$ where $m_{\gamma}$ is the mass acquired by
the gauge field. Magnetic confinement then follows from $\nabla \cdot B = 0$ outside the poles.
Thus, except for off-diagonal configurations in group space, electric and magnetic
confinement are expected to be quite similar with electric, $E$, and magnetic, $H$, fields
interchanged. Of course these off-diagonal configurations must ultimately be taken
into account as the full global colour symmetry is essential to ensure that colour
singlets and not only colour neutral objects are confined. Also, in the limit of vanish-
ing quark mass, the confinement scale is not the arbitrary length $m_{\gamma}^{-1}$ but should be
related to the hadron mass scale through dimensional transmutation [10]. Neverth-
less, a simple semi-classical description of electric confinement in terms of electric flux tubes joining quarks should be available as in the magnetic case and such a description might be useful to construct a correct dual-string model in a semi-classical limit.

However, this semi-classical description does not follow in a straightforward way from the mathematical framework of gauge field theory. In specific models, the difficulty will be traced back to the fact that the quantum action of a gauge field may not be applicable in the classical limit because of topological effects. Once this is cured, the comparison between electric and magnetic confinement can be done in semi-classical terms. One is then led to simple physical pictures whether electric confinement follows from particle motion or from instanton-like effects. These two cases are illustrated in sects. 2 and 3; it is our hope that our analysis may eventually be extended to QCD and therefore be of some help in clarifying the nature of quark confinement.

2. Electric confinement from particle motion: the magnetic superconductor

2.1. The relativistic superconductor

A relativistic model which shares all the static magnetic properties of a superconductor can be constructed by taking the Lorentz-covariant generalization of the Ginzburg-Landau \[11a\] Lagrangian density. We shall review here the main features of this model which are discussed in refs. \[3,4\].

The Lagrangian is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + [(\partial_{\mu} + ie A_{\mu}) \phi^*] [(\partial^\mu - ie A^\mu) \phi] - \mathcal{V}(|\phi|),
\]

(2.1)

\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu},
\]

(2.2)

\[
\mathcal{V}(|\phi|) = -\mu^2 \phi^* \phi + \frac{1}{2} \lambda (\phi^* \phi)^2, \quad \lambda > 0.
\]

(2.3)

We use the notations

\[
g^{00} = +1, \quad g^{kk} = -1, \quad A^1 = A_x, \ldots,
\]

\[
F^{12} = -B_z, \ldots, \quad F^{01} = -E_x, \ldots.
\]

As is well known, the Lagrangian (2.1) leads to spontaneous symmetry breaking of the underlying U(1) symmetry. The photon acquires a gauge-invariant mass \(m_v\).

* For the application of the Ginzburg-Landau formalism to the type-II superconductors considered here, and in particular to the flux tubes discovered by Abrikosov \[11b\], see \[11c\].
which in the semi-classical tree approximation is given by
\[ m_Y^2 = 2e^2 \phi_c^2 , \tag{2.4} \]
where \( \phi_c = \mu / \sqrt{\lambda} \) is the value of \( |\phi| \) which renders \( \mathcal{V}(|\phi|) \) minimum \[12\]. For static magnetic fields, the non-vanishing photon mass implies the Meissner effect. The scalar field \( \phi \) describes the condensate and its mass \( m_S = (\partial^2 \mathcal{V}/\partial \phi \partial \phi^*)|_{|\phi|=\phi_c} = \mu \) measures the inverse coherence length of the condensed superfluid.

If \( m_S \gg m_V \) one obtains a type-II superconductor \[11\]; in this case, a sufficiently large external magnetic field penetrates the (finite) superconductor through tubes of quantized flux \( \Phi \):

\[ \Phi = \frac{2\pi n}{e} , \quad n \text{ integer}. \tag{2.5} \]

These tubes have a core in which \( |\phi| \) rises from zero to approximately its bulk value \( \phi_c \) within a range \( m_S^{-1} \) and the transverse size of the tube is the penetration depth \( m_Y^{-1} \gg m_S^{-1} \). Such a phenomenon occurs because the loss of condensation energy in the core of the tube is overcompensated by the decrease in the kinetic energy required for expelling the magnetic induction; it thus becomes energetically more favorable for the magnetic induction to penetrate the superconductor along thin threads than to run along its surface. The quantization condition (2.5) then follows from the single-valuedness of the \( \phi \) field.

For the infinite-size superconductors considered here, one may construct semi-classical solutions of confined flux tubes which either close on themselves or are of infinite length. If Dirac monopoles are added into the superconductor, they will act as sources of flux because the Dirac quantization condition

\[ eg = 2\pi n \tag{2.6} \]

matches the flux quantization (2.5). In particular, for the smallest available magnetic charges \(|n| = 1\), one obtains the Abelian model of magnetic confinement discussed in the introduction: at each pole a total amount of flux \( \Phi = 2\pi/e \) emerges or recedes spherically up to a distance of order \( m_Y^{-1} \) whereupon it becomes canalized along a flux tube with constant energy per unit length.

In what follows we shall restrict ourselves to the extreme type-II superconductor, that is to the “London limit” \( m_S/m_V \to \infty \). Thus \( |\phi| = \phi_c \) everywhere except along the axis of the flux tube where \( |\phi| = 0 \). In this case the relevant equation of motion for the static flux tubes becomes, in the absence of monopoles,

\[ \Delta A - \nabla (\nabla \cdot A) = m_Y^2 (A - (1/e) \nabla \chi) , \tag{2.7} \]

where \( \chi \) is the phase of \( \phi \). This yields for the supercurrent

\[ J_S = -m_Y^2 \left( A - \frac{1}{e} \nabla \chi \right) . \tag{2.8} \]

In regular gauges \( \nabla \chi \) is singular along the axis of the tube; to ensure the validity of
Stokes’ theorem, it is therefore necessary to pose

\[ \mathbf{V} \times \mathbf{V} = -\mathbf{B}_s \neq 0 , \]

\[ \mathbf{B}_s(r) = \Phi \int \delta^3(r - r'). \]

The line integral in (2.10) defines a fictitious induction \( \mathbf{B}_s \) running along the axis \( s' \) of the flux tube in the opposite sense to that of the quantized flux \( \Phi \). \( \mathbf{V} \times \mathbf{B}_s \) plays the role of a source term driving the supercurrent \( \mathbf{J}_s \). Taking the curl of \( \mathbf{J}_s \) in (2.8), one indeed obtains the London equation in the form

\[ \mathbf{V} \times \mathbf{J}_s = -m_s^2 (\mathbf{B} + \mathbf{B}_s), \]

or equivalently

\[ \Delta \mathbf{J}_s - m_s^2 \mathbf{J}_s = m_s^2 \mathbf{V} \times \mathbf{B}_s, \]

\[ \mathbf{V} \cdot \mathbf{J}_s = 0 . \]

In the presence of monopoles, one easily shows that (2.11) and (2.12) remain valid provided the line integral in (2.10) is still taken along the axis of the flux tube where \( |\phi| = 0 \); this axis now joins an antipole to a pole. The solution of (2.12) can be obtained by quadratures once the source term \( \mathbf{B}_s \) is known; its uniqueness follows from the vanishing of \( \mathbf{J}_s \) at large distances from the axis. Thus (2.12) solves completely the static confinement problem in the presence of monopoles in the semiclassical limit when \( m_S/m_V \to \infty \). One finds a logarithmic divergence in \( \mathbf{B} \) along the axis of the tube but this divergence is removed for \( m_S/m_V \) finite, the scalar mass providing the cut-off.

The static solutions discussed above are valid both for non-relativistic and relativistic superconductors. The new feature introduced by relativity is that, while previously the mean charge density \( \rho \) of the condensate was non-vanishing it now remains in general zero. This is so because \( \rho \) was proportional to \( |\phi|^2 \) in the non-relativistic case, but in a relativistic superconductor both charges exist and the charge operator is now \( \rho = -ie(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*) \) in the absence of a scalar potential. This difference clearly does not alter the static solution but, as pointed out by Nambu [4], it deeply affects the motion of the flux tubes. In particular any solution obtained by boosting the static solutions is now permitted.

2.2. The axiovector potential

We have just reviewed the mechanism by which an Abelian relativistic superconductor exhibits magnetic confinement of Dirac monopoles. Maxwell equations are symmetrical with respect to \( e \) and \( g \) with \( F_{\mu \nu} \) and \( F_{\mu \nu}^* \) interchanged\([|F^{\pm \mu \nu}| = \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} F_{\rho \lambda} (\epsilon^{0123} = +1)]\); one thus expects that such a superconductor can be
described in terms of monopoles minimally coupled to $F^\mu_\nu$ through an axiovector potential $C_\nu$. In the conventional description, confinement was conveniently characterized by the gauge-invariant mass $m_V$ acquired by the vector field $A_\mu$. How would one describe magnetic confinement in terms of the field $C_\mu$? The answer to that question is of interest because in such terms the superconductor may be viewed as a model for electric confinement with $g$ playing the role of an electric charge; this model may then suggest mechanisms for an electric confinement scheme in a more realistic situation.

We now show how to formulate the action for the relativistic Abelian superconductor in terms of the $C_\mu$ field; our study is restricted to the London limit and will be valid only in first-quantized theory. The latter restriction is not new as the Dirac monopoles previously used were considered in that context. To introduce the $C_\mu$ field we must describe the charge $e$ of the condensate by Dirac strings [5] carrying fictitious electric fluxes; thus we pose

\begin{align}
F^\mu_\nu &= C^\mu_\nu - E^\mu_\nu, \\
C^\mu_\nu &= \partial_\mu C_\nu - \partial_\nu C_\mu,
\end{align}

(2.13)

(2.14)

\begin{align}
E^\mu_\nu(x) &= -\sum_i e \int \! \! \! d\sigma_i \, d\tau_i [y_\mu, y_\nu], \delta^4(x - y),
\end{align}

(2.15)

\begin{align}
[y_\mu, y_\nu]_i &= \frac{\partial y_\mu}{\partial \tau_i} \frac{\partial y_\nu}{\partial \sigma_i} - \frac{\partial y_\mu}{\partial \sigma_i} \frac{\partial y_\nu}{\partial \tau_i}.
\end{align}

(2.16)

The sum in (2.15) is over all Dirac strings $i$ attached to the electric charges of the condensate; $y^\mu$ are the Minkowskian coordinates of the world-sheet of the strings parametrized by $\sigma_i$ and $\tau_i$. One has

\begin{align}
\partial_\mu E^\mu_\nu(x) &= \sum_i e \int \! \! \! d\sigma_i \, d\tau_i [y^\mu, y^\nu], \frac{\partial}{\partial y^\mu} \delta^4(x - y) \\
&= \sum_i e \int \! \! \! d\sigma_i \, d\tau_i [\delta^4(x - y), y^\nu]_i \\
&= -\sum_i e \int \! \! \! d\sigma_i \, d\tau_i \delta^4(x - y) dy^\nu_i = -j^\nu.
\end{align}

(2.17)

The last line follows from Stokes theorem: the integral over the rim of the $i$th sheet reduces to the line integral of the world-line $dy^\nu_i$ of the $i$th charge; $j^\nu$ is thus the electric current. From (2.13) and (2.17) one gets, using $(F^\mu_\nu)^+ = -F^\mu_\nu$,

\begin{align}
\partial_\mu F^\mu_\nu = j^\nu.
\end{align}

(2.18)

To obtain the remaining field equations for a superconductor in the presence of monopoles with charges $g_i$, we may try the following action $S$:

\begin{align}
S = S^0 + S^{\text{int}} + S^g,
\end{align}

(2.19)
where

\[
S^0 = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \, d^4x ,
\]

(2.20)

\[
S^{\text{int}} = -\frac{1}{2m_v^2} \int \delta_\mu E^{\mu\alpha} \partial_\nu E^{\nu\alpha} \, d^4x ,
\]

(2.21)

\[
S^g = -\sum_i m_i \int ds_i(z) - \sum_i g_i \int C_v(z_i) \, dz_i ,
\]

(2.22)

Note that the sign in (2.20) which is required to treat \( C_v \) as a vector field is opposite to the sign of the usual free electromagnetic action as \( F_{\mu\nu} F^{\mu\nu} = -F_{\mu\nu} F^{\mu\nu} \). The necessity of such a flip in sign is best understood if one notices that the Lagrangian density in (2.20) is \( \frac{1}{2}(E^2 - B^2) \) which correctly reduces to the potential energy term \(-\frac{1}{2}B^2\) for static monopoles. We shall indeed verify below that this sign yields the correct Lorentz force for the magnetic charges minimally coupled to the \( C_v \) field in \( S \). Note also that the sign in (2.22) is imposed as the kinetic energy of the monopoles \( g_i \) has to be positive. The interaction (2.21) is from (2.17) the required local current-current interaction which is the gauge-invariant manifestation of an induced-mass term for the \( A_\mu \) field in the London limit; this is seen for instance from the relation (2.8) and here also the sign of the interaction term had to be changed for consistency. No kinetic-energy term has been introduced for electric charges as the properties of the condensate are already contained in \( S^0 + S^{\text{int}} \).

If the action (2.19) yielded the correct classical equations of motion for the relativistic superconductor, the quantized theory could be described from the vacuum fluctuations \( Z^* \)

\[
Z = \int e^{iS} \mathcal{D}(\chi) ,
\]

(2.23)

where the functional integration is over all fields \( C_\mu \) and over all distinguishable electric string configurations including all values of \( N^+ \) and \( N^- \); \( N^+ (N^-) \) is the number of particles (antiparticles) of charge \( e \). As we are interested in flux-line formation we shall not vary the number of monopoles.

It will turn out that while (2.23) is indeed correct, the action (2.19) is false in the classical limit when it is considered as a c-number in that limit. This apparent paradoxical situation arises because we missed in (2.19) a term necessary to ensure gauge invariance of the action in the \( C_\mu \) field language. We shall see however that the missing term in \( S \) is a topological integral which for all string configurations takes values \( 2\pi n \) (\( n \) integer); therefore it does not affect (2.23) which is indeed gauge-invariant while \( S \) itself is gauge-dependent. We now introduce this topological integral which will play an essential role in our subsequent study.

* We ignore the gauge-fixing terms in the functional integration as those terms do not affect our results.
2.3. A topological integral

Consider a magnetic charge \( g \) describing a closed path \( C \) in space-time corresponding to the motion in space of a monopole-antimonopole pair with charges \(+g\) and \(-g\). From Stokes' theorem one has

\[
g \oint_C C_\nu \, dz^\nu = \frac{1}{2} \int C_{\mu\nu} M^{\mu\nu} \, d^4x \, , \tag{2.24}
\]

where \( M^{\mu\nu} \) is a Dirac string joining the pair. \( M^{\mu\nu} \) is related to the monopoles with charges \( g_i \) in the same way as \( E^{\mu\nu} \) is related to electric charges, that is

\[
M^{\mu\nu}(x) = -\sum_i g_i \int \delta^4(\sigma_i - z) \, d^3z \, , \tag{2.25}
\]

where we have labeled by \( \sigma_i \) the coordinates of the magnetic string. From (2.25) the magnetic analog of (2.17) is

\[
\partial_\mu M^{\mu\nu}(x) = \sum_i g_i \int \delta^4(x - z) \, d^3x \, = -k^\nu(x) \, , \tag{2.26}
\]

where \( k^\nu(x) \) is now the magnetic current.

We may rewrite (2.24) as

\[
g \oint_C C_\nu \, dz^\nu = \frac{1}{2} \int F^{\mu\nu}_{\mu\nu} \, d^4x + \frac{1}{2} \int E^{\mu\nu}_{\mu\nu} M^{\mu\nu} \, d^4x \tag{2.27}
\]

It follows from (2.27) that the line integral depends on the choice of the electric strings and is therefore not gauge-invariant. Therefore minimal coupling in (2.22) is insufficient to ensure gauge invariance in the presence of electric strings.

To obtain a gauge-invariant action generalizing (2.19), we define

\[
S^c = S + S^t \, , \tag{2.28}
\]

\[
S^t = \frac{1}{2} \int E^{\mu\nu}_{\mu\nu} M^{\mu\nu} \, d^4x \, , \tag{2.29}
\]

where \( M^{\mu\nu} \) is defined in general by (2.25). In fact we shall include in (2.25), not only strings ending at each monopole entering \( S_g \) in (2.22), but also an arbitrary number of closed strings or strings of infinite space-like extension. The usefulness of such terms will become clear later on.

The use of (2.28) instead of (2.19) in the quantum-mechanical action apparently violates minimal coupling. To prove the consistency of (2.28) we must show that \( S^t \) does not contribute to the functional integral and thus that (2.23) is gauge-invariant as it stands. This is indeed the case because \( S^t \) is for all string configurations equal to \( 2\pi n \) (\( n \) integer) owing to the Dirac quantization (2.6). The proof is as follows.
From (2.15) and (2.25) one has
\[ S^t = \sum_{ij} e_{ij} \int d\sigma_i \, d\sigma_j \, e_{\mu\nu\rho\eta} \frac{\partial y^\rho}{\partial \sigma_i} \frac{\partial y^\eta}{\partial \sigma_j} \frac{\partial z^\mu}{\partial \tau_i} \frac{\partial z^\nu}{\partial \tau_j} \delta^4(y - z). \]  

(2.30)

Contributions to (2.30) may arise only when an electric string meets a magnetic string. If the two strings have a line element in common, the determinant in the integrand is zero because the four homogeneous linear equations \( dy^\mu = dz^\mu \) admit a non-trivial solution in \( d\sigma_i, d\tau_i, d\sigma_j, d\tau_j \). Therefore one has to consider only cases in which an electric string crosses a magnetic string at an isolated point. At this point the determinant cannot vanish and hence has an inverse. This permits a change of variables to \( y^\mu \) (or \( z^\mu \)) and the integral over the \( \delta \) function is thus one.

From the Dirac condition (2.6) we then have
\[ S^t = \sum_{ij} 2\pi n_i \rho_{ij}, \]

(2.31)

where \( \rho_{ij} \) is the number of isolated contact points between an electric string \( j \) and a magnetic string \( i \). This proves our assertion.

Thus, despite the fact that in the quantum formulation one may use equivalently \( S \) or \( S^c \), the two actions themselves are not equivalent. As the usual simple derivation of classical field equations relies directly on the action we shall use the gauge-invariant action \( S^c \) whenever we shall study classical effects.

2.4. The classical equations of motion

In the \( C_\mu \) field description, the first Maxwell equation (2.18) has a kinematical origin. The second Maxwell equation,
\[ \partial_\mu F^{\mu\nu} = k^\nu, \]

(2.32)

however, expresses the stationarity of \( S^c \) (or \( S \)) with respect to variations of \( C_\nu \).

The stationarity of \( S^c \) with respect to variations of electric string variables yields
\[ \delta I = -\frac{1}{2} \int O_{\mu\nu}^+ \delta (E^{\mu\nu}) \, d^4 x = 0, \]

(2.33)

with
\[ O_{\mu\nu}^+ = F_{\mu\nu} + \frac{1}{m^2} (\partial_\mu j_\nu - \partial_\nu j_\mu) - M_{\mu\nu}^+. \]

(2.34)

A straightforward calculation yields
\[ \delta I = \frac{1}{2} \sum_i e \int d\sigma_i \, d\tau_i \, \delta y^\lambda \left[ \frac{\partial O_{\mu\nu}^+}{\partial y^\lambda} + \frac{\partial O_{\nu\lambda}^+}{\partial y^\mu} + \frac{\partial O_{\lambda\mu}^+}{\partial y^\nu} \right] \]
\[ \times [y^\mu, y^\nu]_i + \sum_i e \int O_{\mu\nu}^+ \delta y^\mu_i \, dy^\nu_i, \]

(2.35)
where the last term is the contribution of the string end points. Thus the equation of motion for the \( i \)th electric string is 
\[
\left[ \partial_\lambda O^\nu_{\mu\lambda} + \partial_\mu O^\nu_{\lambda\mu} + \partial_\nu O^\mu_{\lambda\mu} \right] \cdot [y^\mu, y^\nu]_i = 0,
\]
equivalently
\[
\frac{\partial O^{\mu\nu}}{\partial y^\mu} [y^\nu, y^\rho]_i^+ = 0.
\] (2.36)

We see from (2.34), (2.32) and (2.26) that \( \partial_\mu O^{\mu\nu} = 0 \) so that (2.36) is identically satisfied. Hence the electric string has no dynamical content. Note that the absence of any constraint [5] for electric strings is a consequence of the string term \( M^+_{\mu\nu} \) in (2.34); this contribution originates in the topological term (2.29).

On the other hand variations of the magnetic string variables \( z^\mu \) yields in a similar fashion
\[
\partial_\mu E^{\mu\nu}[z^\nu, z^\rho]_i^+ = -\gamma^{\nu \rho \mu \nu} [z^\nu, z^\rho]_i^+ = 0.
\] (2.37)

This expresses that the electric current is tangential to the magnetic sheet or that no electric charge may cross a magnetic string. This is always verified as the magnetic string is constrained to stick on the axis of a flux tube. To see this, consider the variations of the electric string extremities; from (2.35) we get
\[
O^+_{\mu\nu} dy^\rho_{(i)} = 0.
\] (2.38)

Eq. (2.38) is indeed the Lorentz-covariant generalization of the static London equation (2.11). Thus the magnetic string term introduced in \( S^t \) is just the source term \( B_s \) driving the flux tubes. This is why we included in \( M^+_{\mu\nu} \) strings without monopole ends: each such string drives, even in absence of monopoles, an excitation of the superconductor which is classically represented by a quantized flux tube.

Finally, we vary the extremities of the magnetic strings, that is the position of the monopoles. This yields the correct equation of motion for a magnetic charge:
\[
m \frac{d^2 z^\mu_{(i)}}{ds^2_{(i)}} = g_t F^{+\mu\nu} \frac{dz^\nu_{(i)}}{ds_{(i)}}.
\] (2.39)

The fact that the complete \( F^{+\mu\nu} \) enters the Lorentz force and not only \( C_{\mu\nu} \) is due to the term \( -E^{+}_{\mu\nu} \) originating in \( S^t \).

Thus we have proved that (2.28) yields all the classical features of a relativistic superconductor in the London limit. It is the introduction of the topological term \( S^t \) which ensures that the electric strings are free of constraints and at the same time provides the correct source term in the London equation.

### 2.5. Confinement in the \( C_\mu \) field language

Let us see how confinement is described classically in the \( C_\mu \) field language by considering at \( z = 0 \) a cylindrical flux tube whose axis is taken along the \( z \) direction; this flux tube may be thought of as originating from a monopole-antimonopole-
pole pair located at \( z = \pm a, a \gg m v \). Far from the monopole ends, we must have

\[
\begin{align*}
  j^1 &= -\frac{x^2}{\rho} j(\rho), \\
  j^2 &= +\frac{x^1}{\rho} j(\rho), \\
  j^3 &= j^0 = 0;
\end{align*}
\]

\( \rho = [(x^1)^2 + (x^2)^2]^{1/2} \). (2.40)

The currents (2.40) may be viewed as superpositions of circular currents generated by particle-antiparticle pairs created out of the condensate and moving around the magnetic string before they annihilate. We may then choose a gauge in which the strings move in the \( x-y \) plane so that during the motion of the pair the electric string has to cross the magnetic string, thereby contributing a unit “bip” to \( S \). In such a “bip-bip” gauge the only non-vanishing component of \( E_{\mu
u} \), averaged over space and time, is \( E^{12}(\rho) = E^{+03}(\rho) = \phi(\rho) \) because the system is neutral, stationary in time and has cylindrical symmetry. We verify indeed that (2.40) satisfies (2.17) with the identification

\[
j(\rho) = \frac{\partial \phi(\rho)}{\partial \rho}.
\] (2.41)

The equation for \( C^\nu \) in a gauge \( \partial_\mu C^\mu = 0 \) is \( \square C^\nu = \partial_\mu E^{+\mu \nu} = 0 \), so that we may choose \( C^\nu = 0 \). The only non-vanishing component of the flux is thus

\[
B_z = \phi(\rho).
\] (2.42)

To evaluate \( B_z \), we must solve the London equation (2.38) which reduces to \( O^{+12} = 0 \). As \( M^{+12} = -\Phi \delta(x) \delta(y) \) one has \( B_z + (1/m_0^2)(\partial_1 j^2 - \partial_2 j^1) + \Phi \delta(x) \delta(y) = 0 \); using (2.40), (2.41) and (2.42) one gets

\[
\Delta \phi - m_0^2 \phi = \Phi \delta(x) \delta(y).
\] (2.43)

This is indeed the correct equation for a cylindrical flux tube (see for instance (2.12)).

The fact that \( C^\nu = 0 \) is a consequence of the choice of gauge; had we chosen the strings parallel to the \( z \) axis we would have found that \( C^0 \) is non-zero and massive. In that gauge the flux is generated entirely from \( C^0 \) and thus receives no contribution from the topological term *. It follows however from the above example that in the \( C_\mu \) field language it is inconvenient to characterize the confinement by a vector-meson mass as is usually done in the \( A_\mu \) field description. A possible alternate characterization is provided by the Wilson criterion [15] which we now examine in a semi-classical limit.

Consider the creation, followed by its annihilation after a time \( \Delta t \), of a monopole-antimonopole pair in a superconductor. Choose a configuration such that during most of its life, the pair stays separated by a distance \( \Delta L \) on, say, the \( z \) axis. The

* Strictly speaking, the electric strings meet the magnetic string at infinity but this effect can be taken out by using a cut-off in \( C^0 \) for large \( |z| \); for details see ref. [14].
motion of the pair may be described by a closed curve $C$ in the $z-t$ plane; the planar area $A$ enclosed by the curve $C$ tends to $\Delta L \Delta t$ for large $\Delta t$. Because of flux-tube formation the quantity $\mathcal{W}$ defined by

$$e^{-i\mathcal{W}} = \frac{\int e^{i(S^0 + S^{\text{int}} - ig \oint C \mathcal{A} \mathcal{B})} \mathcal{D}(x)}{\int e^{i(S^0 + S^{\text{int}})} \mathcal{D}(x)}$$

(2.44)

is expected to tend for large $\Delta t$ and $\Delta L$ to $\alpha \Delta L \Delta t$, where $\alpha$ measures the flux-tube energy per unit length. The proportionality of $\mathcal{W}$ to $A$ for large loops is Wilson's criterion for confinement, which we write as

$$\lim_{\Delta L \to \infty, \Delta t \to \infty} \langle e^{-i g \oint C \mathcal{A} \mathcal{B}} \rangle_0 = e^{-i\alpha A}.$$  

(2.45)

We shall now show that (2.45) is indeed verified in a semi-classical limit, provided due care is given to gauge invariance in the limiting process.

We write the average in the left-hand side of (2.45) as $\langle \exp(-i\gamma) \rangle_0$ with $\xi = ig$ and $\gamma = \oint C \mathcal{A} \mathcal{B}$. The classical limit of such an expression follows in general from the Gaussian approximation (or equivalently the 1-loop approximation)

$$\frac{1}{2\pi} e^{-\frac{1}{2} \langle \gamma^2 \rangle_0} \approx e^{-\frac{1}{2} \langle \gamma^2 \rangle_0}.$$  

(2.46)

In the present case, however, in order to maintain gauge invariance in this limit it is necessary as in subsect. 2.3, to replace $\xi \gamma$ by $\xi \gamma' = \xi \gamma - i S^{t(g)}$ where $S^{t(g)}$ is the topological term (2.29) with $M_{\mu\nu}$ describing the magnetic charge considered. Thus we write

$$\langle e^{-i\gamma} \rangle_0 \to e^{-\frac{1}{2} \langle \gamma^2 \rangle_0}.$$  

(2.47)

It is now possible to evaluate $\langle \gamma^2 \rangle_0$ in terms of the classical flux tube which appears in the presence of the external source $S^{\text{ext}} = i \xi \gamma'$ which describes the monopole-antimonopole pair. This follows from the identity

$$-i \frac{d\langle \gamma' \rangle_k}{d\xi} \bigg|_{\xi = 0} = \langle \gamma^2 \rangle_0 - \langle \gamma' \rangle_0^2 = \langle \gamma'^2 \rangle_0,$$  

(2.48)

where $\langle \gamma' \rangle_k$ is the value of $\gamma'$ obtained by solving the classical equation of motion in presence of the source term. Note that the "fluctuation-dissipation" theorem (2.48) is valid in the present situation only because $\xi$ (or $g$) can be varied independently of $e$ which enters the action $S^0 + S^{\text{int}}$. This is permitted because $S^0 + S^{\text{int}} + S^{\text{ext}}$ is gauge-invariant for all $g$ so that the Dirac quantization condition can be relaxed when $g$ is varied, despite the fact that the quantum equivalence between $S$ and $S^c$ is lost in the variation.
From (2.47), (2.48) and (2.27) one has

\[ \langle e^{-\frac{i\theta}{2} F_{\mu\nu}} d^{\nu} \rangle \rightarrow e^{-i(\log 2) \int F^{+\nu}_{\mu} dS^{(\mu\nu)}} , \] (2.49)

where \( dS^{(\mu\nu)} = M^{\mu\nu} d^4x = -M^{\nu\mu} d^4x \) is an element of the area \( A \) and \( F^{+\nu}_{\mu} \) the classical flux which is linear in \( g \). For \( \Delta L \) and \( \Delta t \) large enough, \( F^{+\nu}_{\mu} = F_{03}^{+} = B_3 = \varphi(\rho) \) as given by (2.42) and (2.43) and \( dS^{(\mu\nu)} = dt dz \). Thus we do verify (2.45); of course \( \alpha \) is divergent as a consequence of the logarithmic divergence of the field on the axis of the tube in the London limit, but such a divergence is not physically relevant.

To conclude, we see that a superconductor can be characterized in the \( C_\mu \) field language, by the Wilson criterion for confinement. This criterion, applied at a semiclassical level is a consequence of the existence of classical flux tubes. These properties are not immediately apparent from the quantum-mechanical action \( S \) because \( S \) differs from the classical action \( S^c \) by the topological term \( S^t \).

2.6. Electric confinement from monopoles

Under the duality transformation \( e \rightarrow g, C_\mu \rightarrow A_\mu, E^{+\nu}_{\mu} \rightarrow -M_{\mu\nu} \), the action (2.19) becomes

\[ \tilde{S} = -\frac{1}{4} \int (A_{\mu\nu} + M_{\mu\nu})(A^{\mu\nu} + M^{+\mu\nu}) d^4x \]

\[ -\int \frac{1}{2m^2} \int \partial_\mu M_{\mu\nu} \partial_\nu M_{\rho} d^4x - \sum_i m_i \int ds(y(i)) - \sum_i e_i \int A_\nu d y_{(i)}^\nu . \] (2.50)

This action describes a "magnetic" superconductor which confines electric flux tubes; Wilson's criterion is obeyed in its conventional form, that is with \( g_t C_\mu \) replaced by \( e_i A_\mu \). Of course if a classical description of these features is required, one should again add to the action the topological term (2.29).

The interest of the model (2.50) is that it suggests that in some non-Abelian generalization, \( M_{\mu\nu} \) might constitute an approximate description of a regular solution of the monopole type \(^*\); the interaction term might then find its origin in the framework of such non-Abelian gauge field theories. In fact we shall see that such a situation indeed arises when the model (2.50) is taken over in \( 2 + 1 \) dimensions.

3. Electric confinement from instantons: the projected magnetic superconductor

A model resembling a magnetic superconductor which exhibits electric confinement in \( 2 + 1 \) dimensions can be constructed from the action (2.50) by projecting

\(^*\) The suggestion that monopoles could lead to an electric Meissner effect was made by Mandelstam [16].
out one space dimension in the following way. Define the field tensor \((g^{00} = +1, g^{11} = g^{22} = -1, \epsilon^{012} = +1)\),
\[
F^{\mu\nu} = A^{\mu\nu} + \epsilon^{\mu\nu\lambda} M_\lambda ,
\]
\[
A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,
\]
\[
M^\sigma(x) = \sum_j g_j \int_{-\infty}^z \delta^3(x - z) \, dz^\sigma ,
\]
\[
g_j e = 2\pi n_j , \quad (n_j \text{ integer}) .
\]
The string \(M^\sigma\) defines a "monopole" located at the point \(z(j)\) whose charge is a \(\delta\) function in time; indeed if
\[
F^\sigma = \frac{1}{2} \epsilon^{\sigma\mu\nu} F_{\mu\nu} ,
\]
one gets
\[
\partial_\sigma F^\sigma = -\partial_\sigma M^\sigma = \sum_j g_j \delta^3(x - z(j)) = k .
\]
Thus \(k\) plays the role of a magnetic current crossing the 1-2 plane from "outer space". To ensure electric confinement, it then suffices to postulate a "current-current" interaction term and to consider the "projected" action
\[
\mathcal{S} = -\frac{1}{4} \int (A_{\mu\nu} + \epsilon_{\mu\nu\sigma} M^\sigma)(A^{\mu\nu} + \epsilon^{\mu\nu\sigma} M_\sigma) \, d^3 x
\]
\[
+ \frac{1}{2m_V} \int \delta_\sigma M^\sigma \, \partial_\rho M^\rho \, d^3 x
\]
\[
- \sum_i m_i \int ds(i) - \sum_i e_i \int A_\nu(y(i)) \, dy_\nu(i) .
\]
Introducing the topological term
\[
\mathcal{S}^t = -\frac{1}{2} \int \epsilon^{\mu\nu\sigma} E_{\mu\nu} M_\sigma \, d^3 x = 2\pi n , \quad (n \text{ integer}) ,
\]
where \(E_{\mu\nu}\) is the electric string tensor (2.15) in \(2 + 1\) dimensions, one gets the gauge-invariant classical action
\[
\mathcal{S}^e = \mathcal{S} + \mathcal{S}^t ,
\]
from which the classical equations of motion can be deduced. One finds by the same procedures as before
\[
\partial_\mu F^{\mu\nu} = j^\nu ,
\]
\[
\left[ F_\sigma + \frac{1}{2} \epsilon_{\mu\nu\sigma} E^{\mu\nu} - \frac{1}{m_V} \partial_\sigma (\partial_\rho M^\rho) \right] \, dy_\sigma(i) = 0 ,
\]
as well as the Lorentz force for electric charges; no constraint appears for magnetic strings but no magnetic current may cross the electric strings.

To exhibit the static electric flux tube engendered by two distant electric charges lying on the 2-axis, we select a “bip-bip” gauge such that only $M^1(x^1)$ is non-vanishing. Then it follows from (3.9) that $A_\mu = 0$ and that the only non-vanishing component of the field is $F^{02} = M^1 = \varphi(x^1)$. The “London equation” (3.10) then shows that $\varphi(x^1)$ falls off as $\exp(-m_\varphi |x^1|)$ from its maximum value on the $x^2$ axis. Thus electric charges are confined and the Wilson criterion follows. This is clear from the method used in the sect. 2. Again it is the use of a topological term which has permitted the description of an electric confinement in classical terms.

The action $\mathcal{S}$ is a formal construct as the “monopoles” (3.3) are not really particles. However we can find an interesting interpretation for the analytic continuation of $\mathcal{S}$ in a 3-dimensional Euclidian space. Defining $x_k \equiv (x^1, x^2, i\xi^0)$, $\bar{A}_k \equiv (A_1, A_2, -iA_0)$,

\begin{align}
\bar{M}_k &= \sum_i g_i \int_{-\infty}^{\infty} \delta^3(x - z) \, dz_k , \\
\bar{F}_{kl} &= \bar{A}_{kl} + e_{klm} \bar{M}_m , \\
\bar{A}_{kl} &= \partial_k \bar{A}_l - \partial_l \bar{A}_k ,
\end{align}

one has in absence of electric sources

\begin{align}
\bar{\mathcal{S}} = i\mathcal{S} &= -\frac{1}{4} \int (\bar{A}_{kl} + e_{klm} \bar{M}_m)(\bar{A}_{kl} + e_{klp} \bar{M}_p) \, d^3x \\
&- \frac{1}{2m_\varphi^2} \int \partial_k \bar{M}_k \partial_l \bar{M}_l \, d^3x .
\end{align}

One recognizes in $\bar{M}_m$ strings due to singular instantons [17]*. It may well be that (3.14) provides an approximate description of the regular instanton fluid which appears if the U(1) group considered here is the unbroken subgroup of a larger SO(3) group, the finite instanton size being described by the “interaction”

\begin{align}
- \frac{1}{2m_\varphi^2} \int k^2(\xi) \, d^3\xi .
\end{align}

The action (3.14) is indeed quite similar to the one proposed by Polyakov to describe in such a situation the instanton confinement mechanism [17]. In our approach, confinement follows directly from the classical equation of motion deduced from $\mathcal{S}^c$. Note that the semi-classical approach used in sect. 2 to verify the Wilson criterion can be carried out directly in Euclidean space.

* The second paper in [17] also explains the mechanism of symmetry restoration by instantons.
Thus in the present case, the whole confinement phenomenon may be attributed to instantons instead of quasi-particles (monopoles) as in sect. 2. This comes about because conserved magnetic supercurrents engendered by magnetic monopoles are no longer available in $2 + 1$ dimensions. Confinement is still possible, however, if projected supercurrents obtained by projecting out a third spatial dimension are introduced. It follows from (3.3) or (3.11) that instantons precisely describe such currents.

4. Concluding remarks

The models for electric confinement described in sects. 2 and 3 do not include any non-Abelian effect; therefore their relevance for quantum chromodynamics resides in the hypothesis that, at least in the Regge region, the main effect of non-Abelian configurations is to symmetrize in group space the Abelian flux-tube structure.

If this turns out to be the case, then our analysis leads to two suggestions.

(i) Confinement arises through formation of electric flux tubes at a semi-classical level; these tubes are approximately described in terms of classical equations of motion, once relevant topological effects have been explicitly accounted for in the action principle.

(ii) Instantons or related objects may lead to confinement in much the same way as particles do. The fact that in the above example the instanton mechanism leads to something like a "projected magnetic superconductor" may then be no accident and is perhaps more relevant than statistical analogies with symmetry restoration to a disordered phase [17]. Indeed, in $3 + 1$ dimensions, instantons [18] and the related merons [13] exist in the framework of unbroken Yang-Mills field theories; such theories are, at a classical level, invariant under conformal transformations which are often conveniently formulated in a projective space.

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References

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