STRANGE DIBARYONS IN THE SIMPLIFIED SKYRME MODEL

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Abstract: We show that the bound-state approach to the strange dibaryons in the Skyrme model can be considerably simplified by omitting the Skyrme stabilizing term proportional to $e^{-2}$ and using the regularized version of the quantum stabilization proposed by Mignaco and Wulck. The strange dibaryons are obtained as systems consisting of bound antikaons in a (variational) axially symmetric SU(2)-skyrmion background field with baryon number $B > 1$.

1. Introduction

Callan and Klebanov \textsuperscript{1}) showed that a fairly good description of the hyperon spectrum in the Skyrme model \textsuperscript{2}) is obtained, if the hyperons are treated as bound kaon-soliton systems. Callan, Hornbostel and Klebanov (CHK) \textsuperscript{3}) successfully completed this program. The basic idea of their model is to treat strangeness separately from the isospin in the Skyrme model, assuming that the vacuum is approximately SU(3) symmetric i.e. $F_K \approx F_\pi$. The strange baryons are generated by binding kaons in the field of "rotating" SU(2) solitons. Since there is no static field associated with the strangeness quantum number, it is essential in this picture that there exist bound states in the kaon-soliton complex that give rise to hyperons. CHK showed that such bound states do exist. A remarkable property of the kaons in the CHK model is that after quantization they look like s-quarks, due to topological effects. This leads to a spectroscopy of hyperons quite similar to that of quark models.

In the CHK approach the kaon-soliton field is written in the form

$$U = (U_\pi)^{1/2} U_K (U_\pi)^{1/2}, \quad (1.1)$$

where $U_\pi$ is a SU(3)-extension of the usual SU(2) skyrmion field used to describe the nucleon spectrum \textsuperscript{4}), and $U_K$ is the field describing the kaons

$$U_\pi = \exp (2iF_\pi^{-1}\lambda_j \pi^j), \quad j = 1, 2, 3, \quad (1.2)$$

$$U_K = \exp (2iF_\pi^{-1}\lambda_a K^a), \quad a = 4, 5, 6, 7. \quad (1.3)$$

The $\lambda$-matrices are the familiar SU(3) matrices.

In this paper we apply the CHK approach to the baryon number $n$ sector of the simplified Skyrme model \textsuperscript{5}), with the stabilizing term proportional to $e^{-2}$ omitted, using the quantum stabilization method proposed by Mignaco and Wulck (MW), generalized to the baryon number $n > 1$ sector and regularized using a modified radial ansatz \textsuperscript{6}), in order to avoid the difficulties with the soliton boundary conditions \textsuperscript{8}). We derive the results for the rotational energies of the arbitrary baryon
number \( n > 1 \) states, but the primary objective of the present study is the dibaryon states with \( n = 2 \). This paper consists of four sections. In sect. 2 the effective interaction for the bound kaon-soliton system is derived. In sect. 3 the rotational energies of the strange dibaryons are calculated. In sect. 4 we make the final conclusions.

2. The effective interaction

The lagrangian density for a bound kaon-soliton system in the simplified Skyrme model is given by

\[
\mathcal{L} = \frac{1}{16} F^2 \text{Tr} \partial_\mu U \partial^\mu U^+ + \frac{1}{16} F^2 \left( m_\pi^2 + 2m_K^2 \right) \text{Tr} \left( U + U^+ - 2 \right) + \frac{1}{24\sqrt{3}} F^2 \left( m_\pi^2 - m_K^2 \right) \text{Tr} \lambda_8 (U + U^+) ,
\]

where \( m_\pi \) and \( m_K \) are pion and kaon masses, respectively, and

\[
U_\pi = \begin{bmatrix} u_\pi & 0 \\ 0 & 1 \end{bmatrix}, \quad U_K = \exp \left\{ i \frac{2^{3/2}}{F_\pi} \begin{bmatrix} 0 & K \\ K^+ & 0 \end{bmatrix} \right\}.
\]

In (2.2) \( u_\pi \) is the SU(2)-skyrmion field for which we choose the variational ansatz discussed in ref. 7) with static radial modification phase function \( \phi = \phi(r) \) discussed in ref. 6). Compared to the usual \( n = 1 \) case, this ansatz has a different twist in the isovector field \( \pi \) rather than a different boundary condition for the chiral phase function \( F = F(r) \), i.e.

\[
u_\pi = \exp \left\{ i \pi \cdot \hat{\pi}_n F(r) + i\phi(r) \right\}, \quad \hat{\pi}_n = \begin{bmatrix} \sin \theta \cos n\phi \\ \sin \theta \sin n\phi \\ \cos \theta \end{bmatrix},
\]

\[
\rho \left( \frac{d\phi}{dr} \right)^2 = (2rR + R^2) \left( \frac{dF}{dr} \right)^2 + 2n(2n-1)F^2 - (n^2 + 1) \sin^2 F
\]

where \( F(r) \) is a radial function which (in the absence of the Skyrme stabilizing term and for \( m_\pi = 0 \)) satisfies the non-linear differential equation

\[
\frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) = 2n(2n-1)F
\]

with the boundary conditions \( F(0) = -\pi \) and \( F(\infty) = 0 \) and where \( R \) is a dimensional scale parameter. The presence of the static radial modification phase \( \phi = \phi(r) \) ensures the existence of the solution \( F(r) = -\pi(r/R + 1)^{-2n} \) with proper soliton boundary conditions. The two-dimensional vector \( K \) in (2.2) is the kaon-doublet

\[
K = \begin{bmatrix} K^+ \\ K^0 \end{bmatrix}, \quad K^+ = [K^- \quad K^0].
\]

In addition to the simplified Skyrme model action obtained using the lagrangian density (2.1), the Wess-Zumino action in the form

\[
S = -\frac{iN_c}{240\pi^2} \int d^5x \, e^{\mu\nu\alpha\beta} \text{Tr} \left[ U^+ \partial_\mu U U^+ \partial_\nu U U^+ \partial_\alpha U U^+ \partial_\beta U U^+ \partial_\gamma U \right]
\]

must be included into the total action of a kaon-soliton system. In (2.6) \( N_c \) is the number of colours in the underlying QCD.
Substituting (1.1), with $U_\pi$ and $U_K$ defined by (2.2), into the total action of the kaon-soliton system, expanding $U_K$ to second order in the kaon fields (2.5) and using the following ansatz for stationary antikaon states

$$K(r, t) = \tau \cdot \pi_n k(r, t) = e^{-i\omega t} k_p(r) \tau \cdot \pi_n \chi,$$  \hspace{1cm} (2.7)

where $\chi$ is a two-component spinor, we obtain the effective interaction-lagrangian density for the kaon-soliton system in the lowest bound state (see ref. 5))

$$\mathcal{L} = \hat{k}^+ \hat{k} + \frac{dk^+}{dr} \frac{d\hat{k}}{dr} + i\lambda(r)(\hat{k}^+ \hat{k} - \hat{k} \hat{k}^+) - k^+[m_K^2 + V_{\text{eff}}(r)]k,$$  \hspace{1cm} (2.8)

where

$$V_{\text{eff}}(r) = -\frac{1}{4} \left[ \left( 1 + \frac{R}{r} \right)^2 \left( \frac{dF}{dr} \right)^2 + 2n(2n-1) \frac{F^2}{r^2} \right] + \frac{n^2 + 1}{r^2} \cos^4 \frac{1}{2} F,$$  \hspace{1cm} (2.9)

$$\lambda(r) = -\frac{N_c}{2\pi^2 F^2} \frac{\sin^2 F}{r} \frac{dF}{dr}.$$

The hamiltonian density corresponding to the lagrangian density (2.8) is given by

$$\mathcal{H} = \Pi^+ \hat{k} + \hat{k}^+ \Pi - \mathcal{L},$$  \hspace{1cm} (2.11)

where

$$\Pi^+ = \frac{\partial \mathcal{L}}{\partial \dot{\hat{k}}} = \hat{k}^+ + i\lambda(r)k^+,$$  \hspace{1cm} (2.12)

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{k}} = \hat{k} - i\lambda(r)k.$$  \hspace{1cm} (2.13)

Substituting (2.7), (2.11) and (2.12) into (2.10) we obtain

$$\mathcal{H} = \Pi^+ \Pi - \frac{d\hat{k}^+}{dr} \frac{dk}{dr} - i\lambda(r)(\hat{k}^+ \Pi - \Pi^+ k) + k^+[m_K^2 + V_{\text{eff}}(r)]k$$

$$+ k^+ \lambda(r)k.$$  \hspace{1cm} (2.14)

From (2.14) we obtain the radial wave equation for the lowest bound-state antikaon wave function $u_0 = rk_p(r)$ in the form

$$\frac{d^2 u_0}{dr^2} - v_{\text{eff}}(r)u_0 + [\omega^2 - m_K^2 + 2\omega\lambda(r)]u_0 = 0,$$  \hspace{1cm} (2.15)

where $\omega$ is the lowest bound-state energy, and the antikaon modes are normalized according to

$$8\pi \int_0^\infty dr r^2 [\omega + \lambda(r)]k_p^2(r)k_p(r) = 1.$$  \hspace{1cm} (2.16)

If the differential equation (2.4) is solved and the phase $F(r)$ is known it is possible to solve (2.15) with (2.9)-(2.10) and find the lowest bound-state energy $\omega$. 
3. The rotational energies of the strange dibaryons

In order to calculate the spectrum of the strange dibaryons we must take into account the rotational modes of the soliton. We consider the spatial ($R$) and isospin ($A$) rotations, where the corresponding angular velocities $\alpha$ and $\beta$ with respect to the body fixed axes are given by

\[
(R^{-1} \partial_0 R)_{ab} = \varepsilon_{abc} \alpha_c, \quad (3.1)
\]
\[
A^{-1} \partial_0 A = \frac{1}{2} i \vec{\beta} \cdot \tau, \quad (3.2)
\]

where $\varepsilon_{abc}$ ($a, b, c = 1, 2, 3$) is the totally antisymmetric tensor with $\varepsilon_{123} = 1$. The kaon and the soliton fields are rotated according to

\[
K \rightarrow RAK = (RA\tau \cdot \hat{\pi}_A A^{-1}) A k, \quad (3.3)
\]
\[
u_\pi = RAu_\pi A^{-1}. \quad (3.4)
\]

Substituting (2.2) with (2.3), (2.5), (3.3) and (3.4) into (2.1) and keeping in the expansion of the kaon field only the two eigenmodes in (2.7) with coefficients $a_1$ and $a_2$ for the up and down spinor $\chi$, respectively, we obtain the following expression for the total lagrangian of the kaon-soliton system

\[
L = L(u_\pi) + L(K) - N \cdot \vec{\beta} + (1 - c)
\times [(N_1 \beta_1 + N_2 \beta_2) - (N_1 \alpha_1 + N_2 \alpha_2) \delta_{n_1} + N_3 (\beta_3 - n \alpha_3)]
+ \frac{1}{2} \Omega [\beta_1^2 + \beta_2^2 + \frac{1}{2} (n^2 + 3)(\alpha_1^2 + \alpha_2^2) - 2(\alpha_1 \beta_1 + \alpha_2 \beta_2) \delta_{n_1} + (\beta_3 - n \alpha_3)^2] \quad (3.5)
\]

with

\[
\Omega = \frac{2}{3} \pi F_{\pi}^2 \int_0^\infty dr r^2 \sin^2 F, \quad (3.6)
\]
\[
c = 1 - \frac{8}{3} \omega \int_0^\infty dr r^2 k_p^*(r) \cos^2 \frac{1}{2} F_k p(r), \quad (3.7)
\]

where $N$ is the spin of the antikaon modes given by

\[
N = \frac{1}{2} \hat{a}_i^+ \tau_{ij} \hat{a}_j \quad (3.8)
\]

and $\hat{a}_i^+ (\hat{a}_i)$ are creation (annihilation) operators of antikaon modes.

The quantization of the rotational energy leads to

\[
H_{\text{rot}} = \frac{1}{2 \Omega} \left[ \frac{4}{n^2 + 3} (J^2 - J_3^2) + I^2 - I_3^2 + c^2 (T^2 - T_3^2) + (I_3 + cT_3)^2 \right.
+ 2c (I \cdot T - I_3 T_3) \left. \right], \quad (3.9)
\]

where $J$ is the total angular momentum of the kaon-soliton system, $I$ is the collective angular momentum of the rotating $n > 1$ soliton and $T$ is the angular momentum
of the kaons. In the special case when \( n = 1 \), i.e. for the spherically symmetric soliton, (3.9) reduces to the well-known form \(^5\)

\[
H_{\text{rot}} = \frac{1}{2\Omega} (I + cT)^2, \quad n = 1. \tag{3.10}
\]

The eigenfunctions of \( H_{\text{rot}} \) are product states of the spatial and isospin rotation matrices with the kaon eigenstates, given by

\[
\langle \alpha, \beta, r, t | I, I_3; J, J_3; I_3^{bf} = Q - T_3, J_3^{bf} = -nQ, T_3 \rangle = \frac{1}{8\pi^2} \frac{(2I + 1)(2J + 1)}{D_{3n,-nQ}(\alpha)D_{I_3,Q-T_3}(\beta)} k_{I, J}(r, t), \tag{3.11}
\]

where we introduced the angular momenta \( J^{bf} \) and \( I^{bf} \) relative to body-fixed axes as

\[
J^{bf} = \frac{\partial L}{\partial \alpha}, \quad I^{bf} = \frac{\partial L}{\partial \beta}, \tag{3.12}
\]

such that the axial symmetry of the classical solution gives rise to the constraint

\[
J_2^{bf} = -n\left( J_3^{bf} + T_3 \right). \tag{3.13}
\]

For non-strange dibaryons \((T = \frac{3}{2}|S| = 0)\), the states of the form \(|I, I_3; J, J_3; I_3^{bf} = Q, J_3^{bf} = -nQ, T_3 = 0\rangle\) are the eigenstates of \( H_{\text{rot}} \) and the rotational energy is given by

\[
E_{\text{rot}} = \frac{1}{2\Omega} \left[ \frac{4}{n^2 + 3} J(J+1) + I(I+1) - \frac{4n^2}{n^2 + 3} Q^2 + c^2 \right]. \tag{3.14}
\]

For strange dibaryons \((T = \frac{3}{2}|S| > 0)\), the eigenstates are obtained after the diagonalization of \( H_{\text{rot}} \). For dibaryons with strangeness \( S = -1 \) the isospin of the bound antikaons is \( T = \frac{1}{2} \) and the Hamiltonian is a \( 2 \times 2 \) matrix.

If the basis states are \(|I, I_3; J, J_3; I_3^{bf} = Q \mp \frac{1}{2}, J_3^{bf} = -nQ, T_3 = \mp \frac{1}{2}\rangle\), the rotational energy is given by

\[
E_{\text{rot}} = \frac{1}{2\Omega} \left[ \frac{4}{n^2 + 3} J(J+1) + I(I+1) - \frac{4n^2}{n^2 + 3} Q^2 + c^2 \right] + \frac{c}{2\Omega} \begin{bmatrix}
Q^{-\frac{1}{2}} + \frac{\sqrt{2}}{4} c & [I(I+1) - Q^2 + \frac{1}{4}]^{1/2} \\
[1(I+1) - Q^2 + \frac{1}{4}]^{1/2} & -Q^{-\frac{1}{2}} + \frac{\sqrt{2}}{4} c
\end{bmatrix}. \tag{3.15}
\]

For dibaryons with strangeness \( S = -2 \) the isospin of the bound antikaons must be \( T = 1 \), since the antikaons are bosons, and the rotational Hamiltonian has the form of a \( 3 \times 3 \) matrix. If the basis states are chosen as \(|I, I_3; J, J_3; I_3^{bf} = Q, J_3^{bf} = -nQ, T_3 = 0\rangle\) and \(|I, I_3; J, J_3; I_3^{bf} = Q \mp 1, J_3^{bf} = -nQ, T_3 = \pm 1\rangle\), ordered according to increasing \( T_3 \) values, the rotational energy is given by

\[
E_{\text{rot}} = \frac{1}{2\Omega} \left[ \frac{4}{n^2 + 3} J(J+1) + I(I+1) - \frac{4n^2}{n^2 + 3} Q^2 + c^2 \right] + \frac{c}{2\Omega} \begin{bmatrix}
2(Q-1) + c & [2I(I+1) - 2Q(Q-1)]^{1/2} & 0 \\
[2I(I+1) - 2Q(Q-1)]^{1/2} & c & [2I(I+1) - 2Q(Q+1)]^{1/2} \\
0 & [2I(I+1) - 2Q(Q+1)]^{1/2} & -2(Q+1) + c
\end{bmatrix}. \tag{3.16}
\]
In (3.14)-(3.16) the values of $I$ and $J$ are subject to constraints $I \geq |I^{(0)}_2|$ and $J \geq |J^{(0)}_2|$. For $n = 2$ the parity of the states is $P = (-)^Q$ and $Q$ must be an integer.

We now apply the quantum stabilization method due to MW, generalized to the baryon number $n > 1$ sector and regularized using the modified radial ansatz $^6$), to the energy of the non-strange dibaryon (3.14). Thus we minimize the energy of the Skyrme soliton only, and not the energy of the kaon–soliton system as a whole. The MW method gives the following expression for the moment of inertia (3.6) occurring in (3.14)-(3.16)

$$\Omega = \frac{\pi}{4F_\pi} b \left\{ \frac{3}{2} \left( \frac{4}{\pi} \right)^2 \frac{1}{a b} \left[ \frac{4}{n^2 + 3} J(J + 1) + I(I + 1) - \frac{4n^2}{n^2 + 3} Q^2 \right] \right\}^{3/4}$$  \hspace{1cm} (3.17)

with

$$a = 2 \int_0^\infty dx \left[ (x + 1)^2 \left( \frac{d \Phi}{dx} \right)^2 + 2n(2n - 1) F^2 \right], \hspace{1cm} (3.18)$$

$$b = \int_0^\infty dx \frac{8}{3} x^2 \sin^2 F(x), \hspace{1cm} (3.19)$$

where $x = r/R$ is a dimensionless variable.

Thus the rotational energies of dibaryons in the simplified CHK approach are given by eqs. (3.14)-(3.16), where the moment of inertia $\Omega$ is given by eq. (3.17) together with eqs. (3.18) and (3.19). The numerical results for $E_{rot}$ in MeV for some dibaryon states ($n = 2$) in the $S = -1$ and $S = -2$ sector, for $N_c = 3$ and using the rotational energy of the NN state in ref. $^7$) as the base value, are given in table 1.

The predicted dibaryon masses are in qualitative agreement with the values calculated with the complete CHK model (including the Skyrme stabilizing term proportional to $e^{-2}$) in ref. $^7$). The states with $n = 2$ are classified and compared with the corresponding quark model states in ref. $^7$) and we do not discuss their classification in the present paper.

The differences between our results and those given in ref. $^7$) can be explained by the considerable simplification of the employed model, where only the soliton energy (and not the energy of the kaon–soliton system as a whole) is minimized.

<table>
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<th>Dibaryon</th>
<th>$I$</th>
<th>$S$</th>
<th>$E_{rot}$</th>
<th>$E_{rot}$ [ref. $^7$]</th>
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<td>147</td>
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<td>$-2$</td>
<td>213</td>
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</table>
and the dynamical term in the rotational perturbation is only evaluated to first order. As in ref. 7) the pion mass $m_\pi$ was assumed to be zero.

4. Conclusions

The present paper shows a possibility to use the Skyrme model, or more precisely the CHK model, for calculation of the dibaryon spectrum without use of the Skyrme stabilizing term, proportional to $e^{-2}$, which makes the practical calculations very lengthy and tedious. The results are in qualitative agreement with those of the complete CHK model (with the stabilizing term included). No attempt to calculate the rotational energy of NN state has been made. It was simply adopted as a base value. Furthermore, the pion mass was assumed to be zero ($m_\pi = 0$).

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