THE THEORY OF THE ELEMENTARY PARTICLES

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§1. HISTORICAL INTRODUCTION

The concept of the elementary physical particles as a few types of simple physical entities which, by their interplay, generate the entire physical world may be said to have started its development in 1897, when the independent existence of the negative electron was established, notably by J. J. Thomson. The next important advance in the development was made in 1911, when Rutherford revealed the essential structure of the atom by his experiment on the scattering of $\alpha$ particles. By the middle of the third decade of this century, the picture of the atom as we know it now, though not including a knowledge of internal structure of the nucleus, had taken shape. It was known that the atom had a heavy nucleus whose radius was of the order $10^{-12}$ cm., and in which was concentrated all but about one part in two thousand of the total mass of the atom. The charge on the nucleus was invariably positive, and an exact integral multiple of the charge of the electron, while the masses of proximate nuclei differed by very nearly an integral multiple of the mass of the proton, the lightest nucleus known.\(^*$

This period marks a certain fairly definite stage in the development of the idea of the elementary particles. All known physical entities belonged to one of two types. On the one hand there were the electromagnetic and gravitational fields, whose state could only be described by giving the value of the field quantities at all points of space. On the other, there were the material particles, of which only two were elementary, the electron and the proton. All atoms and their nuclei were considered to be built up by putting together varying numbers of these two types of elementary particles. Moreover, it was believed that two particles of the same type were indistinguishable, and each one permanent and immutable. For example, the helium atom had two orbital electrons, while its nucleus, being about four times as heavy as the proton, was considered to be made up of four protons, the electric charge of two of these being neutralized by assuming two further electrons to be bound up inside the nucleus. The fact that the nuclei of certain radioactive elements emitted electrons was considered as evidence in favour of this picture.

However, this simple picture was soon seen to require modification. First, the discovery of the fact that in certain circumstances material particles, like electrons, show some of the typical behaviour of waves, necessitated the well-known step from Newtonian mechanics to quantum mechanics. Conversely, the fact that in certain circumstances light shows some of the characteristics of discrete

\(^*$ The mass of every nucleus is very nearly an integral multiple of one-sixteenth the mass of the oxygen (16) atom, and this integer is known as the mass number. As is well known, the mass of the proton is 1.00813 in terms of this unit, and hence the masses of heavy nuclei expressed in terms of the proton mass may differ considerably from the mass number.
particles necessitated the quantization of the Maxwell equations, a step which had been foreshadowed as far back as 1900 by Planck. Thus the sharp distinction between particles and fields had already become blurred, though, as we shall see presently, the complete fusion of the wave and particle aspects only takes place within the framework of relativistic field theory coupled with second quantization.

Similarly, the rapid development of theory, as well as the accumulation of further experimental knowledge of the nucleus, soon showed that all matter could not be thought of as built up from electrons and protons only. First, the formulation of the relativistic wave equation of the electron by Dirac in 1927 showed that it was theoretically impossible to bind an electron in a space as small as that of the nucleus, however strong the force might be. Secondly, the measurement of the spin of several nuclei and the type of statistics they obey showed that if there were electrons in the nucleus, these did not contribute either to the spin or the statistics. For both the electron and the proton have a spin angular momentum \( \frac{1}{2} \) in units of Planck's constant \( h \) and obey the Fermi-Dirac statistics. Hence the deuteron, for example, which has a unit charge and twice the mass of the proton, should, if it were made up of two protons and one electron, obey Fermi statistics and have half integral spin, the orbital angular momentum due to the motion of these particles in a nucleus being always an integer. Actually the deuteron has a spin 1 and obeys Bose-Einstein statistics. The same holds for the nitrogen nucleus of mass 14. The general experimental result is that the spin of the nucleus is half odd integral or integral, depending on whether its mass number is an odd or even integer, and the statistics is then respectively Fermi-Dirac or Bose-Einstein. Thus the assumption that the nucleus is made up of protons and electrons does not lead to agreement with the results of experiment.

The difficulty was resolved by the discovery of the neutron by Chadwick in 1932, a particle which has nearly the same mass as the proton, but no charge. One could then unify all the facts about nuclei on the assumption that the neutron also had a spin \( \frac{1}{2} \) and obeyed Fermi-Dirac statistics, by supposing all nuclei to be built up of protons and neutrons only. The neutron must be looked upon as another elementary particle, for the same arguments as apply to nuclei prevent its being considered as made up of a proton and an electron.

The very admission of the neutron as an elementary particle, however, necessitated the abandonment of one of the properties hitherto attributed to an elementary particle, namely its permanence. For the existence of radioactive nuclei emitting electrons leads to the conclusion that in the act of \( \beta \)-emission a neutron must convert itself into a proton and the emitted electron be born. Since the spin of each of these particles is \( \frac{1}{2} \) and the orbital spin can only be an integer, one must assume with Pauli that yet another electrically uncharged elementary particle, known as a neutrino, probably having a spin \( \frac{1}{2} \), must be emitted with the electron. Moreover, the neutrino also allows one to assume that energy is conserved in the process of \( \beta \)-emission. For the energy states of the parent and newly formed nuclei are sharply defined and the change in energy of the nucleus is therefore fixed, whereas the electrons in \( \beta \)-decay are known to come out with a continuous range of energies less than this fixed value. One
may suppose the difference in energy to be carried off by a neutrino, which, because of its weak interaction with matter, eludes observation. The $\beta$-decay thus appears as the reaction

$$N \rightarrow P + e^- + n,$$

where $N$, $P$, $e^-$ and $n$ stand for the neutron, proton, electron and neutrino respectively.

In relativity theory the energy $E$ and momentum $p$, being the components of a vector, are connected by the formula

$$E^2 = p^2 + M^2,$$

where $M$ is an arbitrary constant which can be interpreted as the mass of the particle. (The velocity of light $c$ can be taken to be unity by a suitable choice of the unit of time. Similarly, Planck's constant $\hbar$ may also be taken to be unity.) Thus for every value of the momentum there must correspond two values of $E$, and this feature must also occur in any relativistic quantum theory. As is well known, the Dirac equation allows states of the electron in which its total energy in a vacuum is negative. This difficulty, which would rob the theory of any physical meaning, was solved by Dirac by the assumption that all states of negative energy were already occupied, each with one electron. The further assumption that electrons obey the Fermi-Dirac statistics, as is known from experiment, now prevents any electron in a positive energy state from falling into a negative energy state. The (infinite) density of electrons in negative energy states which uniformly pervades the whole of space is assumed to be physically unobservable. Any departure from this regularity would, however, be observable. Thus, it can be shown quite easily that a "hole" made by removing one of the (negatively charged) electrons from a state of negative energy would behave exactly like an electron of positive energy and positive instead of negative charge. That the "hole" would behave like a particle of the same rest mass merely follows from the fact that the connection between the energy and momentum of the hole is the same as that for the electron whose absence creates it. Thus the necessity of interpreting the Dirac equation in a physically intelligible way leads to two important conclusions. First, the electron must obey Fermi statistics, for without the operation of Pauli's exclusion principle electrons in positive energy states are not prevented from falling into states of negative energy even if these are occupied. Secondly, electrons of positive charge, or positrons as they are now called, must exist in nature and be capable of being created in pairs (each with a negative electron) if enough energy is available for the purpose. Thus the Dirac theory for the first time explained a fundamental fact about the elementary particles, namely that particles of spin $\frac{1}{2}$ must obey the Fermi-Dirac statistics. It also provided for the first time an actual mechanism by which energy could be transformed into matter. For example, just as a quantum of light is absorbed by an electron in an atom, so a quantum whose energy was greater than $2M$ could be absorbed by an electron in a negative energy state, which would then jump to a positive energy state and become physically observable, at the same time leaving a hole behind which would manifest itself as a positron. In this process, which may be represented by the equation

$$h\nu \rightarrow e^- + e^-,$$
where $e^+$ stands for the positive electron, the energy of the quantum is converted into the rest mass and kinetic energy of the two particles.

The predictions of the Dirac theory have been remarkably confirmed by the discovery of the positron by Anderson in 1932, and by the observation of the process of pair creation in cosmic-ray showers by Blackett and others. Not only has the theory been quantitatively confirmed by experiment at energies of a few million volts, but the formulae for radiation loss and pair creation can be shown to be valid to the highest energies. For these processes form the basis of the cascade theory put forward in 1937 by Bhabha and Heitler and Carlson and Oppenheimer, and its predictions are in good agreement with experiments on the creation of cosmic-ray showers and the absorption of the soft component in cases where the energy of the particles and quanta involved is many thousand times the rest mass of the electron.

The Dirac theory brings about a further development of our concept of the elementary particles. The positron must certainly be considered as an elementary particle in the sense that it is not made up of other elementary particles. And yet its existence is indissolubly connected with that of the electron. One cannot make a physically satisfactory theory of electrons which does not at the same time lead to the existence of positrons and vice versa. One must therefore look upon both electrons and positrons as manifestations of one elementary physical entity, the quantized electron-field, which in the course of its interaction with other physical entities takes on states which manifest themselves by the existence of a certain number of electrons and positrons.

We now come to the latest stage in the development of the theory of the elementary particles which is intimately bound up with the theory of nuclear forces. It has been found from a study of nuclei that the forces which act between any two nuclear particles have the following general characteristics:

1. The force is a short-range force much stronger than the coulomb force when the two particles are within a distance of about $2 \times 10^{-13}$ cm. of each other, and falls rapidly to zero for larger distances, so that, for example, the nuclear force between the nuclei of two neighbouring atoms in a molecule is entirely negligible;
2. The magnitude of the force between like particles (two protons or two neutrons) and unlike particles (a neutron and a proton) appears to be of the same magnitude in corresponding states;
3. The force between a proton and a neutron appears to be an "exchange force" involving the transformation of the neutron into a proton and vice versa whenever interaction takes place.

The first successful effort to explain some of the features of nuclear forces was made by Yukawa in 1935. Starting from the fact that the equation

$$\left\{ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \chi^2 \right\} U = 0$$

has a static singular solution of the form

$$U = e^{-\chi r}$$

where $r = (x^2 + y^2 + z^2)^{1/4}$, a circumstance which had already been made use of

* This expression refers to the wave-field describing the electron, and must be distinguished from the electromagnetic field produced by the electron.
when Einstein introduced the so-called $\lambda$-term in his gravitation theory,* Yukawa assumed that the nucleons (a convenient word which is used to denote either a proton or a neutron) generate a field which satisfies equation (4). The resulting interaction between two nucleons is then a short-range one as required, and its magnitude can be adjusted by a proper choice of the value of the constant connecting the nucleons with the $U$-field. Equation (4) is, however, just the relativistic wave equation of a particle whose energy and momentum are connected by the relation (2) with $\chi \hbar$ written in place of $M$. It therefore describes a particle of mass $M = \chi \hbar$. The observed range of nuclear forces requires $\chi$ to have the value $0.5 \times 10^{13}$ cm.$^{-1}$, and $M$ has then a value of about 170 times the mass of the electron. (In anticipation of the later development, we shall straightway describe a particle of roughly this mass as a meson.) Further, in order to take account of the exchange character of the interaction between a proton and neutron, one must assume that the meson carries a unit positive or negative electric charge. Denoting the meson by $\mu$, we thus have two processes,

\[
\begin{align*}
N & \leftrightarrow P + \mu^- , \\
P & \leftrightarrow N + \mu^+ .
\end{align*}
\]

Yukawa fitted the $\beta$-decay into this picture by assuming that the meson had an interaction with an electron and a neutrino exactly analogous to (6). We write it in the more suitable form,

\[
\begin{align*}
\mu^+ & \leftrightarrow e^+ + n, \\
\mu^- & \leftrightarrow e^- + n .
\end{align*}
\]

By choosing the constant connecting the meson-field with the electron-field to be very small it is possible to make the theory give the rate of $\beta$-decay correctly. The process (1) must now be considered to take place in two virtual stages, namely, first (6) and then (7).

No elementary particle of rest mass intermediate between those of the proton and electron was known in 1935. But as a result of the clarification that had been introduced by the cascade theory in cosmic-ray phenomena, it soon became evident that a particle of this type must be present in cosmic rays, and by 1938 its existence had been definitely established. However, there was yet no experimental information about the spin of the meson, and as far as theory was concerned it could be either 0 or 1.

For a discussion of the meson theories of nuclear forces we refer the reader to a Report by R. Peierls (1939) in this series. One may sum up the present position by saying that the best account of nuclear forces is given by the theory of Möller and Rosenfeld (1939, 40). In this theory positively and negatively charged mesons of spin 0 and 1 play a part, as well as neutral mesons. It will appear below that it is precisely with these assumptions that the best account of cosmic-ray phenomena can also be given.

The discovery of the meson introduces yet a new feature into our concept of the elementary particles. Since the mass of the meson is greater than that of

\* It is interesting to note that the introduction of this term gives the gravitation quantum a rest mass $\hbar / \lambda$, which is now directly connected with the size of the finite (but unbounded) universe, since the latter is determined by $\lambda$. 

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the electron and neutrino, it is possible for the process (7) to take place in the absence of other matter. The meson must therefore be unstable, and it can be computed, by using constants chosen to fit β-ray data, that its average time of decay is of the order $10^{-8}$ sec. The spontaneous decay of the meson makes the meson a natural clock and provides a means of testing a prediction of the special theory of relativity in an extreme case. If $\tau$ is the mean time of decay at rest, then a meson of energy $E$ must have a mean decay time $\tau E/\mu$, where $\mu$ is the rest mass (Bhabha, 1938). Experiments on the decay time of mesons of different energies have been made which confirm the slowing down of clocks in rapid motion in a striking way (cf. Euler and Heisenberg, 1938; Blackett, 1938; Rossi and Hall, 1941). Rossi and Hall find, however, that the mean proper lifetime of cosmic-ray mesons at sea-level is about $2.4 \times 10^{-6}$ sec., a discrepancy with the calculated lifetime which had already been noticed earlier. Møller, Rosenfeld and Rozental (1939) have pointed out that the disagreement can be made to disappear on the basis of Møller and Rosenfeld’s theory of nuclear forces by assuming different decay times for mesons of spin 0 and 1. Hamilton, Heitler and Peng (1943) have found that a quantitatively consistent picture of cosmic-ray and nuclear phenomena can be given by assuming that the primary cosmic rays are protons which in the very topmost layer of the atmosphere lose their energy by the rapid creation of mesons of spin 0 and 1. They further assume, following Møller and Rosenfeld, that mesons of spin 1 have a life of $10^{-8}$ sec., while mesons of spin 0 have a life of $10^{-6}$. The latter constitute most of the hard component in the atmosphere, and it is their lifetime which is measured in experiments. The former decay almost at the moment they are formed, and are responsible for creating the soft component.

All the elementary particles so far known in nature have a spin of 0, $\frac{1}{2}$ or 1 unit.* The question now arises whether elementary particles of higher spin can exist. The only theoretical approach to this problem is to investigate whether consistent equations for particles of higher spin can be set up by taking as our guide some general principles underlying theories of particles of spin 0, $\frac{1}{2}$ and 1, which we may be reasonably certain are correct. The latest work on this question is reviewed in the next two sections.

As is well known, there are certain difficulties which still encumber the present field theories. They are of two types. First, there are the infinities which have an analogue in classical theory, as, for example, the infinite electromagnetic field energy of a point electron. Secondly, there are infinities of a specifically quantum nature for which there is no classical analogue. Such difficulties are, for example, concerned with what is known as the dynamical or transverse self-energy of the electron, and the fluctuation of the infinite zero-point energy of a quantized field. Considerable progress has been made in recent years in overcoming these difficulties, and this will be reviewed in § 4. The problems have not, however, been completely solved yet.

* The spins of the proton and neutron have only been observed in non-relativistic circumstances. The scheme put forward in § 3 (ii) would allow the proton or neutron to have any half odd integral spin greater than $\frac{1}{2}$, while, nevertheless, manifesting the spin $\frac{1}{2}$ only in their normal condition.
§ 2. PARTICLE-FIELDS OF SPIN 0, ½ AND 1

One of the primary requirements of every theory of the elementary particles is that it shall be in keeping with the principles of the special theory of relativity. Any mathematical quantities which appear in the equations must therefore have certain well-defined transformation properties for changes in the frame of reference, and these must be combined in the fundamental equations in such a way that the latter remain unaltered by any transformation of the Lorentz group involved in going from one Cartesian frame to another moving relative to it with uniform velocity. The only quantities of this description are tensors and spinors,* and one should expect that only those would occur in the theory of an elementary particle which are irreducible, that is, which cannot be split into two or more parts in a relativistically invariant way. Correspondingly, we should expect the equations describing an elementary particle-field to be irreducible, that is, incapable of being split into separate sets of equations in a relativistically invariant way. Indeed, the fact that the elementary particles interact with each other and in some processes one type may disappear, giving place to another variety, as, for example, in the reactions (6) or (7) mentioned above, raises the question as to what precisely constitutes an elementary particle-field. One can answer this question as follows: An elementary particle-field is one described by an irreducible set of equations. In this sense the proton and neutron, despite the fact that they can transform into each other, must be regarded as the manifestation of two different particle-fields, while the positron and electron are manifestations of the same irreducible particle-field. A spinor has two types of indices which transform according to different inequivalent representations of the Lorentz group, and it is usual to distinguish these by putting a dot over indices of one type and leaving those of the other type undotted. We denote a spinor index by a Greek letter which can take on the values 1 and 2. Tensor indices will be denoted by Latin letters and take on the values 0, 1, 2, 3. Spinor indices can be lowered and raised just like tensor indices, and a repeated Latin or Greek index automatically implies summation over the values 0 to 3 in the former case and 1 and 2 in the latter. A spinor  with having  undotted and  dotted indices is irreducible if it is symmetric in all the  undotted indices and symmetric in all the  dotted ones. It has only \((k+1)(l+1)\) independent components, and it is said to transform according to the irreducible representation \(\mathcal{D}(k, l)\) of the proper Lorentz group. The transformation of the Lorentz group which merely reverses the direction of the three space axes turns it into the spinor  with  undotted and  dotted indices, which accordingly transforms according to the representation \(\mathcal{D}(l, k)\). Both must appear in any equation for an elementary particle in order that it should be invariant under all transformations of the full Lorentz group including reflections. We denote the co-ordinates by \(x^k\) and take the metric tensor \(g_{kl}\) in the form 
\[
g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \quad g_{kl} = 0, \quad k \neq l. 
\]
A transformation \(t\) of the Lorentz group is one whose coefficients \(t^k_l\) by definition satisfy the equation
\[
g_{kl} = g_{rs} t^s_r t^t_k. \quad \ldots \ldots (8)
\]

* Tensor and spinor calculus are essential for this work, and their use cannot be avoided altogether. However, the reader unfamiliar with them will probably be able to pick up some general ideas by reading the text.
An infinitesimal transformation is given by
\[ t^k = \delta^k_l + \epsilon^k_l, \]
where the \( \epsilon^k_l \) are arbitrary small quantities and \( \delta^k_l = 0, k \neq l \). \( \delta^0_0 = \delta^1_1 = \delta^2_2 = \delta^3_3 = 1 \). It follows from (8) that \( \epsilon^k_l = -\epsilon^l_k \). This infinitesimal transformation is represented in any representation, irreducible or otherwise, by a matrix
\[ T = 1 + \frac{1}{2} I^r \epsilon_{rs} \]
where 1 stands for the unit matrix. The infinitesimal transformations \( I^r \) completely determine the representation, and they satisfy certain commutation relations which only depend on the group and not on the representation concerned. These are
\[ [I^{mn}, I^{rs}] = I^{mn} I^{rs} - I^{rs} I^{mn} = -g^{mn} I^{rs} + g^{ms} I^{nr} + g^{nr} I^{ms} - g^{ns} I^{mr}. \]

The simplest relativistic wave equation is (4), the so-called Klein-Gordon equation. It is obviously equivalent to the set of linear differential equations
\[ \begin{align*}
U_k &= \frac{\partial}{\partial x^k} U, \\
\frac{\partial}{\partial x_k} U_k + \chi^2 U &= 0,
\end{align*} \]
as can be seen by eliminating the \( U_k \) from the second equation by using the first set. This equation describes a particle of spin zero since everything about the particle is known if the function \( U \) depending only on the co-ordinates is known.

The generalization of the Maxwell equations given by Proca (1936) is
\[ \begin{align*}
\chi G_{kl} &= \left( \frac{\partial}{\partial x^k} U_l - \frac{\partial}{\partial x^l} U_k \right), \\
\frac{\partial}{\partial x_k} G_{kl} + \chi U_l &= 0.
\end{align*} \]

These have since been used to describe the meson. There are four wave functions \( U_k \) with one relation between them, namely \( \frac{\partial}{\partial x_k} U_k = 0 \), which can be deduced from (12), so that for every plane wave there are three different solutions. It therefore corresponds to a particle of spin 1 (in units of \( \hbar \)).

Finally, the electron is described by the Dirac equation, which we write in the form
\[ (\gamma^k p_k + \chi) \psi = 0, \]
where \( p_k = -i \frac{\partial}{\partial x^k} \) and the \( \gamma^k \) are four matrices satisfying
\[ \gamma^k \gamma^l + \gamma^l \gamma^k = 2g^{kl} \]
As is well known, the relativistic invariance of (13) requires the \( \gamma^k \) to satisfy the commutation relations
\[ [\gamma^k, I^{mn}] = g^{kl} \gamma^m - g^{km} \gamma^l, \]
irrespective of whether the \( \gamma^k \) satisfy (14) or not. The \( I^r \) in this case are the infinitesimal transformations which determine the transformation property of the wave function \( \psi \), and must satisfy (10). They can be interpreted as the operators connected with the spin of the particle. The algebra defined by (14)
The theory of the elementary particles has only one irreducible representation, namely, the usual one by matrices of four rows and columns.

Now both (11) and (12) can obviously be written in the form (13) if we take for the $x^k$ matrices whose elements can be read off from (11) and (12) respectively. It was first noted by Duffin that in both cases the matrices so obtained satisfy the commutation rule

$$x^k x^m + x^m x^k = g^{kl} x^l + g^{km} x^k.$$ \hspace{1cm} \ldots \ldots (15)

Conversely, Kemmer (1939, 43) has investigated the algebra generated by (15) and shown that there are only two non-zero irreducible representations, namely, those by square matrices of five and ten rows respectively. Thus (13) with the $x^k$ satisfying (15) is mathematically equivalent to (11) or (12), depending on which representation is taken. This proof of equivalence, however, rests on taking a suitable representation of the matrices satisfying (15) and then showing that the resulting equations are (11) or (12). This procedure also has the disadvantage that the matrix method cannot be used in the calculation of processes involving interaction with nucleons, for there the field quantities such as $U_k$ appear explicitly in the different interaction terms, and not $\phi$.

This lacuna has recently been filled by Harish-Chandra (1946 b). He has shown that when the $x$ matrices satisfy (15), a matrix $\Gamma_k$ of one row and four columns always exists such that

$$\Gamma_k x_i = -\Gamma_k x_i,$$

$$\Gamma_k x_i x_m = \Gamma_k g_{lm} - \Gamma_l g_{km}.$$ \hspace{1cm} \ldots \ldots (16)

A transition from (13) to (12) is now immediately possible. Defining two quantities $U_k$ and $G_{kl}$ by

$$U_k = \Gamma_k \psi,$$

$$G_{kl} = i\Gamma_k x_l \psi = -i\Gamma_l x_k \psi = -G_{lk},$$ \hspace{1cm} \ldots \ldots (17)

and multiplying (13) by $\Gamma_i$ on the left, we get

$$-ip_k G^k + \chi U_i = 0.$$ 

Multiplying (13) by $\Gamma_i$ gives

$$p_i \Gamma_i x_m x^k \psi + \chi \Gamma_i x_m \psi = 0$$

or, by (16),

$$p_m U_i - p_l U_m - i\chi G_{lm} = 0.$$

These equations are just (12). An immediate correspondence between the "wave" formulation (12) and the "particle" formulation (13) can therefore be established, allowing a transition from one to the other at any stage of the calculation.

Precisely the same can be done for the five-rowed representation of the $x$'s. The relations (16) have then to be replaced by certain others. The introduction of the $\Gamma_k$ and a study of their algebra also allows general expressions to be written down immediately for the spurs ("traces") of the products of an arbitrary number of $x$'s. The formalism should therefore prove of assistance in actual calculations.

It is important to note that in the case of all the spins considered above, the
infinitesimal transformations $I^{kl}$ which determine the transformation properties of the wave function, and hence the spin of the particle, are given by

$$I^{kl} = [x^k, x^l].$$

(18)

Insertion of (18) into (13 a) shows that the $x$'s satisfy

$$[x^k, [x^l, x^m]] = g^{kl}x^m - g^{km}x^l,$$

(19)

which is, therefore, a relation satisfied by the matrices for particles of spin 0, $\frac{1}{2}$ and 1.

It is fairly certain that the equations (11), (12) and (13) correctly describe the behaviour of particle-fields of spin 0, $\frac{1}{2}$ and 1. In order to develop equations for higher spin values one must find some general principles common to all of them. These are:

A. It can be deduced from the equations that each component of the wave function satisfies the second-order wave equation (4). This is physically equivalent to the statement that the particle described by the field has in each case only one value of the rest mass (except for sign).

B. The particle-field is completely described by an equation of the form (13) without the help of any further subsidiary conditions. The transformation properties of the wave function, and hence the spin of the particle, are determined entirely by the infinitesimal transformations $I^{kl}$ defined by (18). Equivalently, the $x$ matrices satisfy (19) for all spins.

§3. PARTICLE-FIELDS OF SPIN GREATER THAN 1

(i) The Dirac-Fierz-Pauli scheme

In 1936 Dirac gave a scheme of equations for particles of higher spin, taking as his guiding principle the postulate that the statement A shall hold for all spins. His equations connect a symmetric spinor $a_{\mu\nu...}^{\alpha\beta...}$ in $2k$ undotted and $2l$ dotted indices, with a symmetric spinor $b_{\mu\nu...}^{\alpha\beta...}$ in $2k+1$ undotted and $2l-1$ dotted indices, where $k, l$ are integers or half odd integers. The differential operator $p_k$ in spinor notation takes the form $p^{\lambda\mu...}$. Dirac's equations for a particle of spin $n = k + l$ are

$$p^{\lambda\mu...} a_{\mu\nu...}^{\alpha\beta...} = \chi b_{\lambda\mu...}^{\alpha\beta...},$$

(20)

and the equations obtained from these by a reflection. Either $a$ or $b$ can be eliminated from the equations, and it can be seen at once that each component separately satisfies the second-order wave equation (4). Further, contracting, say, the indices $\lambda$ and $\mu$ in the second equation, it immediately follows, on account of the symmetry of $a$ in all its indices, that

$$p^{\lambda}_{\lambda\mu...} b_{\lambda\mu...}^{\alpha\beta...} = 0.$$

(21)

A similar equation holds for $a$. Now if we define another spinor $\epsilon_{\nu\mu...}^{\alpha\beta...}$ by the equation

$$\chi \epsilon_{\nu\mu...}^{\alpha\beta...} = p^{\lambda\mu...} b_{\lambda\mu...}^{\alpha\beta...},$$

(22 a)

* This follows from the fact that the spinors $\epsilon^{\alpha\beta...}$ and $\epsilon_{\nu\mu...}$ used for raising and lowering spinor indices are antisymmetric.

† See note added in proof, page 271.
then this spinor is symmetric in all its indices on account of (21), and we can immediately deduce from (22 a) that the equation
\[ \mathbf{p}_\mu \, \sigma_{\mu\nu...} = \chi^\nu_{\mu\nu...} \] 
also holds. (22) are therefore equivalent to (20). This process can be continued till one arrives at a spinor which has only dotted or undotted indices. There are, therefore, \( n \) different equations, if \( n \) is an integer, which are equivalent in the force-free case. For \( n \) half an odd integer there are \( n + \frac{1}{2} \) sets of equivalent equations, which all describe a particle of the same spin \( n \).

Fierz (1939) has quantized the scheme in the force-free case. It can be shown that for particles of integral spin the quantization can only be carried through so as to lead to Bose-Einstein statistics for the particles. On the other hand, for half-integral spins the total energy of the field is positive in states of positive mass and negative in states of negative mass. A physically sensible interpretation therefore requires that the field for particles of half odd integral spin shall be so quantized that the particles obey Fermi-Dirac statistics, in order that one may then overcome the difficulty about states of negative energy in the same way as in Dirac's electron theory. The essence of the connection between spin and statistics has been brought out by Pauli (1940). It has therefore been possible to find an explanation for the long suspected and empirically found connection between spin and statistics.

The Dirac scheme does not include all the equations which are consistent with the assumption A. For example, the scalar wave equation (11) for a particle of spin 0 cannot be written in the form (20). In fact, there is a corresponding set of equations for higher spins, namely,
\[ \begin{align*}
\hat{p}^\mu_a z^a_{\mu...} &= \chi^{\nu b}_{\mu...} \\
\hat{p}^\mu_i b^a_{\mu...} &= \chi^{a b}_{\mu...} \end{align*} \]  
(23)

The Dirac scheme also has several serious disadvantages. The equations cannot be derived from a variation principle. This leads to the difficulty that when interaction with an electromagnetic field is introduced in the usual way by writing
\[ \Pi_k = -i \frac{\partial}{\partial x^k} + e A_k \]  
(24)
in place of
\[ \hat{p}_k = -i \frac{\partial}{\partial x^k}, \]
where \( A_k \) are the electromagnetic potentials, the equations become inconsistent, as has been shown by Fierz and Pauli (1939). The difficulty is in fact that the equations derived from a simple variation principle can only be reduced to (20) if the spinors satisfy the subsidiary conditions (21), and these do not come directly out of a variation principle. Moreover, these conditions are necessary in order that the statement A should be satisfied.

It has been shown by Fierz and Pauli (1939) that it is possible to derive the equations (20) as well as the subsidiary conditions (21) from a Lagrange function.
by a special and rather complicated procedure. For example, a particle of spin $\frac{3}{2}$ can be described in Dirac's scheme by two spinors $a^x_\mu$ and $b^y_\mu$. To derive these equations from a variation principle Fierz and Pauli introduce two subsidiary spinors $c^\alpha$ and $d^\beta$ in the Lagrange function. The variation of all these spinors independently leads to 16 equations. The arbitrary constants in the Lagrangian are then so adjusted that in the force-free case these equations lead to the identical vanishing of the spinors $c$ and $d$ on the one hand and to the equations (20), with the subsidiary conditions (21), on the other. Interaction with an electromagnetic field can then be introduced by the usual procedure (24). In the presence of interaction, however, the subsidiary spinors do not vanish identically, but there are, nevertheless, just so many equations that for any given value of the momentum of the particle there are only four independent states, the number which a particle of spin $\frac{3}{2}$ must have.

The procedure becomes progressively more complicated for larger spins: for example, for a particle of integral spin $n$ which can be described by a tensor $A_{k_1k_2...k_n}$ symmetric in its $n$ indices and satisfying $A_{k_1k_2...k_n}=0$, one has to introduce one subsidiary field $A_{k_1...k_n-2}$ of rank $n-2$, one of rank $n-3$, two of rank $n-4$, three of rank $n-5$ and so on to $n-2$ fields of rank 0. The Lagrangian is then built up as a sum of invariant terms quadratic in these fields multiplied by arbitrary coefficients. These coefficients are then adjusted so that as a result of the equations all the subsidiary fields vanish in the force-free case, and lead to the equations of Dirac together with the required subsidiary conditions for the principal field. Since the number of coefficients is larger than the number of conditions to be imposed, there remains an arbitrariness in the formulation which would manifest itself when interaction with other particle-fields is present.

It can also be proved (Bhabha, 1945 b) that for the equations of this scheme the statement B is certainly not satisfied. Moreover, this extended scheme has not been quantized, and it remains to be seen whether this can be carried through in a satisfactory manner.* In any case, we should expect a fundamental physical entity like an elementary particle-field to be described by equations which are simple and elegant, like those for particles of spin 0, $\frac{1}{2}$ or 1, and it therefore seems to me unlikely that the complicated scheme given above could be the correct one for describing particles of higher spin, if such exist.

(ii) The alternative scheme

An alternative scheme of equations was recently put forward by the author (Bhabha, 1945 a-45c) which takes as its point of departure the postulate that the statement B shall hold for all spins. The condition (18) plays a vital rôle now. One introduces a new index 4 and defines certain new quantities $f^{4k}$ and $g^{k4}$ by

$$f^{4k} = -f^{4k} = \varepsilon^k, \quad g^{44} = -1, \quad g^{kk} = g^{4k} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (25)$$

Then both (10), (13 a) and (18) can be combined into the one equation (10) by

* I have been informed by Prof. W. Pauli, in a letter, that the quantization of the scheme in the presence of electromagnetic interaction is involved and difficult, and that unpublished calculations by Weinberg and Kusaka exist on this subject.
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letting the indices take on the values 0 to 4 instead of 0 to 3.* The resulting ten matrices $I^{EL}$, where a capital index runs from 0 to 4, are then seen to satisfy the same commutation rules as the nucleus of a representation of the Lorentz group in five dimensions, and the problem of finding an irreducible representation of the $z$'s satisfying (19) can be solved immediately by using the theory of the representation of the Lorentz group in five dimensions, which is all known. Each irreducible representation gives one irreducible equation for an elementary particle-field. Every irreducible representation of the Lorentz group in five dimensions is characterized by two numbers $n$ and $m$, which are both positive integers (or zero) or both half odd integers such that $n > m$. We denote this representation by $\mathcal{R}(n, m)$. For this representation the eigenvalues of all the $z$'s and $I$'s (multiplied in some cases by $i$) are $n, n-1, \ldots, -(n-1), -n$, which may occur more than once. Hence the four $z$'s and six $I$'s, multiplied in some cases by $i$, satisfy the algebraic equation

$$\{X^2 - n^2\} \{X^2 - (n-1)^2\} \{X^2 - (n-2)^2\} \ldots = 0.$$  

(26)

which has just these roots. The last term is $X$ or $X^2 - \frac{1}{4}$, depending on whether $n$ is an integer or half odd integer. This equation is of importance. Denote the operator which occurs in the wave equation (13) by $P = x^k p_k$. The $z$'s being matrices of finite degree, $P$ must satisfy an algebraic equation of the same degree in which the $p_k$ occur as coefficients. The Lorentz invariance of this algebraic equation now requires that the $p_k$ should only occur in the invariant form $p^2 = p^k p_k$. Now considering the equation in the system in which $p_0 \neq 0$, $p_k = 0$ for $k = 1, 2, 3$, $P$ reduces to $p_0 x_0$, and since $x_0$ satisfies the equation (26), $P$ satisfies the equation

$$\{P^2 - n^2 p_0^2\} \{P^2 - (n-1)^2 p_0^2\} \ldots = 0.$$  

Operating on the wave function $\psi$ with this equation, and replacing every $P$ by $-\chi$, by a repeated use of the original wave equation (13), we find that every component of the wave equation satisfies

$$\{n^2 p_0^2 - \chi^2\} \{(n-1)^2 p_0^2 - \chi^2\} \ldots \psi = 0,$$  

(27)

the last term being $\chi$ or $\{\frac{1}{2} p_0^2 - \chi^2\}$, depending on whether $n$ is an integer or half an odd integer. The physical meaning of this circumstance is that the equation (27), and hence (13), has solutions in which energy and momentum are connected by the equation (3), with $M$ given respectively, for integral or half odd integral $n$, by

$$\pm \frac{\chi}{n-1}, \pm \frac{\chi}{n-2}, \ldots, \pm \chi \text{ or } \pm 2\chi.$$

(28)

The particle-field therefore describes particles with several different values of the rest mass, but these must all be considered as different states of the same physical entity, just like the positron and electron, since the equations are irreducible.

For $n = \frac{3}{2}$ there are two different values of the rest mass $M = \frac{2 \chi}{3}$ and $3M$.

* I have been informed by Prof. W. Pauli, in a letter, that this circumstance has been noted by Lubansky (1942). There appear to be other papers along somewhat similar lines by Kramers, Belinfante and Lubansky (1941) and Lubansky and Rosenfeld (1942). Since these were published in the Dutch Physica during the war, I have been unable to see any of them.
Only for \( n = \frac{1}{2} \) or 1 is there a unique value of the mass, and the statement \( A \) is then also satisfied.

It can be shown (Bhabha, 1945 c) that the equations (13) can be derived from a Lagrangian

\[
L = \psi^\dagger \mathbf{D} \{ \dot{x}^k p_k + \chi \} \psi, \quad \ldots \ldots \quad (29)
\]

where

\[
\mathbf{D} = \text{const. } e^{i\pi \alpha_0}, \quad \ldots \ldots \quad (30)
\]

these expressions holding for all spins. \( \psi^\dagger \) is the Hermitian conjugate of \( \psi \). \( \mathbf{D} \) commutes with \( x^0 \) and anti-commutes with the other three \( x^k \). The algebraic equation (26) satisfied by \( x^0 \) depends on the maximum value of the spin \( n \), and hence the Hermitian matrix \( \mathbf{D} \) reduces to a polynomial in odd or even powers of \( x^0 \) with the leading term \( (x^0)^m \). (30) reduces to the well-known expressions found by Dirac and Kemmer if \( n = \frac{1}{2} \) or 1 respectively. Interaction with an electromagnetic field can now be introduced by the procedure (24). The current is given by

\[
s^k = - e \psi^\dagger \mathbf{D} x^k \psi \quad \ldots \ldots \quad (31)
\]

and the energy tensor by

\[
T^{kl} = \psi^\dagger x^k p^l \psi. \quad \ldots \ldots \quad (32)
\]

This tensor can be symmetrized as usual. The formal analogy with the Dirac equation of the electron is therefore complete.

The two numbers \( n \) and \( m \), characterizing an irreducible representation \( \mathcal{R}(n, m) \) of the Lorentz group in five dimensions, and hence a possible elementary particle-field, can be given a physical interpretation. \( n \) must be considered to be the maximum value of the spin of a particle described by the field, since it is the maximum eigenvalue of the spin operator. The physical significance of \( m \) becomes clear in the non-relativistic approximations of the equation. Thus, it can be shown that non-relativistically in the state of lowest rest mass the wave equation reduces to

\[
\left\{ \pi_0 - \frac{1}{2M} \pi^k x^k + \frac{e}{2nM} H_I K^I(m) \right\} \psi(n) = 0, \quad \ldots \ldots \quad (33)
\]

where a bar below an index denotes summation over 1, 2 and 3 only. \( M = \chi/n \) is the lowest value of the mass, \( H_I \) are the three components of the magnetic field and \( K^I(m) \) are the three matrices representing the three components of angular momentum belonging to the total spin value \( m \). The particle therefore behaves exactly like a particle of spin \( m \) and magnetic moment \( e/2nM \) in its lowest mass state. \( m \) is therefore the spin of the particle in its stable state of lowest mass. The formula (33) includes at one stroke all the non-relativistic approximations that have so far been given for the electron and the meson, besides those for particles of higher spin in their state of lowest mass. For example, the Dirac equation is described in this scheme by \( n = m = \frac{1}{2} \), and (33) shows that its magnetic moment is \( e/M \). The Klein-Gordon equation is described by \( n = 1, \quad m = 0 \), and (33) shows that its spin is zero. Finally, the vector meson equations are described by \( n = m = 1 \), and (33) then shows that the magnetic moment is \( e/2M \) with a spin 1. Thus in this scheme of equations the different values of \( m \) for the same \( n \) describe physically different fields, unlike Dirac’s theory for higher spins, where there are a number of mathematically equivalent alternatives for the same value of the spin \( n \).

It has been pointed out (Bhabha, 1945 c) that the equation given by the scheme \( \mathcal{R}(\frac{3}{2}, \frac{1}{2}) \) describes a particle which behaves non-relativistically in its state of
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lowest rest mass like a particle of spin \( \frac{1}{2} \), as equation (33) shows, and it may therefore describe the behaviour of the proton. The experimental evidence so far at our disposal does not exclude this possibility, since all direct and indirect evidence of the spin of the proton has been obtained in non-relativistic conditions. Moreover, for this equation the particle has only one other value of the rest mass, and this is three times the normal value. Only in cosmic-ray phenomena is there enough energy to throw a proton into this state, and the chance of its observation even then would be very small because of its very short life. Other equations which are also not excluded for describing the proton are \( \mathcal{A}(n, \frac{1}{2}) \) (\( n \) half an odd integer). The number of different values of the rest mass is then \( n + \frac{1}{2} \) and the ratio of the second lowest to the lowest value is \( n/(n - 1) \). Thus a very large value of \( n \) is certainly not permissible. The behaviour of the proton in the relativistic region would be different for all these equations, and it would be possible to decide in favour of one of these against the others by experiments on cosmic radiation, where it is believed that primary protons are responsible for creating mesons by a process analogous to the emission of radiation.

In the irreducible representation \( \mathcal{A}(n, m) \) the \( \alpha \)'s have the eigenvalues \( n, n - 1, \ldots, -n \), which occur, say, \( \nu_n, \nu_{n-1}, \ldots, \nu_1 \) times, the actual values of the \( \nu \)'s depending on the value of \( m \). It is convenient to take \( \alpha^0 \) in the diagonal form and bring equal eigenvalues next to each other by a rearrangement of the rows and columns thus:

\[
\alpha^0 = \begin{bmatrix}
    nE(\nu_n) \\
    (n - 1)E(\nu_{n-1}) \\
    \vdots \\
    \vdots \\
    -nE(\nu_1)
\end{bmatrix}
\]

where \( E(\nu) \) stands for the unit matrix of \( \nu \) rows and columns. All the other elements are zero. \( D \) than takes the form

\[
D = -\begin{bmatrix}
    E(\nu_n) \\
    -E(\nu_{n-1}) \\
    \vdots \\
    \vdots \\
    (-1)^{2n+1}E(\nu_1)
\end{bmatrix}
\]

For solutions in which the particle is at rest (\( p_k = 0, k \neq 0 \)) \( \psi \) has the form

\[
\psi = \phi e^{i \frac{X}{\hbar} t}
\]

\( \ldots (36) \)
(where $t$ has been written in place of $x^0$). The $\phi$'s are independent of the co-ordinates and describe the state of the spin. Substitution of (34) and (36) into (13) then shows that only those components $\phi(j)$ of $\phi$ are non-zero which correspond to the eigenvalue $-j$ of $x^0$ in (34). This solution describes a state in which the particle has rest mass $\chi j$ and is at rest. Since the sign of $D$ alternates in the successive squares in (35) we have the following situation. If $n$ is an integer, the sign of $Dx^0$ is opposite for the eigenvalues $j$ and $-j$ of $x^0$, while if $n$ is half an odd integer the sign is the same. Hence, for integral spins, for every state of rest mass $\chi j$ there is another state with the opposite sign of $j$ in (36) for which the charge density is opposite, while the total energy given by (32) is the same. The theory therefore leads to particles of positive and negative charge for each value of the rest mass exactly as in the case of spin 0 and 1. For half odd integral spins, for every state of rest mass $\chi j$ there is another in which the charge is the same, but the energy opposite. Thus the same connection exists here between spin and statistics as exists for the equations of Dirac.

The quantum conditions for the wave function are

$$[(\psi(x')Dx^0)_{\mu}, \psi^*(x)] = \delta_{\mu}^\nu \delta(x'^1 - x^1) \delta(x'^2 - x^2) \delta(x'^3 - x^3),$$

where, on the left-hand side, $[\ ]$ stands for a commutation bracket for integral spins and an anticommutation bracket for half-integral spins. The quantization of this scheme requires a special and somewhat unusual procedure. Consider integral spins. In the state of lowest mass, $\chi / n$, the energy of the field is always positive. In the next lowest mass state, $\chi / (n - 1)$, however, the energy is always negative for both signs of the mass. This difficulty can be avoided as follows. When second quantization is introduced, one assumes that the metric in Hilbert space corresponding to the next lowest state of the mass is negative definite for all momentum states belonging to this mass value. Since a Lorentz transformation cannot change one mass value into another, this procedure is relativistically invariant. In the case of half-integral spins the difficulty is more serious. The alternation in sign of $Dx^0$ has the effect that the coefficients of the Fourier components in the analysis of $\psi$ belonging to the lowest but one value of the mass have to satisfy commutation relations of the type

$$-[b* b + b b*] = 1,$$

which are mathematically impossible if $b^*$ is the Hermitian conjugate of $b$. The mathematical difficulty can be overcome by adopting a procedure recently given by Dirac (1942, cf. Pauli, 1943) in another context. We assume $b^*$ to be the adjoint of $b$ with an indefinite metric in Hilbert space. With a proper choice of metric, $b^*$ then becomes $-b^*$ or minus the Hermitian conjugate of $b$, and (37) can be satisfied. This procedure does not lead here to the same difficulties of physical interpretation as when it is applied to quantization in accordance with the Bose-Einstein statistics. For, due to the Fermi-Dirac statistics, in each state of a given momentum, energy and spin there can be either one particle or none, and one can so arrange the metric that the "negative probability" connected with an indefinite metric only occurs for the situation where no particle is present in that state. The negative probability then has no effect on the
charge or energy, since both are zero in this state. The theory therefore leads to physically reasonable results in every case.

The above discussion shows that the statements A and B are both fulfilled only in the case of spins 0, $\frac{1}{2}$ and 1. The postulate that they should both hold would therefore exclude any elementary particles of spin greater than 1. It has, however, been pointed out by Fierz and Pauli that the equations given by them for a particle of spin 2 and rest mass zero are exactly the same as Einstein's approximation to the gravitation equations for weak fields. Particles of spin 2 but zero rest mass do therefore appear to exist in nature (see also footnote on page 257 (§ 1) regarding the rest mass of the "graviton").

§ 4. THE DIVERGENCE DIFFICULTIES OF QUANTIZED PARTICLE-FIELDS

It now remains to consider briefly the limitations and difficulties of the quantized field theories for particles of spin 0, $\frac{1}{2}$ and 1. It is well known that the attempt to calculate a process to a higher approximation than the first one in which the process appears leads to infinities. In the case of the interaction of the electron with the electromagnetic field the second approximation is small compared with the first for most physical processes, and the difficulty is not serious. It remains, however, in principle and leads to such theoretically unsatisfactory results as an infinite self-energy of the electron. In the case of mesons and their interaction with nucleons, the higher approximations are not small compared with the first, and some solution of these difficulties is a practical necessity. As mentioned in the introduction, some of these difficulties exist also in classical theory, while others have no classical analogue. We consider them separately.

The classical difficulties are essentially of the following nature. The energy of the electromagnetic field produced by, say, an electric charge $e$ spread over a sphere of radius $r$ is of the order $e^2/r$, and, therefore, tends to infinity as $r \to 0$. The moment therefore that we deal with point electrons, the field energy must become infinite. It has, however, been pointed out (Bhabha, 1940) that this infinity has a physical meaning and corresponds to the fact that the work done in compressing a finite electric charge to a point is infinite. There is however no logical reason for regarding a point charge as the limit of a finite distribution, and it would be contrary to our concept of the elementary particles to think of them as made up in this way. It was first shown by Dirac (1938) that classically consistent equations of motion of a point charge can be found, and Pryce (1938) showed that the field energy could be redefined to agree with these equations. A general proof that this can be done for all types of fields and for point particles having a charge, dipole or higher multipole moment can be given, based on the inflow theorem (Bhabha and Harish-Chandra, 1945) which holds for every energy tensor which is conserved in empty space. Thus consistent equations for a point particle can always be found. For all fields satisfying the condition A or equation (4) it can be proved further (Bhabha and Harish-Chandra, 1946; Harish-Chandra, 1946 a) that a modified definition of the energy tensor can be given which is free from the worst singularities of the usual tensor. It can be shown that the equations for the particle so obtained can be put into Hamiltonian form by using the $\lambda$-limiting process of Wenzel and Dirac. Thus classically
the difficulties may be considered now to be completely removed with one exception—the methods do not immediately work in cases where the world lines of two particles cross.

Quantum mechanically, therefore, the λ-limiting process will remove all singularities where the corresponding classical singularities have been removed by it. It will not work in processes like pair creation (Pauli, 1943), but there is no difficulty in dealing with a process where the electric charge does not change, as, for example, in (6), where the charge of the proton is carried off by the newly-born meson.

Coming now to the specifically quantum-mechanical difficulties, these are due to the virtual emission and re-absorption of very energetic quanta, as, for example, in the transverse self-energy of the electron. An attempt to remove these has been made by Dirac (1942). His procedure requires the introduction of an indefinite metric in Hilbert space for certain momentum states of the radiation field. This has the result that those states in which an odd number of quanta are present in these momentum states appear with a "negative probability", a circumstance which has no direct physical meaning. A different scheme for interpreting his equations has, therefore, been given by Dirac. But other difficulties remain in the Dirac scheme. Although the "negative energy" quanta cancel out the usual infinite transverse self-energy of the electron, they also cancel out several higher approximations which should be of physical significance. For example, Eliezer * has shown that higher-order corrections to the Klein-Nishina formula, which should represent the effect of the classical radiation damping, are also cancelled out.

Finally, attempts have been made (Heitler, 1941; Wilson, 1941) to derive from the usual perturbation equations of quantum mechanics another set which is devoid of the singular terms by simply leaving these terms out. The resulting equations then allow the calculation of higher-order processes. They thus allow radiation damping to be taken into account for processes involving the scattering of mesons by nucleons (Heitler and Peng, 1942). As far as scattering is concerned, the cross-sections obtained are essentially the same as those obtained on the basis of the classical theory with radiation damping (Bhabha, 1941). The quantum theory also permits the calculation of processes where one meson produces several others in collision with a proton.

To sum up, one may say that although it cannot yet be claimed that the problem of the divergence difficulties in the quantum theory of particle-fields has been completely solved, very considerable progress has been made in several directions and the theoretical position has definitely become clearer. One may reasonably hope that the remaining problems will be solved by a straightforward extension of the present formalism.

Our present picture of the physical world and the elementary particles may be formulated as follows. The permanent things in nature are certain generalized concepts like energy, momentum, angular momentum and electric charge, which are always conserved, while the actual elementary particles themselves are but the transitory embodiments of their metamorphoses. The underlying elementary particle-fields must, however, be conceded a permanent existence, for

* Unpublished work.
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although at some given instant no particle of that field, as, for example, an electron, may be in existence, one could always demonstrate the continuing existence of the underlying particle-field by performing an experiment in which a particle of that type is created. In the equations for these particle-fields certain quantities enter as arbitrary mathematical constants, as, for example, the constants associated with the masses of the particles of the different fields, the electronic charge $e$, the “mesic charge” of a nucleon, and so on, which have quite definite values in nature. The explanation of why these constants have the empirically found ratios to each other appears to be completely outside the scope of the present scheme. To explain these ratios some radically new idea is required.

REFERENCES


Note added in proof (see footnote to page 262). It would be consistent with this standpoint of statement B to assume that the particle-field is completely described by an equation of the more general form

$$(\mathbf{x} \cdot \mathbf{p}_k + \mathbf{\beta} \chi) \psi = 0,$$

where $\beta$ is an invariant matrix which commutes with all the $I^{kl}$. The connection (18) between the $I^{kl}$ and the $\chi$'s can then be generalized by making use of the matrix $\beta$. 