PARTONS IN QUANTUM CHROMODYNAMICS

Guido ALTARELLI

Istituto di Fisica, Università di Roma,
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy

NORTH-HOLLAND PUBLISHING COMPANY-AMSTERDAM
PARTONS IN QUANTUM CHROMODYNAMICS

Guido Altarelli
Istituto di Fisica, Università di Roma,
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy

Received 20 July 1981

Contents:

1. Introduction 3
2. Gauge theories and the QCD Lagrangian 5
3. Asymptotic freedom 10
   3.1. Asymptotic scale invariance and the renormalization group equations 11
   3.2. The running coupling in asymptotically free theories 15
   3.3. The beta function in one and two loops 18
   3.4. Asymptotic expansions 21
4. Inclusive leptoproduction in the leading logarithmic approximation 25
   4.1. The QCD improved parton model 26
   4.2. The general evolution equations 32
   4.3. The splitting functions and their physical interpretation. The factorization theorem 34
   4.4. Explicit solution near $x = 1$ 42
   4.5. Some properties of moments 44
   4.6. Polarized parton densities 47
5. Leptoproduction beyond the leading logarithmic approximation 50
6. The photon structure functions 64
7. The Sudakov form factor of partons 68
8. Jets in leptoproduction and their transverse momentum 70
9. $e^+e^-$ annihilation 77
   9.1. The total hadronic cross section 77
   9.2. Scaling violations for fragmentation functions 80
   9.3. Annihilation into real photons 90
   9.4. Jets 91
10. Heavy quarkonium decays 100
11. Drell–Yan processes 104
12. Electro-weak form factors of hadrons 113
13. QCD effects in weak non leptonic amplitudes 116
14. Outlook 120

References 121

Abstract:
An overall view of the physics of QCD in the perturbative domain is presented in a form that could be of use both as an introduction to the subject with its main lines of current development and as a reference review text for more expert readers as well.
1. Introduction

Quantum Chromodynamics (QCD) as a theory of strong interactions ([216, 203, 204, 413, 242a]; for a general review see [319]) is the final outcome of two decades of continuous developments, both experimental and theoretical, in many areas of particle physics including hadron spectroscopy, the phenomenology of deep inelastic phenomena, the theory of relativistic quantum fields and the study of the structure of electroweak interactions. The fundamental notion of coloured [235, 251] fractionally charged quarks ([215, 434]; for a review see [236]), really revolutionary because of the apparent confinement of these objects, emerged out of the study of hadron spectroscopy and was later established by the successes of the (so to say) naive parton model ([185, 73] for reviews: [186, 285]) in electron and neutrino scattering and finally in e+e− annihilation. The demise of hadrons from the rank of elementary particles to the more modest status of composite objects has brought back the possibility of a description of strong interactions in terms of a renormalizable quantum field theory. A parallel remarkable progress in the theory of quantum fields was taking place on the formulation [428], the quantization ([180], see also [184, 424, 318, 197, 404]) and the renormalization ([260, 305]; see also [302]) of non Abelian gauge theories on the development of operator expansion techniques [420, 82, 202, 433] and of renormalization group equation methods ([393, 214, 77, 344a, 96, 394], see also for reviews [316, 359]) leading in particular to the notion of asymptotic freedom [261, 242a, 361] and the realization of the unicity [122] in this respect of non Abelian gauge theories among renormalizable theories. At the same time the construction and the impressive experimental success of the gauge theory of electroweak interactions ([412, 376, 224]; for recent reviews see for example: [11, 207]) imposed gauge symmetry as the unifying principle in the theory of fundamental interactions, gravity included, thus directly pointing to an extension of the same structure to the strong interactions as well. The gauging of an exact colour symmetry is then the simplest and most natural possibility.

The selection of SU(3) as colour gauge group is unique in view of (a) the fact that the group must admit complex representations because it must be able to distinguish a quark from an antiquark (there are meson states made up of q̅q but not similar qq bound states). (b) There must be colour singlet (because we see no colour replicas of known hadrons), completely antisymmetric barionic states made up of qqq in order to solve the statistics puzzle for the lowest lying baryons of spin 1/2 and 3/2 in the 56 of (flavour) SU(6) (see for example [120]). (c) The number of colours for each kind of quarks must be in agreement with the data on the total hadronic e+e− cross-section and on the π⁰→2γ rate ([5, 65], see also [80, 241]) (which are proportional to the number of colour replicas and its square respectively) and also, to some extent, on other processes like lepton pair production and the semi-leptonic branching ratio of the τ lepton. Within simple groups (a) restricts the choice to SU(N) with N ≥ 3, SO(4N + 2) with N > 1 (SO(6) has the same algebra as SU(4)), and E6, and then (b) and (c) directly lead unambiguously to SU(3) with each flavour of quarks in a fundamental representation of the group. Too many coloured quarks not only would violate (c) but could also spoil asymptotic freedom, as we shall see later.

Thus QCD as a gauge theory of quarks and gluons is unique among renormalizable theories in providing a basis for the parton model within the principles of relativistic quantum field theory and it stands today as a main building block of the standard model based on SU(3)⊗SU(2)⊗U(1) for the chromo-electro-weak interactions (at down to earth energies).

Although QCD emerges as essentially the only possible theory for the strong interactions within reach of the weapon arsenal of present theoretical physics, yet QCD is far from being established even as a viable theory. Two main questions are indeed pending. First this theory is founded on the
Guido Altarelli, Partons in quantum chromodynamics

conjecture that colour symmetry is exact. Second in its present formulation one is not yet able to prove that confinement is implied by the theory as a necessary consequence. On this last crucial question some insight has been accumulated in the last few years from the study of gauge theories on a lattice and from the analysis of the structure and the topological properties of non Abelian gauge theories (see for example [82a] for a summary). The results are encouraging and make confinement perhaps more believable today than in the past. On the other hand the observation of colour liberation would also impose a radical revision of our views.* Therefore QCD must still be considered as a tentative theory. Thus testing QCD is particularly important in that it represents the less established sector of the standard model. Testing QCD is however difficult. It is more difficult than testing the electro-weak sector, because there the interaction is so weak that perturbation theory is almost always reliable and moreover the leptons are at the same time the fields in the Lagrangian and the particles in our detectors. On the contrary QCD is a theory of quarks and gluons while the real world is made up of hadrons. Also perturbative methods, our almost unique tools, are only applicable in those particular domains of strong interaction physics where the freedom, which is only asymptotic, can actually be reached.

Deep inelastic phenomena immediately emerge as the natural testing ground for QCD. In fact, on one hand, the hadronic non perturbative unknowns are reduced to a minimum when restricting to interactions of hadrons with currents. On the other hand the conditions of high energy and deep inelasticity make perturbation theory applicable. The difficulty of testing QCD is reflected in the fact that no single process provides by itself a clearcut and definite experimental proof of the theory, at least when practical limitations in the experimental possibilities are taken into account. In view of a number of ifs and buts that can be raised against any given experiment it is clear that our confidence in QCD rests at present on the overall picture as is emerging from several processes and many different kinds of tests.

The present article is devoted to a review of QCD in the perturbative domain. Given the enormous amount of literature on this subject my aim is not completeness in any sense. Rather the idea is to sketch an overall view of the field with a description of the main methods and applications. The discussion is limited to the most fundamental aspects of hard processes. Unless explicitly stated only those results are described in detail which are considered as established in QCD. Those ramifications which are not followed to the end are however listed and reference to a few key papers is given. More speculative extensions, less direct applications involving some degree of model dependence and issues still subject to debate are in general left out.

I am not aiming at rigour or at proving every single statement. Rather I tried to present all important ideas and results in the most intuitive and physically motivated terms I was capable to find, without however indulging in deceptive oversimplifications. At the same time I made an effort to collect all results in their most practical forms (hopefully with all correct factors and signs) with mention of the approximations involved so that this article can possibly serve as a reference to more expert readers as well.

The readers who are interested in a first orientation in the field with particular emphasis on its phenomenological aspects, without entering too soon into technical matters, are urged to start the reading directly from section 4 and to go back to the field theoretic preliminaries in the first sections on a second reading.

This is mainly a theoretical review and I did not attempt to include a systematic review of the experimental literature. I feel I have no qualifications for that. However statements on the comparison

* The observation of fractionally charged particles \( q = \pm 1/3, \pm 2/3 \) has been reported in one single experiment [301a, b], out of many others so far attempted. Similar, although not identical experiments, led to negative results [319a]. We stress, in any case, that a fractionally charged particle is not necessarily coloured.
between theory and experiment and some references are scattered all over the paper. A few illustrative examples of experimental data are shown for the most relevant quantities.

There are already in the literature a number of fine reviews on the subject written at different times and with different purposes (for example: Politzer [362]; Gross [240]; Gaillard [213]; Ellis, Sachrajda [168]; Brodsky, Lepage [87]; Ross [371]; Llewellyn-Smith [311]; Field [187]; Buras [90b]; Dokshitzer, Dyakonov, Troyan [151]; Reya [368]; Mueller [332]; see also Llewellyn-Smith [314]).

2. Gauge theories and the QCD Lagrangian

Consider a Lagrangian density \( \mathcal{L}[\phi, \partial_\mu \phi] \) which is invariant under a \( D \) dimensional continuous group of transformations:

\[
\phi' = U(\theta^A)\phi \quad (A = 1, 2, \ldots D).
\]

For \( \theta^A \) infinitesimal \( U(\theta^A) = 1 + ig \sum_A \theta^A T^A \) where \( T^A \) are the generators of the group \( \Gamma \) of transformations (2-1) in the (in general reducible) representation of the fields \( \phi \). We restrict here to the case of internal symmetries, so that \( T^A \) are matrices independent of the spacetime coordinates. The generators \( T^A \) are normalized in such a way that for the lowest dimensional non trivial representation of the group \( \Gamma \) (we denote by \( t^A \) the generators in this particular representation) we have:

\[
\text{tr}(t^A t^B) = \frac{1}{2} \delta^{AB}.
\]

The generators satisfy the commutation relations:

\[
[T^A, T^B] = iC_{ABC} T^C.
\]

In the following for each quantity \( V^A \) we define

\[
V = \sum_A T^A V^A.
\]

If one now makes the parameters \( \theta^A \) to depend on the spacetime coordinates \( \theta^A = \theta^A(x_\mu) \), \( \mathcal{L}[\phi, \partial_\mu \phi] \) is in general no more invariant under the gauge transformations \( U(\theta^A(x_\mu)) \), because of the derivative terms. (For an introduction to gauge theories see for example [2, 399, 267].) Gauge invariance is recovered if the ordinary derivative is replaced by the covariant derivative:

\[
D_\mu \phi = \partial_\mu + ig G_\mu,
\]

where \( G_\mu^A \) are a set of \( D \) gauge fields (in one to one correspondence with the group generators) with the transformation law:

\[
G_\mu^A = U G_\mu U^{-1} - \frac{1}{ig} (\partial_\mu U) U^{-1}.
\]

For constant \( \theta^A \), \( G \) reduces to a tensor of the adjoint (or regular) representation of the group:

\[
G_\mu^A = U G_\mu U^{-1} = G_\mu + ig[\theta^A, G_\mu].
\]
which implies:

\[ G'^C_\mu = G^C_\mu - gC_{ABC}\theta^A G^B_\mu \]  

(2-8)

where repeated indices are summed up.

As a consequence of eqs. (2-5, 2-6) \( D_\mu \phi \) has the same transformation properties as \( \phi \):

\[ (D_\mu \phi)' = U(D_\mu \phi). \]  

(2-9)

Thus \( \mathcal{L}[\phi, D_\mu \phi] \) is indeed invariant under gauge transformations. In order to construct a gauge invariant kinetic energy term for the gauge fields \( G^A \) we consider:

\[ [D_\mu, D_\nu] \phi = ig[\partial_\mu G_\nu - \partial_\nu G_\mu + ig[G_\mu, G_\nu]] \phi \equiv igF_{\mu\nu} \phi \]  

(2-10)

which is equivalent to:

\[ F^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - gC_{ABC} G^B_\mu G^C_\nu. \]  

(2-11)

From eqs. (2-1, 2-9, 2-10) it follows that the transformation properties of \( F^A_{\mu\nu} \) are those of a tensor of the adjoint representation:

\[ F^A_{\mu\nu} = UF_{\mu\nu}U^{-1}. \]  

(2-12)

The complete Yang–Mills Lagrangian invariant under gauge transformations can be written down in the form:

\[ \mathcal{L}_{YM} = -\frac{1}{4} \sum_A F_{\mu\nu}^A F^{A\mu\nu} + \mathcal{L}[\phi, D_\mu \phi]. \]  

(2-13)

For an Abelian theory, as for example QED, the gauge transformation reduces to \( U[Q(x)] = \exp(ieQ\theta(x)) \) where \( Q \) is the charge generator. The associated gauge field (the photon), according to eq. (2-6) transforms as:

\[ G'_{\mu} = G_{\mu} - \partial_{\mu} \theta(x). \]  

(2-14)

In this case the \( F_{\mu\nu} \) tensor is linear in the gauge field \( G_{\mu} \) so that in absence of matter fields the theory is free. On the other hand in the non Abelian case the \( F^A_{\mu\nu} \) tensor contains both linear and quadratic terms in \( G^A_{\mu} \), so that the theory is non trivial even in absence of matter fields.

The Lagrangian of QCD is given by:

\[ \mathcal{L}_{QCD} = -\frac{1}{4} \sum_A F_{\mu\nu}^A F^{A\mu\nu} + \sum_{i=1}^f \bar{q}_i(i\hat{D} - m_i)q_i \]  

(2-15)

with

\[ D_{\mu} = \partial_{\mu} + ig \sum_{A=1}^8 t^A G^A_{\mu}. \]  

(2-16)
In eq. (2-15) $j$ is a flavour index, $f$ is the number of quark flavours (we did not show the colour indices of quarks, transforming according to the fundamental representation $3$ of SU(3)colour) and $\hat{D} = \gamma_\mu D^\mu$. The matrices $r^A$ satisfy eqs. (1-2, 1-3). The QCD gauge fields $G^A_\mu$ are called “gluons”.

In QED the photon is coupled to all charged particles in proportion to their charges. In scalar QED there are also sea gull terms, of order charge squared, bilinear in the photon field, which arise from the covariant derivative in the quadratic kinetic energy term of boson fields. In QCD the gluons are coupled to all coloured particles but they themselves carry colour so that they are self coupled. The quartic gluon vertex of order $g^2$ is then the analogue of the sea gull term.

Throughout this article we shall denote by $g$ and $\alpha = g^2/4\pi$ the QCD coupling, while the QED analogue is called here $\alpha_{em} = 1/137$.

If the masses of $L \leq f$ light quarks are neglected $\mathcal{L}_{QCD}$ is also invariant under a global $U(L)_L \otimes U(L)_R$ chiral group. Since no approximate parity doubling of light bound states is observed this symmetry must be either spontaneously or dynamically broken. The pseudoscalar mesons are obvious candidates for would-be-Goldstone bosons [335, 227] associated to the breakdown of the axial group. In particular for $L = 2$ the pion is thought to be the approximately massless Goldstone particle associated with the breaking of $U(2)_L \otimes U(2)_R$ down to $SU(2)_V \otimes U(1)_V \otimes U(1)_A$. $SU(2)_V$ corresponds to the observed isospin symmetry, $U(1)_V$ to baryon number conservation. The so called “U(1) problem” is the problem of identifying the breaking mechanism for the remaining $U(1)_A$. For a review see [134]. A state in the $\eta - \eta'$ space cannot be the associated Goldstone particle because the masses are proven to be too large [414]. However in presence of instantons [64] the associated current $j^A_\mu$ has an anomaly proportional to the topological charge density [263, 268, 98]:

$$\partial_\mu j^A_\mu = I(x) = \frac{g^2}{32\pi^2} \sum_{A=1}^8 F^{A}_\mu\nu \tilde{F}^{A\mu\nu} \tag{2-17}$$

with:

$$\tilde{F}^{A}_\mu\nu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{A}_{\rho\sigma}. \tag{2-18}$$

Although this is the most promising way out of the U(1) problem some objections have been advanced against first generation explanations based on this mechanism [133] and the study of the U(1) problem is still under way [425, 405, 123].

The existence of instantons in QCD also leads to a different problem: the “$\theta$-problem”. For a review see for example [356]. The complicated structure of the quantum vacuum state in presence of non trivial topological effects can be shown to be equivalent [263] to including in the QCD Lagrangian an additional term given by:

$$\mathcal{L}_\theta = -\theta I(x) \tag{2-19}$$

($\theta$ being a fundamental constant associated with the vacuum state). According to the previous discussion (see eq. (2-17)) this term is a total derivative. However because of the non trivial topology of the vacuum state it cannot be ignored. This term induces $P$, $T$ and $CP$ violations in the strong interactions. When the electroweak interactions are added the effective $\theta$ becomes $\tilde{\theta} = (\theta + \arg \det m)$ where $m$ is the quark mass matrix. The $\theta$ problem is the problem of finding a natural explanation for the empirical smallness of $\tilde{\theta} (\tilde{\theta} \approx 10^{-8})$. Various tentative solutions have been proposed [355, 415, 418] (degeneracies in the quark mass matrix, axions) but none seems established at the moment.
These topological effects are negligible in the perturbative domain of interest here and will be ignored in the following.

$\mathcal{L}_{\text{QCD}}$ as given by eq. (2-15) is not sufficient for a direct quantization of the theory. This feature, common to all gauge theories including QED, can be guessed by observing that the free equation of motion for $G^A_\mu$ as obtained from eq. (2-15) is given by:

$$\Box g_{\mu\nu} - \partial_\mu \partial_\nu G^A = 0. \quad (2-20)$$

The propagator of the gauge field should be determined by the inverse of the operator $\Box g_{\mu\nu} - \partial_\mu \partial_\nu$ which however has no inverse, being a projector over the transverse gauge vector states. This difficulty, related to the gauge symmetry of the theory, is removed by fixing a specific gauge (for an alternative method see [354]).

If one chooses a covariant gauge condition $\partial^\mu G^A_\mu = 0$, then a gauge fixing term has to be added to the Lagrangian, which has the form:

$$\Delta \mathcal{L}_{\text{GF}} = -\frac{1}{2\lambda} \sum_A (\partial^\mu G^A_\mu)^2 ; \quad (2-21)$$

$1/\lambda$ acts as a Lagrangian multiplier. The free equations of motion are now modified as follows:

$$\Box g_{\mu\nu} - (1 - 1/\lambda) \partial_\mu \partial_\nu G^A = 0. \quad (2-22)$$

This operator has an inverse whose Fourier transform is given by:

$$D^{AB}_\mu(q) = \frac{i}{q^2 + i\epsilon} \left[ -g_{\mu\nu} + (1 - \lambda) \frac{q_{\mu} q_{\nu}}{q^2 + i\epsilon} \right] \delta^{AB} \quad (2-23)$$

which is the general form of the vector propagator in this class of gauges. $\lambda$ can take any value and disappears in the final expression of all gauge invariant, physical quantities. Important particular cases are $\lambda = 1$ (Feynman gauge) and $\lambda = 0$ (Landau gauge).

While in an Abelian theory the gauge fixing term is all that is needed for a correct quantization, in a non Abelian theory the formulation of complete Feynman rules involves a further subtlety. This is formally taken into account by introducing a set of $D$ fictitious ghost fields that only propagate in closed loops [180]. It is clear that gauge fields connected by a gauge transformation describe the same physics. Ghosts appear, in the form of a transformation Jacobian, in the process of elimination of the redundant variables associated with fields on the same gauge orbit. The ghost contributions are formally generated by an additional term in the Lagrangian density. This is obtained by considering an infinitesimal gauge transformation for the gauge fixing condition $\partial^\mu G^A_\mu = 0$ (see eqs. (2-6, 2-7)):

$$\partial^\mu G^C_\mu = \partial^\mu G^C_\mu - g C_{ABC} \partial^\mu (\theta^A G^B_\mu) - \Box \theta^C = -g C_{ABC} G^B_\mu \partial^\mu \theta^A - \Box \theta^C \quad \text{ (2-24)}$$

where the gauge condition has been taken into account. The ghost Lagrangian is then given by:

$$\Delta \mathcal{L}_{\text{Ghost}} = \eta^C [\Box \delta_{AC} + g C_{ABC} G^B_\mu \partial^\mu] \eta^A . \quad (2-25)$$
The set of $D$ ghost fields $\eta^A$ are to be treated as scalar fields except that a factor $(-1)$ for each closed loop has to be included as for fermion fields.

Starting from a non-covariant gauge fixing condition one can construct ghost free gauges. An example, also important in other respects, is provided by the set of "axial" gauges $n^\mu A^\mu = 0$ (actually for spacelike $n$ one has an axial gauge proper, for $n^2 = 0$ one has a lightlike gauge, and for $n$ timelike one can talk of a Coulomb-like or temporal gauge [299, 290, 200, 374, 114]).

The gauge fixing term is of the form:

$$\Delta \mathcal{L}_{GF} = \frac{-1}{2\lambda} \sum_A (n^\mu A^\mu)^2$$

leading to the propagator, in the limit $\lambda = 0$:

$$D^{\mu\nu}_{AB}(q) = \frac{i}{q^2 + i\epsilon} \left[ -g_{\mu\nu} + \frac{n_{\mu} q_{\nu} + n_{\nu} q_{\mu}}{(nq)} - \frac{n^2 q_{\mu} q_{\nu}}{(nq)^2} \right] \delta^{AB}.$$  \hspace{1cm} (2.27)

In this case there are no ghosts because $(n^\mu A^\mu)'$, obtained by a gauge transformation from $(n^\mu A^\mu)$, contains no gauge fields when the gauge condition $(n^\mu A^\mu) = 0$ is satisfied, as can be inferred by a glance to eq. (2.24).

The Feynman rules of a non Abelian gauge theory are summarized in table 1.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Feynman rules of QCD in covariant or axial gauges</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fermion</strong></td>
<td>$p$</td>
</tr>
<tr>
<td>p</td>
<td>$p - m + i \epsilon$</td>
</tr>
<tr>
<td><strong>Gluon</strong></td>
<td>$A^\mu \overleftrightarrow{B^\nu}$</td>
</tr>
<tr>
<td>q</td>
<td>$\frac{i}{q^2 + i\epsilon} \left[ -g_{\mu\nu} + \frac{n_{\mu} q_{\nu} + n_{\nu} q_{\mu}}{(nq)} - \frac{n^2 q_{\mu} q_{\nu}}{(nq)^2} \right] \delta^{AB}$ (covariant)</td>
</tr>
<tr>
<td><strong>Ghost</strong></td>
<td>$A \overleftrightarrow{B}$</td>
</tr>
<tr>
<td>p</td>
<td>$\frac{i\delta_{AB}}{p^2 + i\epsilon}$ (covariant)</td>
</tr>
<tr>
<td><strong>Fermion</strong></td>
<td>$A^\mu$</td>
</tr>
<tr>
<td>Vertex</td>
<td>$p \mu A$</td>
</tr>
<tr>
<td><strong>Gluon</strong></td>
<td>$[p \cdot q + r - o]$ $g C_{ABC} (g_{aB}(q - p)<em>A + g</em>{aA}(p - r)<em>B + g</em>{aC}(r - q)_C)$</td>
</tr>
<tr>
<td><strong>Triple</strong></td>
<td>$i \lambda A \overleftrightarrow{B \nu}$</td>
</tr>
<tr>
<td>Vertex</td>
<td>$q \nu B$</td>
</tr>
<tr>
<td><strong>Gluon</strong></td>
<td>$\nu C \overleftrightarrow{B \nu}$</td>
</tr>
<tr>
<td><strong>Quartic</strong></td>
<td>$\nu C \overleftrightarrow{D \nu}$</td>
</tr>
<tr>
<td>Vertex</td>
<td>$g C_{ABC} \delta^A$ (covariant)</td>
</tr>
<tr>
<td><strong>Ghost</strong></td>
<td>$r \mu B$</td>
</tr>
<tr>
<td>Vertex</td>
<td>$q B$</td>
</tr>
<tr>
<td><strong>Ghost</strong></td>
<td>$(r \cdot q = p)$</td>
</tr>
<tr>
<td>$g C_{ABC} \delta^A$ (covariant)</td>
<td></td>
</tr>
</tbody>
</table>
The quantum theory so constructed has been proven to be renormalizable and the renormalized theory is gauge invariant ([260, 305], see also [302, 2]).

A number of Casimir factors enter all practical computations. Let $R$ be a representation of the group (either reducible or irreducible) with generators $T^A$. One sets the definitions:

$$\delta_{ab} C_2(R) = \sum_A (T^A T^A)_{ab} \quad (2.28)$$

$$\delta_{AB} T(R) = \text{tr}(T^A T^B) \quad (2.29)$$

$$\delta_{AB} C_2(A) = \sum_{C,D} C_{ACD} C_{BCD} \quad (2.30)$$

where $A$ in $C_2(A)$ stands for the adjoint representation and $a, b$ are colour indices. Note that

$$C_2(R) = \frac{d(A) T(R)}{d(R)} \quad (2.31)$$

with $d(R)$ being the dimension of the representation $R$. In particular $d(A) = D$ is the dimension of the group.

In the following we set for brevity

$$C_A = C_2(A). \quad (2.32)$$

Also if $R = R_q$ with $R_q$ being the reducible representation containing $f$ flavours of quarks, we define

$$C_F = C_2(R_q) \quad (2.33)$$

$$T = T(R_q). \quad (2.34)$$

For $SU(N)_{\text{colour}}$ with each flavour of quarks in the $N$ dimensional (fundamental) representation one has (recall eq. (2.2))

$$C_A = N \rightarrow 3; \quad C_F = \frac{N^2 - 1}{2N} \rightarrow \frac{4}{3}; \quad T = \frac{f}{2} \quad (2.35)$$

where the QCD value obtained for $N = 3$ is also shown.

3. **Asymptotic freedom**

Non Abelian gauge theories are unique among renormalizable theories (in 4 spacetime dimensions) in being asymptotically free. This is the crucial property that makes QCD so prominent a candidate for the theory of strong interactions in that it provides a solid basis for incorporating and extending the extremely successful parton description of deep inelastic phenomena. This section is dedicated to a
simple introduction to the problem of asymptotic scale invariance and its breaking, the renormalization
group equations (RGE), the concept of running coupling constant, the definition and the physical
implications of asymptotic freedom. (For a pedagogical introduction see [121], see also [267].)

3.1. Asymptotic scale invariance and the renormalization group equations

Consider a renormalizable theory with a single (dimensionless) coupling constant $g$, as is the case for
QCD. We shall mostly work with $\alpha$ defined by:

$$\alpha = \frac{g^2}{4\pi}. \tag{3-1}$$

At large values of all external momenta one would expect that physical observables become
approximately independent of masses and exactly so in the limit of neglecting all terms of order mass
divided by the scale of external momenta. Of course this can possibly be true only away from thresholds
and for quantities which are infrared finite in the limit of zero mass. A sufficient, but often too
restrictive, condition is to consider large spacelike external momenta ($p^2 = -x_i Q^2$, $Q^2 \to \infty$) with all
possible quadratic invariants being non vanishing and of the same order as $Q^2$ [411]. When suitable
conditions of this sort are satisfied the asymptotic behaviour of interesting physical quantities can be
studied in the massless theory. Thus let us consider in the following the massless theory.

In the massless theory a naive expectation is that scaling according to canonical dimensions should be
ture on the grounds that no dimensional scale parameters are left in the theory. It is well known that
this naive expectation is false. The reason is that at the quantum level the theory is not completely
specified by the bare Lagrangian but a regularization procedure and a set of renormalization prescrip-
tions must also be added. In the process of giving sense to the massless theory through the
renormalization procedure a finite mass scale must necessarily be introduced for a meaningful definition
of the renormalized coupling $\alpha$, the wave function renormalization constants and so on.

The renormalized coupling $\alpha$ can be defined by some three (or four) point Green function at
specified values of the external momenta. For example in QCD one can take the renormalized (one
gluon vertex function $\Gamma_{3G}(p^2, q^2, r^2)$ at given symmetric values of $p^2, q^2, r^2$
the virtual squared masses of the external gluons) as the quantity defining the renormalized coupling $g$.
A possible choice is $p^2 = q^2 = r^2 = -\mu^2$. The corresponding coupling $\alpha$ is thus a function of $\mu^2$. Note
that $\mu^2 = 0$ is not allowed because of the infrared singularities of $\Gamma_{3G}$ for on shell massless gluons. In
this respect the interplay of infrared and ultraviolet singularities is crucial and in fact will play an
important role in the following. Similarly the definition of the scale for the renormalized fields (after
factorization of the wave function renormalization factor) requires to consider the inverse propagator at
$p^2 = -\mu^2 \neq 0$. It is precisely the appearance of $\mu$ that introduces a scale for momenta, destroys the
naive scaling laws and makes the problem of the asymptotic behaviour of physical quantities much
subtler.

Note that while in the massless theory the introduction of $\mu$ is forced, in a massive theory the
physical masses provide natural scales for the definition of renormalized quantities. But, in massive
theories as well, a definition in terms of $\mu$ is perfectly legitimate. Actually the introduction of $\mu$ is
indeed convenient, because the requirement that physical results do not depend on $\mu$ provides us with a
powerful constraint on the momentum dependence of physical quantities ([393, 214, 77, 344a, 96, 394], see
also for reviews [316, 359]). In fact a change of $\mu$, within a specified definition of $\alpha$ and of the wave
function renormalization constants $Z_i$, in brief within a given renormalization prescription, cannot affect
physical results and is therefore compensated by the corresponding, well defined, change of $\alpha$ and $Z_i$. Note that the renormalized masses can always be defined in terms of the position of the pole in the appropriate renormalized propagator, so that a change of $\mu$ does not affect $m_i$.

Consider the simplest case of an adimensional physical quantity $F$ which depends on a single energy parameter $Q$ in the massless theory. We also assume, for purposes of illustration, to know that for $F$ a change of $\mu$ can be compensated by only a change of $\alpha$. This is seldom the case but examples can be given of physical quantities that indeed satisfy both the above requirements (the ratio $R$ in $e^+e^-$ annihilation is one of these examples, as we shall see). Naive scaling would predict a constant value for $F$ because of its canonical dimensions. However, according to the previous discussion, $F$ (after renormalization) can actually be a function of $Q^2/\mu^2$ and $\alpha$ (itself a function of $\mu^2$). By introducing the variable

$$t = \ln(Q^2/\mu^2)$$

we can in general set:

$$F = F(t, \alpha).$$

(3-3)

The independence of $F$ on $\mu$ strongly constrains the functional form of $F$. In fact we now show that $F$ must depend on $t$ and $\alpha$ through a single variable: the “running coupling constant”.

The invariance under a change of $\mu$ implies the validity of the RGE:

$$\left[ \frac{\partial}{\partial \ln \mu^2} + \frac{\partial \alpha}{\partial \ln \mu^2} \frac{\partial}{\partial \alpha} \right] F(t, \alpha) = 0 \quad (3-4)$$

where the first term takes into account the dependence on $\mu$ through $t$ while the second term is from the dependence through $\alpha$. We introduce the beta function:

$$\beta(\alpha) = \frac{\partial \alpha}{\partial \ln \mu^2}. \quad (3-5)$$

Note that $\beta(\alpha)$ is determined by the theory and it is independent of the particular physical quantity $F$ one is considering. We cast the RGE in the form:

$$\left[ -\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] F(t, \alpha) = 0. \quad (3-6)$$

In order to solve the RGE we introduce a function $\alpha(t)$ defined by:

$$t = \int_0^{\alpha(t)} \frac{dx}{\beta(x)} \quad (3-7)$$

which implies the following properties of $\alpha(t)$:

$$\alpha(Q^2 = \mu^2) = \alpha(\mu^2) \equiv \alpha \quad (3-8)$$
\[ \frac{d\alpha(t)}{dt} = \beta[\alpha(t)] \]  

(3-9)

\[ \frac{d\alpha(t)}{d\alpha} = \beta[\alpha(t)]/\beta(\alpha). \]  

(3-10)

The last two equations immediately follow from eq. (3-7) by taking derivatives of both sides with respect to \( t \) and \( \alpha \) respectively.

The general solution of eq. (3-6) with the boundary condition eq. (3-8) at \( t = 0 \) is given by

\[ F(t, \alpha) = F(0, \alpha(t)). \]  

(3-11)

In fact on any function of the single variable \( \alpha(t) \) the operator relation holds:

\[ \left[ -\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] \left[ -\frac{\partial \alpha(t)}{\partial t} + \beta(\alpha) \frac{\partial \alpha(t)}{\partial \alpha} \right] \frac{\partial}{\partial \alpha(t)} = 0 \]  

(3-12)

where the last equality follows from eqs. (3-9, 3-10). \( \alpha(t) \) (or rather \( g(t) \), see eq. (3-1)) is commonly referred to as the "running coupling constant". Thus the RGE implies that the whole \( Q^2 \) dependence of \( F \) arises through the running coupling \( \alpha(t) \). In turn the dependence on \( \alpha(t) \) of \( F(t) \) is fixed if one knows \( F(0, \alpha) \).

For a general Green function the RGE is more complicated than eq. (3-6) because a change of \( \alpha \) is not by itself sufficient to compensate for a variation of \( \mu \), but a possible change of the wave function (or analogue) factors must also be taken into account. The relation between unrenormalized (as computed from the appropriate Feynman diagrams in terms of bare quantities and some cut-off \( M \)) and renormalized Green functions is of the form:

\[ \Gamma_{\text{UNR}}(Q/M, x, \alpha_0) = Z_T \Gamma_{\text{REN}}(Q/\mu, x, \alpha). \]  

(3-13)

Here \( Q \) is the energy scale and \( x \) are fixed ratios of invariants (scaling variables). Appropriate powers of \( Q \) can always be factored out in order to reduce to an adimensional quantity as in eq. (3-13). The cut-off \( M \) is assumed to carry the dimensions of a mass; formal changes take place in dimensional regularization which are not important to point out now. \( \alpha_0(\alpha) \) is the bare (renormalized) coupling. \( Z_T \) is in general a product of several wave function renormalization factors \( Z_{T_i}^{1/2} \), one for each external field \( \phi_n \), and of factors \( Z_O \) for each local operator \( O \) if \( \Gamma \) refers to the matrix element of a product of local operators. In presence of local operators, \( Z_T \) is in general a matrix mixing together two or more operators. For simplicity we assume here no mixing, so that \( \Gamma \) is multiplicatively renormalizable and \( Z_T \) is a number. In the right hand side of eq. (3-13) all cut-off dependence is contained in \( Z_T \) and \( \alpha \). These quantities are fixed by the renormalization conditions which is where \( \mu \) enters as a subtraction point or a reference mass.

Because \( \Gamma_{\text{UNR}} \) is clearly independent of \( \mu \), so must be the right hand side of eq. (3-13) and one can write the RGE:

\[ \left[ \frac{\partial}{\partial \ln \mu^2} + \frac{\partial \alpha}{\partial \ln \mu^2} \frac{\partial}{\partial \alpha} + \frac{1}{Z_T} \frac{\partial Z_T}{\partial \ln \mu^2} \right] Z_T \Gamma_{\text{REN}}(t, x, \alpha) = 0 \]  

(3-14)
where the additional term with respect to eq. (3-4) takes the $\mu$ dependence of $Z_r$ into account. By removing $Z_r$ we end up with the general RGE:

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_r(\alpha)\right] \Gamma_{\text{REN}}(t, x, \alpha) = 0 \quad (3-15)$$

where $\beta(\alpha)$ was introduced in eq. (3-5) and

$$\gamma_r(\alpha) = \frac{1}{Z_r} \frac{\partial Z_r}{\partial \ln \mu^2} \quad (3-16)$$

is the "anomalous dimension" function, which contrary to $\beta(\alpha)$ clearly depends on the particular quantity $\Gamma$ under consideration. Note that $\beta(\alpha)$ and $\gamma_r(\alpha)$ are independent of $M$ because so is $\Gamma_{\text{REN}}$. In gauge theories, for quantities depending on the gauge choice, there is in general still another term which takes into account the possible variation with $\mu$ of the gauge parameter.

We recall as an example and also for later application that the coefficient functions in a short distance [420] or light cone [82, 202, 433] operator expansion satisfy RGE [395, 97, 113]. Consider the light cone operator expansion for the product of two currents:

$$J(x) J(0) \sim E(x^2) \sum_n C_n(x^2) x^{n_1} x^{n_2} \cdots x^{n_k} O_{n_{1 \cdots k}}^{n_{1 \cdots k}}(0) + \cdots \quad (3.17)$$

where all Lorentz and internal indices are omitted for brevity and one single string of terms is kept in the expansion. $O_{n_{1 \cdots k}}^{n_{1 \cdots k}}(0)$ is a series of local operators, $E(x^2)$ is a singular c-number function whose singularity is dictated by canonical dimensions and $C_n(x^2)$ are c-number coefficient functions that describe the deviations from the canonical behaviour of free field theory.

According to the previous discussion renormalized Green functions with local operator insertions satisfy RGE. Thus one has:

$$(D + 2 \gamma_r + \gamma_1 + \gamma_F) \langle F| J(x) J(0)| I \rangle = 0 \quad (3-18)$$

$$(D + \gamma_{O^s} + \gamma_1 + \gamma_F) \langle F| O_{n_{1 \cdots k}}^{n_{1 \cdots k}}(0)| I \rangle = 0 \quad (3-19)$$

where

$$D = \frac{\partial}{\partial \ln \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} \quad (3-20)$$

and the anomalous dimension functions refer to the two currents, the external states I and F the local operators $O_{n_{1 \cdots k}}^{n_{1 \cdots k}}$. From eqs. (3-17, 3-18) it follows that

$$0 = (D + 2 \gamma_r + \gamma_1 + \gamma_F) \langle F| JJ| I \rangle$$

$$0 = (D + 2 \gamma_r + \gamma_1 + \gamma_F) E(x^2) \sum_n C_n(x^2) x^{n_1} x^{n_2} \cdots x^{n_k} \langle F| O_{n_{1 \cdots k}}^{n_{1 \cdots k}}(0)| I \rangle \quad (3-21)$$

By taking into account the independent tensor structure of each term in the sum and that $E(x^2)$ is
independent of \( \mu \) and \( \alpha \) one obtains the anticipated result:

\[
(D + 2 \gamma_J - \gamma_{\alpha\nu}) C_n(x^2 \mu^2, \alpha) = 0.
\]

(3-22)

Note that the anomalous dimensions of the external fields drop off in the difference, in agreement with the independence of the coefficient functions from the particular matrix element which is considered.

Of particular importance is the case of conserved currents in the massless limit. The conservation of the corresponding charge implies that the anomalous dimension of the current is zero, or better, that there exists a class of renormalization prescriptions where \( \gamma_J \) is zero. In fact the charge commutes with the interaction and as a consequence the interaction does not renormalize the charge. In this case the coefficient functions obey RGE with only the operator anomalous dimensions:

\[
(D - \gamma_{\alpha\nu}) C_n(x^2 \mu^2, \alpha) = 0.
\]

(3-23)

From now on \( \Gamma \) stands for \( \Gamma_{\text{REN}} \). The general solution of eq. (3-15) with the appropriate boundary condition at \( t = 0 \) is given by

\[
\Gamma(t, x, \alpha) = \Gamma(0, x, \alpha(t)) \exp \int_\alpha^{\alpha(t)} \frac{\gamma_r(x)}{\beta(x)} \, dx.
\]

(3-24)

In fact we already know that \( \Gamma(0, x, \alpha(t)) \) is a solution of the equation with \( \gamma_J = 0 \), while the exponential factor is immediately verified to be a solution of the complete equation (by using eqs. (3-9, 3-10) for \( \alpha(t) \)). Thus, provided that \( \Gamma(0, x, \alpha) \) is well defined in the massless theory, the \( t \) dependence of \( \Gamma(t, x, \alpha) \) appears through the running coupling in a form which is in principle prescribed.

In practice however one should explicitly know \( \beta(\alpha), \gamma_r(\alpha) \) and \( \Gamma(0, x, \alpha) \) in order to evaluate eq. (3-24). Actually we are only able to compute these quantities in perturbation theory. This necessarily limited knowledge is however sufficient to construct an asymptotic expansion for \( \Gamma \) in the important case that \( \alpha(t) \) vanishes in the limit \( t \to \infty \) (and only in this case) [421]. This is what is meant by “asymptotic freedom”. In the following we shall concentrate on this case.

3.2. The running coupling in asymptotically free theories

Asymptotically free theories (AFT) are defined by the property that the running coupling vanishes for \( t \to \infty \). We are concerned here with freedom in the ultraviolet region; a completely analogous discussion can also be made for the infrared region \( t \to -\infty \).

From eq. (3-9) we see that, when \( \beta(\alpha(t)) < 0 \), the non negative quantity \( \alpha(t) \) decreases as \( t \) increases. Starting from a value of \( t \) where \( \beta(\alpha(t)) < 0 \), \( \alpha(t) \) decreases with increasing \( t \) down to a value \( \alpha(\infty) \) determined by the condition \( \beta(\alpha(\infty)) = 0 \). A zero in \( \beta \) (with \( \beta \) analytic around the zero point) causes a divergence of the integral in eq. (3-7) so that \( \alpha(\infty) \) corresponds in fact to \( t \to \infty \). On the other hand \( \beta(0) = 0 \) because the \( \mu \) dependence of \( \alpha \) is induced by diagrams with at least one loop. The necessary and sufficient condition for asymptotic freedom is therefore that \( \beta(\alpha) < 0 \) for \( 0 < \alpha \leq \alpha(t = 0) \). For small \( \alpha \) one can compute \( \beta(\alpha) \) in perturbation theory and establish whether or not the theory is asymptotic-
ally free. Once the theory has been proved to be asymptotically free one still needs to assume that the physical values of $\alpha$, for $t \equiv 0$, are in the domain of attraction of the origin, i.e. that there is no other zero of $\beta(\alpha)$ between $\alpha(\mu^2)$ and the origin.

For $g$ defined in terms of trilinear vertices, as in gauge theories, $\beta(\alpha)$ starts at order $\alpha^2$. In fact from eqs. (3-1, 3-5) it follows that

$$\beta(\alpha) = \frac{g}{2\pi} \frac{\partial g}{\partial \ln \mu^2}$$

(3-25)

and $\partial g/\partial \ln \mu^2$ is of order $g^3$ because the $\mu$ dependence is developed when dressing up the lowest order vertex of order $g$ by at least the exchange of one line between the external legs. One can thus write down in general:

$$\beta(\alpha) = -b \alpha^2(1 + b' \alpha + \cdots).$$

(3-26)

In this notation AFT correspond to $b > 0$.

In AFT we can choose $\mu$ large enough that the perturbative expansion for $\beta(\alpha)$, given in eq. (3-26), can be inserted in eq. (3-7) which defines $\alpha(t)$. One then obtains:

$$\frac{1}{\alpha(t)} = \frac{1}{\alpha} + bt - b' \ln \frac{\alpha(t)}{\alpha}.$$  

(3-27)

By neglecting for a moment the $b'$ term one has:

$$\alpha(t) = \frac{\alpha}{1 + b \alpha t} = \frac{1}{b \ln(Q^2/\Lambda^2)}$$

(3-28)

where the last equality stems from the position

$$\ln \frac{\mu^2}{\Lambda^2} = \frac{1}{b \alpha}.$$  

(3-29)

In AFT $\alpha(t)$ decreases to zero logarithmically. The first form in eq. (3-28) corresponds to the leading logarithmic approximation (LLA) for $\alpha(t)$: for $\alpha(t)/\alpha$ only terms of the form $(\alpha t)^n$ are kept at each order $n$ in perturbation theory. On the other hand the second expression in terms of $\Lambda$ is the correct asymptotic behaviour independent on the smallness of $\alpha$.

By restoring the $b'$ term one obtains from eq. (3-27):

$$\alpha(t) = \alpha_0(t) \left[1 + b' \alpha_0(t) \ln \frac{\alpha_0(t)}{\alpha} + o(\alpha_0^2(t))\right]$$

(3-30)

where $\alpha_0(t)$ is given by the expression in eq. (3-28). By a suitable redefinition of $\Lambda$ one can cast this equation in the asymptotic form [90c]:

$$\alpha(t) = \alpha_0(t) \left[1 - b' \alpha_0(t) \ln \frac{Q^2}{\Lambda^2} + o(\alpha_0^2(t))\right]$$

(3-31)
with \( \alpha_0(t) \) given by the same functional form as in eq. (3-28) but in terms of the new \( \Lambda \):

\[
\ln \frac{\mu^2}{\Lambda^2} = \frac{1}{b \alpha} + \frac{b'}{b} \ln b \alpha .
\]  

(3-32)

In comparing the LLA expression for \( \alpha(t) \) in eq. (3-28) with the improved form in eq. (3-31) one must keep \( \alpha(\mu) \) fixed by changing \( \Lambda \). If one compares the two formulae at fixed \( \Lambda \), then the two expressions for \( \alpha(t) \) only coincide at infinite \( Q \) (for \( Q \gg \Lambda \)). On the other hand the physical measurements that fix \( \alpha(\mu) \) are at finite energy, and it is clear that the comparison is to be made at fixed physics.

In the previous discussion we have been working within a given definition of the renormalized coupling \( \alpha = \alpha(\mu) \). However the renormalized coupling can be defined according to infinitely many different prescriptions. For example besides the class of prescriptions that we have already mentioned, i.e. the momentum subtraction schemes, also definitions based on dimensional regularization and minimal subtraction are being widely used. We shall come back shortly on these very practical methods. For the moment we want to consider the relation between two different definition schemes in general terms.

The relation between the couplings \( \alpha \) and \( \alpha' \) in two definitions at the same scale \( \mu \) is in general given by an expansion of the form:

\[
\alpha = \alpha'(1 + k^{(1)} \alpha' + O(\alpha'^2))
\]  

(3-33)

provided that \( \alpha \) and \( \alpha' \) reduce to the same bare coupling in trivial order. Although the value of any physical quantity is unaffected by a change of prescription the functional form of any observable in terms of different couplings is obviously different. In particular all terms in the perturbative expansion of any physical quantity, except for the first non trivial term, depend on the definition adopted for the coupling. For example if \( \rho(\alpha) \) is measurable and its perturbative expansion in terms of \( \alpha \) is given by

\[
\rho(\alpha) = \rho^{(0)} + \rho^{(1)} \alpha + \rho^{(2)} \alpha^2 + \cdots
\]  

(3-34)

then its expansion in terms of \( \alpha' \), by inserting eq. (3-33), results to be:

\[
\rho(\alpha') = \rho^{(0)} + \rho^{(1)} \alpha' + (\rho^{(2)} + k^{(1)} \rho^{(1)}) \alpha'^2 + \cdots
\]  

(3-35)

In this respect it is remarkable that in the expansion of \( \beta(\alpha) \), eqs. (3-5, 3-26), both \( b \) and \( b' \) are independent of the renormalization prescription for \( \alpha \). This is of course due to the close relation between \( \beta(\alpha) \) and \( \alpha \) which follows from eq. (3-5). In fact on one hand by replacing eq. (3-33) into eq. (3-26) one finds:

\[
\beta(\alpha) = -b \alpha^2 (1 + b' \alpha + \cdots) = -b \alpha'^2 (1 + b' \alpha' + \cdots) (1 + 2k^{(1)} \alpha' + \cdots) .
\]  

(3-36)

On the other hand from the definition of \( \beta(\alpha) \) in eq. (3-5) one has:

\[
\beta(\alpha) = \frac{\partial \alpha}{\partial \ln \mu^2} \frac{\partial \alpha'}{\partial \ln \mu^2} \frac{\partial \alpha}{\partial \alpha'} = \beta(\alpha') (1 + 2k^{(1)} \alpha' + \cdots) .
\]  

(3-37)

By comparison one concludes that the first two terms of \( \beta(\alpha) \), and only these, are indeed independent of the definition of \( \alpha \) in the massless theory.
This fact in turn implies that the functional form of the running coupling in the asymptotic region, also including the first non leading term as in eq. (3-31), is identical for all definitions of \( \alpha \). Therefore different prescriptions, leading to different values of \( \alpha(\mu) \) for the same physics, correspond to different values of \( \Lambda \), according to eq. (3-32) [41, 104, 147].

3.3. The beta function in one and two loops

In this section we sketch the calculation of the one loop beta function in non Abelian gauge theories with massless spin 1/2 matter fields [361, 242a]. We also state the corresponding result in two loops [103, 269]. Recently a calculation of the beta function in the minimal subtraction scheme in three loops has also been completed [398]. We shall work with dimensional regularization [267, 37, 78] and minimal (MS: [262]) or modified minimal subtraction [MS: [55]), so that these convenient prescriptions will also be defined.

In \( d \neq 4 \) spacetime dimensions the gauge coupling carries dimensions. The reference mass \( \mu \) is introduced as a scale for the coupling:

\[
g_{\text{dim}} = \mu^e g_{\text{dim}} = \mu^e g
\]

with

\[
2e = 4 - d.
\] (3-38)

Consider the one loop corrections of order \( \alpha \) to some quantity which we take to start with 1 in trivial order. Massless external lines (or at least some of them) are taken off-shell by an amount \( p^2 \). The unrenormalized quantity \( \Gamma_U \), obtained from computation of one loop Feynman diagrams in \( d \) dimensions leads to a result of the form:

\[
\Gamma_U = 1 + \alpha \left( \frac{\mu^2}{-p^2} \right)^e \left[ \frac{B}{\epsilon} + A + o(\epsilon) \right].
\] (3-39)

The factor \( (\mu^2)^e \) corresponds to eq. (3-37) while the factor \( (p^2)^{-e} \), forced by dimensions, follows from the increased degree of infrared singularity at \( \epsilon > 0 \). Renormalization by MS consists in the prescription of subtracting only the pole part in \( 1/\epsilon \) with no finite terms:

\[
\Gamma_R = 1 + \alpha \left[ \left( \frac{\mu^2}{-p^2} \right)^e \left( \frac{B}{\epsilon} + A \right) - \frac{B}{\epsilon} \right] = 1 + \alpha \left[ B \ln \left( \frac{\mu^2}{-p^2} \right) + A \right].
\] (3-40)

Renormalization by MS prescribes to subtract only the pole part in \( 1/\hat{\epsilon} \) where:

\[
1/\hat{\epsilon} = 1/\epsilon + \ln 4\pi - \gamma_E
\] (3-41)

with \( \gamma_E \) being the Euler–Mascheroni constant (\( \gamma_E = 0.5772 \ldots \)). This superficially curious definition is motivated by the systematic appearance of this combination of constants from the loop integrals in \( d \) dimensions. Note that the coefficient of \( \ln \mu^2 \) is independent of these details which only affect the finite part.
Fig. 1. Diagrams for the one loop beta function and their contributions to $b$ in the Feynman gauge.

The Ward identities of gauge invariance are preserved by dimensional regularization and MS or $\overline{\text{MS}}$ renormalization. As a consequence one can compute the beta function starting from any trilinear vertex in the Lagrangian. As an illustration let us consider the gluon–quark–antiquark vertex. We denote by $\Gamma^{\text{qgG}}$ the one particle irreducible three point vertex function with massless off-shell quarks (by an amount $p^2$) and a gluon of vanishing four momentum $k$. The renormalized vertex function is determined at order $\alpha$ by the diagrams in fig. 1-a and is of the form (either in MS or MS):

$$F_{\text{qG}} = g + \frac{a}{B} \ln \left( \frac{\mu^2}{-p^2} \right) + \cdots \tag{3-42}$$

The adimensional quantity $\Gamma^{\text{qgG}}$ satisfies the RGE:

$$\left[ \frac{\partial}{\partial \ln \mu^2} + \beta(\alpha) \frac{\partial g}{\partial \alpha} \right] \Gamma^{\text{qgG}} = 0 \tag{3-43}$$

$\gamma_{q,G}(\alpha)$ are the anomalous dimension functions connected with the external legs ($\sqrt{Z_i}$ for each external leg). Strictly speaking eq. (3-43) as it stands is only valid in the Landau gauge. In other gauges an additional term should be included which takes into account the variation with $\mu$ of the gauge parameter, because the off-shell vertex function is not gauge invariant, but in any case this term is unimportant at one loop accuracy (see for example [242a]). $\gamma_{q,G}(\alpha)$ start at order $\alpha$:

$$\gamma_{q,G}(\alpha) = \alpha \gamma_{q,G}^{(1)} + \cdots \tag{3-44}$$

At one loop accuracy, by inserting eqs. (3-1, 3-26, 3-42, 3-44) into eq. (3-43), one obtains:

$$b = 2(B^{\text{qgG}} + \gamma_{q}^{(1)} + \gamma_{G}^{(1)}) \tag{3-45}$$

In turn $\gamma_{q,G}^{(1)}$ are determined by considering the inverse propagators of the quark and the gluon respectively (one particle irreducible two point functions). Precisely we consider the coefficient of $\partial$
from the diagram in fig. 1-b and the coefficient of $-g_{\mu\nu}$ from the diagrams in fig. 1-c and denote these quantities by $\pi_{q,G}$. After renormalization:

$$\pi_{q,G} = 1 + \alpha \left[ B_{q,G} \ln \left( \frac{\mu^2}{-p^2} \right) + A_{q,G} \right].$$  \hspace{1cm} (3-46)

At one loop accuracy $\pi_{q,G}$ satisfy the RGE:

$$\left[ \frac{\partial}{\partial \ln \mu^2} + \alpha \gamma_{q,G}^{(1)} \right] \pi_{q,G} = 0 .$$ \hspace{1cm} (3-47)

Note that the term from $\beta(\alpha)$ can be omitted here since it would contribute to order $\alpha^2$. Thus one obtains from eqs. (3-45, 3-46, 3-47):

$$\gamma_{q,G}^{(1)} = -B_{q,G}$$ \hspace{1cm} (3-48)

and

$$b = 2(B^{qG} - B_q - \frac{1}{2}B_G) .$$ \hspace{1cm} (3-49)

As an example consider the diagram of fig. 2. In the Feynman gauge it is proportional to the quantity:

$$\frac{-\gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\omega}}{h^2 (p-h)^2} = \frac{2(1-e)(2h_{\mu}h - h^2 \gamma_{\mu})}{h^2 (p-h)^2}$$ \hspace{1cm} (3-50)

(for Dirac algebra and integration over loop momenta in $d$ dimensions see for example [318a]).

After integration over momenta, collecting all factors, the contribution of this diagram to the unrenormalized $\gamma_{\mu}$ vertex is given by (after removing $\gamma_{\nu}$):

$$i\mu^{-1} \Gamma_{qG}^{(a)} = g \left\{ 1 + \frac{\alpha}{4\pi} (C_F - \frac{1}{2}C_A) \left( \frac{4\pi \mu^2}{-p^2} \right)^{\epsilon} \left( \frac{1}{\epsilon} + 1 - \gamma_E \right) \right\} .$$ \hspace{1cm} (3-51)

Renormalization by $\overline{\text{MS}}$ leads to

$$i\Gamma_{qG}^{(R)} = g \left\{ 1 + \frac{\alpha}{4\pi} (C_F - \frac{1}{2}C_A) \left[ \ln \left( \frac{\mu^2}{-p^2} \right) + 1 \right] \right\} .$$ \hspace{1cm} (3-52)
Even in this very simple context we appreciate the main advantages of dimensional regularization and renormalization by MS or MS: massless phase space and automatic subtraction diagram by diagram.

The contribution to $b$ of this diagram is simply obtained from the coefficient of the logarithm, according to eq. (3-49):

$$2B^{qqG} = \frac{1}{2\pi} (C_F - \frac{1}{2} C_A).$$

(3-53)

Actually we could have spared some work because for $b$ there is no need of computing the non logarithmic terms (for example the Dirac algebra could have been performed in 4 dimensions, dropping the $\epsilon$ term in eq. (3-50)) but the calculation was completed for illustration.

In fig. 1 we reported the contribution of each diagram to $b$ in the Feynman gauge. The complete result is given by:

$$b = \frac{11C_A - 4T}{12\pi} \rho_{QCD}. 33 - 2f.$$  

(3-54)

In the case of SU(3) the theory is asymptotically free for $f \leq 16$.

A similar analysis can be carried through for all renormalizable theories in 4 dimensions. The result is that AFT must necessarily contain non Abelian gauge fields [122]. It is interesting that in $d = 6$ the toy theory $\lambda \phi^4$ also turns out to be asymptotically free [317, 427].

The value of $b'$, the two loop beta function parameter defined in eq. (3-26), for a non Abelian gauge theory with massless spin 1/2 matter fields has also been computed [103, 269]:

$$b' = \frac{17C_A^2 - 10TCA - 6TC_F}{2\pi(11C_A - 4T)} \rho_{QCD} \frac{153 - 19f}{2\pi(33 - 2f)}.$$  

(3-55)

Note that for small $f$, $b'$ is positive and decreases with $f$ until it is close to zero at $f = 8$ and becomes negative for $f \approx 9$.

When working at large but fixed $Q^2$, with $Q$ much larger than the masses of some light quarks and much smaller than the masses of some heavy quarks, then the relevant number of flavours $f$ is that of light quarks, according to an intuitive decoupling theorem [33]. In fact since the theory obtained by dropping out heavy quarks is also renormalizable, the behaviour of amplitudes with light external particles is dictated in this range of $Q^2$ by the set of diagrams where all internal lines are also light. The precise statement and proof of the decoupling theorem in non Abelian gauge theories involves a number of subtleties (see for example [344]).

A plot of $\alpha(t)$ in QCD according to eqs. (3-31, 3-54, 3-55) is shown in fig. 3. At large $Q^2$, $\alpha(t)$ becomes quite insensitive to the value of $A$. Essentially the same value of $\alpha(t)$ at sufficiently large $Q^2$, results for a wide range of initial values. Thus even a rough determination of $\alpha(t)$ at $Q^2 \sim 1$ GeV$^2$ from precocious scaling is enough to fix a stringent upper bound at $Q^2 \approx 10^2 - 10^3$ GeV$^2$.

3.4. Asymptotic expansions

We consider now the asymptotic behaviour of the general solution of a RGE in an asymptotically free theory.
Fig. 3. The running coupling as a function of $Q^2$ for various values of $\Lambda$ (in MeV). Note that at large $Q^2$ the running coupling becomes more and more insensitive to $\Lambda$.

We start by rewriting eq. (3-24) in a form appropriate for quantities given in terms of a short distance or light cone expansion. A typical example is a moment of a structure function in lepto-production [218, 243]. One has:

$$M_n^\alpha(t, \alpha) = C_n^\alpha[\alpha(t)] \exp\left[ \int_\alpha^{\alpha(t)} \frac{\gamma_n(\beta)}{\beta(\alpha)} \right] Q_n(\alpha). \quad (3-56)$$

Here $n$ is an index selecting a given local operator, $\alpha$ refers to different quantities all expressed in terms of the same local operator matrix element $Q_n(\alpha)$ (for example the $n$th moment of one out of several structure functions $\alpha$).

We start by considering the simplest case of a single multiplicatively renormalizable operator. By series expansion we define:

$$C_n^\alpha(\alpha) = C_n^{(0)\alpha}[1 + \alpha C_n^{(1)\alpha} + \cdots] \quad (3-57)$$

$$\gamma_n(\alpha) = \alpha \gamma_n^{(1)} + \alpha^2 \gamma_n^{(2)} + \cdots \quad (3-58)$$

When these expressions, together with the expansion in eq. (3-26) for $\beta(\alpha)$, are inserted in the exponential in eq. (3-56) one finds:
\[ E_n[\alpha(t), \alpha] = \exp\left[ \int_{\alpha}^{\alpha(t)} \frac{\gamma_n(\alpha)}{\beta(\alpha)} \right] = \left[ \frac{\alpha}{\alpha(t)} \right]^{\gamma_{1b}^{(1)}} \left[ 1 + \frac{\gamma_n^{(2)} - b' \gamma_n^{(1)}}{b} (\alpha - \alpha(t)) + \cdots \right]. \] (3-59)

In ordinary perturbation theory \( \alpha - \alpha(t) \) starts with \( \alpha^2 t \), according to eq. (3-28). Thus the leading logarithmic series, made up of terms of order \( (\alpha t)^n \), is entirely in the first factor, while the second factor resumes subleading corrections of order \( \alpha (\alpha t)^n \). We now recombine the exponential with the expansion of the reduced coefficient \( C_n^\alpha[\alpha(t)] \):

\[ M_n^\alpha(t, \alpha) = C_n^\alpha \left[ \frac{\alpha}{\alpha(t)} \right]^{\gamma_{1b}^{(1)}} Q_n(\alpha) \left[ 1 + \frac{\alpha - \alpha(t)}{b} \left( \gamma_n^{(2)} - b C_n^{(1)\alpha} - b' \gamma_n^{(1)} \right) + \cdots \right]. \] (3-60)

Actually one can include an \( \alpha \)-dependent factor into the matrix element \( Q_n(\alpha) \) which in general is non perturbative. In this way one obtains the asymptotic behaviour in terms of \( \alpha(t) \) for whatever size of \( \alpha \):

\[ M_n^\alpha(t, \alpha) = C_n^{(0)\alpha}[\alpha(t)]^{-\gamma_{1b}^{(1)}} \tilde{Q}_n(\alpha) \left[ 1 - \frac{\alpha(t)}{b} \left( \gamma_n^{(2)} - b C_n^{(1)\alpha} - b' \gamma_n^{(1)} \right) + \cdots \right] \] (3-61)

with:

\[ \tilde{Q}_n(\alpha) = \alpha^{\gamma_{1b}^{(1)}} Q_n(\alpha) \left[ 1 + \frac{\alpha}{b} \left( \gamma_n^{(2)} - b' \gamma_n^{(1)} \right) + \cdots \right]. \] (3-62)

When working at non leading accuracy \( \alpha(t) \) must be taken in the improved form eq. (3-31).

Obviously the orders of perturbation theory are reshuffled in going from an expansion in \( \alpha \) to one in \( \alpha(t) \). For example a two loop quantity \( \gamma_n^{(2)} \) and a one loop quantity \( C_n^{(1)\alpha} \) both contribute to the first correction in \( \alpha(t) \).

In the LLA, valid for \( \alpha(t) \) sufficiently small, eq. (3-61) reduces to the form:

\[ M_n^\alpha(t, \alpha) = C_n^{(0)\alpha}[\alpha(t)]^{-\gamma_{1b}^{(1)}} M_n^\alpha(0) \left[ 1 + o(\alpha(t)) \right]. \] (3-63)

The reduced coefficients \( C_n^{(0)\alpha} \) are the same as in free field theory. Different processes, distinguished by the label \( a \), are described in terms of the same non perturbative quantities \( [\alpha/\alpha(t)]^{\gamma_{1b}^{(1)}} M_n^\alpha(0) \) with coefficients determined by the free field theory. In this way a simple connection is established with the naive parton model, although logarithmic scaling violations are now present. The connection with the parton model will be discussed in detail in the next section.

In fixed point theories, i.e. those with an ultraviolet stable zero of the beta function at a finite value \( \alpha_\infty \) of the coupling, the logarithmic behaviour is replaced by a (in general) non integer power behaviour. In this case the reduced coefficient evaluated at the fixed point \( \alpha_\infty \) appears in the asymptotic expansion.
The connection with the free field coefficients is lost and thus also the asymptotic recovery of the (modified) parton model. The regular behaviour of the expansion coefficients of beta functions in renormalizable theories makes the presence of fixed points at small values of \( \alpha \) quite unplausible.

The next to leading correction in eq. (3-61) depends on the definition adopted for the renormalized coupling \( \alpha \) but not on the renormalization prescription for the operator matrix elements.

When the definition of \( \alpha \) is changed according to eq. (3-33), the quantities \( b, b', C_n^{(1)a}, \gamma_n^{(1)} \) are clearly unaffected (recall the discussion of eqs. (3-34 to 3-37)), but \( \gamma_n^{(2)} \) is modified into:

\[
\gamma_n^{(2)} \rightarrow (\gamma_n^{(2)})' = \gamma_n^{(2)} + k^{(1)} \gamma_n^{(1)}
\]

as follows from eqs. (3-33, 3-58). This change of the anomalous dimensions compensates to second order the variation of \( \alpha'/\alpha(t) \) in eq. (3-60). In fact, from eqs. (3-27, 3-33) one finds:

\[
\frac{\alpha}{\alpha(t)} = \frac{\alpha'}{\alpha'(t)} \left[ 1 + k^{(1)}(\alpha' - \alpha'(t)) \right].
\]

This dependence on the definition of \( \alpha \) is eliminated if one given process is taken as a measure for the value of \( \alpha \) (with that particular prescription). When all other processes are expressed in terms of the defining experiment the prescription dependence of the next to leading coefficient is cancelled.

For example, setting for shorthand eq. (3-61) in the form:

\[
M_n^a(t, \alpha) \sim \left[ \frac{\alpha}{\alpha(t)} \right]^{d_n} \left[ 1 + e_n^a \frac{\alpha(t)}{\pi} + \cdots \right]
\]

with

\[
d_n = \gamma_n^{(1)}/b
\]

\[
e_n^a = \frac{\pi}{b} \left[ b C_n^{(1)a} - \gamma_n^{(2)} + b' \gamma_n^{(1)} \right]
\]

(an overall constant factor in front of each moment is omitted), we can write:

\[
M_n^a(t, \alpha) \sim [M_{n0}^a(t, \alpha)]^{d_n/d_{n0}} \left[ 1 + d_n \left( \frac{e_n^a}{d_n} - \frac{e_{n0}^a}{d_{n0}} \right) \frac{\alpha(t)}{\pi} + \cdots \right]
\]

where \( M_{n0}^a(t, \alpha) \) has been taken as a reference process. It is immediate to verify that the coefficient of \( \alpha(t) \) in the non leading correction is now independent of the definition of \( \alpha \). This is the sense to be given to all formulae which depend on the definition of the coupling: a useful intermediate stage (see for example [346]).

Consider now a change of the renormalization prescription for the operator matrix element \( Q_n(\alpha) \) (for a fixed definition of \( \alpha \)). Provided the two definitions for the operator coincide at trivial order one can set in general:

\[
Q_n'(\alpha) = \tau_n(\alpha) Q_n(\alpha) = [1 + \alpha \tau_n^{a(1)} + \cdots] Q_n(\alpha).
\]
Under this transformation all three factors in eq. (3.56) are modified according to:

\[
C_n^a[\alpha(t)]\, E_n[\alpha(t), \alpha] \, Q_n(\alpha) = C_n^a[\alpha(t)]\, \tau_n^{-1}[\alpha(t)]\, \tau_n(\alpha)\, E_n[\alpha(t), \alpha] \, \tau_n^{-1}(\alpha) \, \tau_n(\alpha) \, Q_n(\alpha)
\]

\[
= C_n^a[\alpha(t)]\, E'_n[\alpha(t), \alpha] \, Q'_n(\alpha)
\tag{3.71}
\]

where the exponential \( E_n[\alpha(t), \alpha] \) was defined in eq. (3.59). Note that \( E_n(\alpha, \alpha) = 1 \). We thus read from the previous equation that:

\[
C_n^a(\alpha) = C_n^a[\alpha(t)]\, \tau_n^{-1}(\alpha)
\tag{3.72}
\]

\[
E'_n[\alpha(t), \alpha] = \tau_n[\alpha(t)]\, E_n[\alpha(t), \alpha] \, \tau_n^{-1}(\alpha).
\tag{3.73}
\]

In particular both non leading terms \( C_n^{n\alpha} \) and \( \gamma^{(2)}_n \) depend on the definition of \( Q_n(\alpha) \) but the combination \( \gamma^{(2)}_n - b C_n^{n\alpha} \) appearing as a coefficient of \( \alpha(t) \) in eqs. (3.60, 3.61) is invariant. Note that the term \( b' \gamma^{(1)}_n \) which also appears in the same equations is invariant by itself, because so are \( \gamma^{(1)}_n \) and \( b' \). It is important to remark that the ambiguous part of \( C_n^a(\alpha) \) must then be independent of the index \( a \); for example the difference of the one loop coefficients \( C_n^{(1)a} - C_n^{(1)b} \) for two different "processes" is well defined.

In the general case of several operators mixing together, \( C_n^a(\alpha(t)) \) is a row vector. An ordered definition of the exponential is to be adopted in order for \( E_n \) to be a solution of the RGE in the general case:

\[
E_n[\alpha(t), \alpha] = P \exp \int_\alpha^\alpha(\alpha') \frac{\gamma_n(\alpha')}{\beta(\alpha')} \, d\alpha' = 1 + \int_\alpha^\alpha(\alpha') \frac{\gamma_n(\alpha')}{\beta(\alpha')} \, d\alpha' + \int_\alpha^\alpha(\alpha') \int_\alpha^\alpha(\alpha''\alpha''') \frac{\gamma_n(\alpha') \gamma_n(\alpha'')}{\beta(\alpha') \beta(\alpha'')} \, d\alpha'' d\alpha''' + \cdots
\tag{3.74}
\]

For example for a two by two anomalous dimension matrix in the basis where \( \gamma^{(1)} \) is diagonal, with eigenvalues \( \gamma^{(1)}_+, \gamma^{(1)}_- \) (we are omitting the index \( n \) for simplicity), \( \gamma^{(2)}_n \) is in general non diagonal with matrix elements \( \gamma^{(2)}_{ij} \) \((i, j = +, -)\) and the expressions of the \( E[\alpha(t), \alpha] \) matrix elements to second order turn out to be:

\[
E_{zz} = \left[ \frac{\alpha}{\alpha(t)} \right]^{\gamma^{(2)}_{bb}} \left[ 1 + \frac{\alpha - \alpha(t)}{b} \left( \gamma^{(2)}_{+} - b' \gamma^+_+ \right) + \cdots \right]
\]

\[
E_{zz} = \frac{\gamma^{(2)}_{zz}}{\gamma^{(1)}_- - \gamma^{(1)}_+ + b} \left[ \left( \frac{\alpha}{\alpha(t)} \right)^{\gamma^{(1)}_{bb}} \alpha - \left( \frac{\alpha}{\alpha(t)} \right)^{\gamma^{(1)}_{bb}} \alpha(t) \right] + \cdots.
\tag{3.75}
\]

Note, finally that eq. (3.70) remains valid with \( \tau_n(\alpha) \) a matrix.

4. Inclusive leptoproduction in the leading logarithmic approximation

We are now ready to start a systematic description of the physical implications of perturbative QCD.
This section is devoted to deep inelastic, totally inclusive leptoproduction. We assume the reader is familiar with the definition of structure functions, the naive parton model and the canonical light cone expansion for this class of processes. For a review of the physics of deep inelastic processes before QCD see [10]; we refer to this article for notations and for all the background to the present section.

A reader interested in a simple introduction to the physical ideas on which perturbative QCD is founded should start the reading of this review from this section and then look back in the previous sections for filling the gaps he will find on his way.

### 4.1. The QCD improved parton model

Consider leptoproduction on a given hadronic target. The current four momentum is spacelike and one can choose the Lorentz frame where $q$ carries no energy:

\[ q = (0; -Q, 0). \]  

(4-1)

The first entry of all four vectors refers to the energy, the second to the longitudinal component and the last specifies the bidimensional transverse momentum vector. Neglecting the mass $M$, the target four momentum is given by:

\[ P = (E; p, 0) = \left( \frac{Q}{2x}, \frac{Q}{2x}, 0 \right) \]  

(4-2)

where

\[ x = \frac{Q^2}{2(Pq)} = \frac{Q^2}{2M^2} \]  

(4-3)

is the familiar Bjorken variable.

In the naive parton model, when terms down by powers of $1/Q^2$ are neglected, a given structure function, e.g. $F_1$, is obtained by convoluting the density of a quark parton in the target, with fraction $y$ of the target longitudinal momentum, with the pointlike cross section for quark-current scattering:

\[ 2F_1 \sim \int \frac{dy}{y} q_0(y) \sigma_{\text{point}}(yP + q) + o(1/Q^2). \]  

(4-4)

In order for this naive parton model recipe to hold the target wave function must produce enough damping in the virtual mass $(\sqrt{k^2})$ and transverse momentum $(k_\perp)$ distributions for a parton so that all terms of order $k^2/Q^2$ and $k_\perp^2/Q^2$ can be neglected in that upon integration over the phase space they indeed produce terms of order $1/Q^2$. If this is true then the parton is quasi real and collinear and the amplitude squared factorizes into two real processes: the probability density of finding one given parton in the target and the parton-current cross section. In the corresponding massless and collinear approximation the parton four momentum $k$ reduces to a fraction of the target four momentum:

\[ k = yP = \left( \frac{y}{x}, \frac{y}{x}, 0 \right). \]  

(4-5)
A second requisite for the validity of the naive parton model is the possibility of inserting the pointlike cross section, at least effectively, i.e. after a suitable redefinition of the parton density. This can only be true in model field theories, either super-renormalizable or softened by an ad hoc cut-off in transverse momentum ([25, 382, 18] for $\lambda \phi^3$; [154] for $k_c$ cut-off theories with fermions; [93, 94, 300]). The pointlike cross section is proportional to a $\delta$ function expressing the vanishing of the squared invariant mass of the initial $q - \gamma^*$ system, as it must be equal to the final parton mass squared. When all factors are properly taken into account one ends up with:

$$2F_1(x) = \int \frac{dy}{y} q_0(y) e^2 \delta \left( \frac{x}{y} - 1 \right) = e^2 q_0(x)$$

(4-6)

where $e^2$ is the charge squared of the quark (or the appropriate coupling squared) and the sum over $q$ and $\bar{q}$ flavours is omitted for simplicity.

In QCD the factorization into a convolution of a parton density times a parton-current cross section is also true provided the same assumptions on the target wave function are kept, but one certainly cannot restrict to the pointlike cross section. In fact the parton cross section (i.e. the structure function on a quark target) receives important contributions from all orders in $\alpha$.

To illustrate this point we consider the correction of order $\alpha$ to the parton cross section in the expression for $F_1$:

$$2F_1(x, Q^2) = \int \frac{dy}{y} q_0(y) \left[ e^2 \delta \left( \frac{x}{y} - 1 \right) + \sigma_\alpha \left( \frac{x}{y}, Q^2 \right) + \cdots \right].$$

(4-7)

Note that our “cross sections” are actually adimensional after extraction of an over all scale factor. Also note that the measure $dy/y$ is a relic of the Lorentz invariant measure $d^4p \delta(p^2) \theta(p_0)$ for the internal parton line and correctly takes into account the energy ratio between parton and target in the determination of the relative normalization of fluxes. The cross section at order $\alpha$ is computed from the diagrams in fig. 4 which include virtual corrections to the basic process

$$q + \gamma^* \rightarrow q$$

(4-8)

![Fig. 4. Diagrams for the quark-current total cross section at order $\alpha$.](image)
and real emission diagrams for the process

\[ q + \gamma^* \rightarrow q + G \]  \hspace{1cm} (4-9)

with G being a gluon. From the expressions of the quark and current momenta in eqs. (4-1, 4-2) it is clear that the cross section can be written in general as a function of \( x/y \) and \( Q^2 \) and \( y \geq x \) because more energy than in the elastic case is needed in the initial state when a gluon is also produced.

The structure of \( \sigma_\alpha \) is as follows:

\[ \sigma_\alpha(x, Q^2) = \frac{\alpha}{2\pi} e^t [ t P(x) + f(x) + o(1/Q^2) ] \]  \hspace{1cm} (4-10)

where \( t \) was defined in eq. (3-2). The presence of \( \log Q^2 \) implies that scaling is violated and the naive parton model cannot be valid in QCD. Its origin is precisely the same as for the \( \log E/m \) term in the total cross section for Compton scattering in QED (in which case \( E \) and \( m \) are the energy and mass of the electron).

Consider in fact the diagram in fig. 5 (in a physical gauge). In a general gauge this is not the only diagram which contributes but it can serve best as an example. Neglecting masses we parametrize the initial quark and the gluon momentum as (see eq. (4-5)):

\[ k = (k; k, 0); \quad k = \frac{y}{x} \frac{Q}{2} \]

\[ p = \left( \eta k + \frac{k_\perp^2}{2\eta k}; \eta k, -k_\perp \right). \]  \hspace{1cm} (4-11)

The gluon carries a fraction \( \eta \) of initial quark momentum and a given \( k_\perp \) (opposite to that of the intermediate quark \( r \) and of the final quark). One has:

\[ r^2 = (k - p)^2 = -k_\perp^2/\eta. \]  \hspace{1cm} (4-12)

The virtual mass and \( k_\perp^2 \) are thus related for the quark \( r \). The differential cross section for \( q + \gamma^* \rightarrow q + G \) behaves as \( 1/k_\perp^2 \) for a gluon emitted near the forward direction with respect to the incoming quark (in the lab. frame). This arises from \( (1/r^2)^2 \sim 1/k_\perp^4 \) from the propagator squared and a factor of \( k_\perp^2 \)

\[ \begin{array}{c}
\gamma^* \\
| \\
\downarrow \\
r \\
| \\
| \\
p \\
| \\
\downarrow \\
G \\
| \\
| \\
k 
\end{array} \]

Fig. 5. One particular diagram contributing to the quark-current cross section at order \( \alpha \).
from the numerator due to the vanishing of the residue at the gluon vertex. In fact helicity conservation along the quark line forbids the emission of a real massless collinear vector gluon. Actually it would be the same for all renormalizable interactions. For example for a scalar gluon the argument is reversed. The quark helicity flips at the gluon vertex but a collinear scalar gluon would not be able to balance the spin component. The upper limit of the integration over $k_\perp$ is given by

$$ (k_\perp^2)_{\text{max}} = Q^2 \left( \frac{1 - z}{4z} \right); \quad z = \frac{x}{y} $$

and one obtains:

$$ \sigma_{\alpha}^{\text{real}} \sim \int \frac{dk_\perp^2}{k_\perp^2} \frac{\alpha}{2\pi} P_{\text{real}}(z) \sim \frac{\alpha}{2\pi} P_{\text{real}}(z) \ln Q^2. $$

We thus realize that the log $Q^2$ term arises from the hard component of the $k_\perp$ distribution of the emitted gluon. Note that the divergence at the lower limit of integration in eq. (4-14) has to be cured either by restoring the physical mass of the quark or by taking off-shell massless quarks or in some other way. To each log $Q^2$ is in fact associated a collinear logarithmic mass singularity so that one can investigate the structure of this latter for learning more about the former. All the intricacies of the infrared regularization can however be lumped inside $f(x)$ in eq. (4-10) by introducing the reference mass $\mu$ which enters in $t$. Thus while $P(x)$ is well defined even in the massless theory, $f(x)$ depends on the infrared regularization. Since the physical mechanism of regularization has to do with bound state effects outside the realm of perturbative methods, we shall only consider in the following those properties of the non logarithmic terms which are independent of that.

The connection with mass singularities is further clarified by considering the complete real diagram contribution of order $\alpha$ to $d\sigma_{\alpha}^{\text{real}}(z)/dc$ in the massless theory, where $c = \cos \theta$ and $\theta$ is the center of mass angle for process (4-9) ($\theta = 0$ for the gluon along the initial quark direction). By recalling that $F_1$ is proportional to the cross section from a transverse virtual photon one obtains by a simple calculation of the diagrams in fig. 4-b the angular distribution:

$$ \frac{d\sigma_{\alpha}^{\text{real}}(z)}{dc} = \frac{\alpha}{2\pi} C_F e^2 \left\{ \frac{1 + \frac{(1 - c)(1 - z)^2}{(1 - c)(1 - z)} + \frac{1 + c}{2} } {1 - z} \right\}. $$

The total cross section $\sigma_{\alpha}^{\text{real}}$, i.e. the quantity which enters as the real diagram contribution in the expression for $2F_1$ in eq. (4-7), should be obtained by integration over $dc$. But we immediately see that the integral of eq. (4-15) diverges logarithmically at $c = 1$. However, any mass regulator $m$ would replace the forward zero of $1 - c$ by a quantity proportional to $m^2/Q^2$, thus leading to a log $Q^2/m^2$ term after integration. Alternatively in dimensional regularization the angular integral leads to a pole $1/\epsilon$ which eventually becomes log $Q^2/\mu^2$ in analogy with eq. (3-40). Note, however, that even without explicitly introducing a cut-off we can directly read the residue of the pole at $c = 1$, that is the coefficient of the logarithm in eq. (4-10):

$$ P_{\text{real}}(x) = C_F \frac{1 + x^2}{1 - x}. $$
Later on we shall see that the contribution of virtual diagrams to $P(x)$ (only present at $x = 1$, because these terms are a simple rescaling of the pointlike cross section) can also be obtained without specifying an explicit regularization.

The problem with the term $at$ in eq. (4-10) and, in higher orders, with the whole leading logarithmic series $(at)^n$ is that this sequence must be resummed to all orders. The problem is handled by factoring out the leading logarithmic series and reabsorbing this factor in a redefinition of the parton density. To order $\alpha$ this is a trivial operation and we can write:

$$2F_1(x, t) = \int_1^1 \frac{dy}{y} q_0(y) e^2 \left[ \delta \left( \frac{x}{y} - 1 \right) + \frac{\alpha}{2\pi} \left( tP \left( \frac{x}{y} \right) + f\left( \frac{x}{y} \right) \right) + \ldots \right]$$

$$= \int_1^1 \frac{dy}{y} \left[ q_0(y) + \Delta q(y, t) \right] e^2 \left[ \delta \left( \frac{x}{y} - 1 \right) + \frac{\alpha}{2\pi} f\left( \frac{x}{y} \right) + \ldots \right]$$

(4-17)

with

$$\Delta q(x, t) = \frac{\alpha}{2\pi} t \int_1^1 \frac{dy}{y} q_0(y) P\left( \frac{x}{y} \right) + \ldots.$$  

(4-18)

The original parton density is replaced by an effective $Q^2$ dependent parton density (as seen by the virtual current):

$$q_0(x) \rightarrow q(x, t) = q_0(x) + \Delta q(x, t) + \ldots$$  

(4-19)

The $t$ dependence of the effective parton density can be understood as due to the fact that a current with larger $Q^2$ probes a wider range of $k_t^2$ in the parton cloud of the target.

The next step is to consider the derivative of the effective parton density:

$$\frac{dq(x, t)}{dt} = \frac{\alpha}{2\pi} \int_1^1 \frac{dy}{y} q(y, t) P\left( \frac{x}{y} \right) + o(\alpha^2 t).$$  

(4-20)

The derivative of the effective parton density contains at most the sequence of next to leading logarithmic terms $\alpha(\alpha t)^{n-1}$. According to the general discussion in section 3, this indicates that the simple replacement of $\alpha$ by the running coupling $\alpha(t)$ transforms eq. (4-20) into a correct asymptotic statement:

$$\frac{d}{dt} q(x, t) = \frac{\alpha(t)}{2\pi} \int_1^1 \frac{dy}{y} q(y, t) P\left( \frac{x}{y} \right) + o(\alpha^2(t)).$$  

(4-21)

Similarly for the structure function once it is expressed in terms of the effective parton density one has:
\[ 2F_i(x, t) = \int dy \frac{q(y, t)}{y} e^{2\left[ \delta \left( \frac{x}{y} - 1 \right) + \frac{\alpha(t)}{2\pi} f \left( \frac{x}{y} \right) + \cdots \right]} \]

\[ = e^{2\left( x, t \right)} + o(t). \quad (4-22) \]

In the case of leptoproduction these results can be proven to all orders in perturbation theory by application of the RGE to the coefficient functions in the light cone expansion of the current product \[ [218, 243] \]. The connection with the general formalism of the previous section is directly seen by introducing the moments of the effective parton densities, defined by:

\[ q_n(t) = \int_0^1 dx \, x^{n-1} q(x, t). \quad (4-23) \]

By taking moments of both sides of the integro-differential eq. (4-21) the convolution is resolved in a product of moments and \( q_n(t) \) satisfy the differential equation:

\[ \frac{d}{dt} q_n(t) = \frac{\alpha(t)}{2\pi} A_n \alpha(P_n(t)) + o(\alpha^2(t)) \]

where the set of constants \( A_n \) are the moments of \( P(x) \) in eq. (4-25):

\[ A_n = \int_0^1 dx \, x^{n-1} P(x). \quad (4-25) \]

The simplicity of the evolution equation for \( q_n(t) \) explains why the results for scaling violations in QCD are so often formulated in terms of moments.

Eq. (4-24) for moments can be readily solved by changing variable to:

\[ Y = \int_{\mu^2}^{Q^2} \frac{dk_1^2}{k_1^2} \frac{\alpha(k_1^2)}{2\pi} = \frac{1}{2\pi b} \ln \frac{\alpha(Q^2)}{\alpha(\mu^2)} \]

where the last equality holds in the LLA according to eq. (3-28). One obtains (we take the notations \( \alpha(t) \) and \( \alpha(Q^2) \) as equivalent):

\[ q_n(t) = q_n(0) \exp(A_n T)[1 + o(\alpha(t))] = q_n(0) \left[ \frac{\alpha}{\alpha(t)} \right]^{A_n/2\pi b} [1 + o(\alpha(t))]. \quad (4-27) \]

By comparison with eq. (3-63) we have the connection between \( A_n \) and the one loop anomalous dimension of the relevant operator:

\[ A_n = 2\pi \gamma_n^{(0)}. \quad (4-28) \]
Note that the moments are not analytic in \( a(t) \); this was expected, because they contain the leading logarithmic sequence in perturbation theory. Their derivatives can instead be expanded in \( a(t) \) and this is the reason for considering the evolution equations and their kernels.

4.2. The general evolution equations

Up to this point we implicitly restricted our attention to non singlet (under the flavour group) structure functions. We now generalize the evolution equations to take into account the presence of gluons in the target and of several flavours of quarks.

At order \( \alpha \) an additional parton process, besides those in eqs. (4-8, 4-9), contributes to the total cross section:

\[
G + \gamma^* \rightarrow q + \bar{q}.
\]

In lowest order the relevant diagrams are shown in fig. 6: a gluon in the target produces a \( q \bar{q} \) pair which interacts with the current. The cross section at order \( \alpha \) for this process also contains a \( \log Q^2 \) term plus a finite term. As a consequence the complete form of eq. (4-17) is given by:

\[
2F_i(x, t) = \int \frac{dy}{y} \left\{ \sum_i q_{oi}(y) e_i y^2 \left[ \delta(\frac{x}{y} - 1) + \frac{\alpha}{2\pi} \left( tP_{qq}(\frac{x}{y}) + f_q(\frac{x}{y}) \right) \right] + \left( \sum_i e_i y^2 \right) G_0(y) \frac{\alpha}{2\pi} \left( tP_{qG}(\frac{x}{y}) + f_G(\frac{x}{y}) \right) \right\}
\]

(4-30)

where \( i = 1, 2, \ldots, 2f \) runs over both \( q \) and \( \bar{q} \) of all flavours and \( G_0(x) \) is the (naive) gluon density in the target. In the massless theory the differential cross section at order \( \alpha \) for the parton process (4-29) (with \( q \bar{q} \) of one specified flavour) is proportional to (after extraction of a scale factor and in notations as for eq. (4-15):

\[
\frac{d\sigma_\alpha^G(z)}{dc} = \frac{\alpha}{2\pi} \frac{z^2 + (1-z)^2}{2} \left[ \frac{1}{1+c} + \frac{1}{1-c} - 1 \right].
\]

(4-31)

By integration over \( dc \) one would obtain a logarithmic singularity associated with either a forward quark or a forward antiquark, with an identical residue

\[
P_{qG}(x) = \frac{1}{2} [x^2 + (1-x)^2].
\]

(4-32)
Note that in eq. (4-30) there are in fact two terms in the sum for each q \bar{q} pair. The factor of $\frac{1}{2}$ is in general $T/\rho$ (see eq. (2-35)).

The quark densities are redefined by a generalization of eqs. (4-18, 4-19) according to:

$$\Delta q_i(x, t) = \alpha(t) \frac{1}{2\pi} t \int \frac{dy}{y} \left[ q_i(y) P_{qq}(\frac{x}{y}) + G(y) P_{qG}(\frac{x}{y}) \right] + \cdots$$  \hspace{2cm} (4-33)

As a consequence the final results in eqs. (4-21, 4-22) are modified into:

$$\frac{d}{dt} q_i(x, t) = \alpha(t) \frac{1}{2\pi} \int \frac{dy}{y} \left\{ q_i(y, t) P_{qq}(\frac{x}{y}) + G(y, t) P_{qG}(\frac{x}{y}) \right\} + o(\alpha^2(t))$$ \hspace{2cm} (4-34)

and

$$2F_1(x, t) = \int \frac{dy}{y} \left\{ \sum_i e_i^2 q_i(y, t) \left[ \delta(\frac{x}{y} - 1) + \frac{\alpha(t)}{2\pi} f_a(\frac{x}{y}) \right] + \left( \sum_i e_i^2 \right) G(y, t) \frac{\alpha(t)}{2\pi} f_G(\frac{x}{y}) \right\} + \cdots$$

$$= \sum_i e_i^2 q_i(x, t) + o[\alpha(t)].$$ \hspace{2cm} (4-35)

The set of evolution equations for the quark densities do not close and an equation for the gluon density is also needed. This can be obtained by suitably extending the same line of reasoning to a gedanken probe sensitive to color charges, for example a virtual gluon. The resulting equation is of the form:

$$\frac{d}{dt} G(x, t) = \alpha(t) \frac{1}{2\pi} \int \frac{dy}{y} \left\{ \sum_i q_i(y, t) P_{qG}(\frac{x}{y}) + G(y, t) P_{GG}(\frac{x}{y}) \right\} + o(\alpha^2(t)).$$ \hspace{2cm} (4-36)

We introduce for convenience the short hand notation:

$$\int \frac{dy}{y} q(y, t) P(\frac{x}{y}) = \int \frac{dy}{y} q(\frac{x}{y}, t) P(y) \equiv [q \otimes P](x, t).$$ \hspace{2cm} (4-37)

By taking the difference of eq. (4-34) for $q_i$ and $q_i$ (with $q_{i,j}$ any quark or antiquark) the gluon term drops out and the simpler non singlet (or valence) equations of the previous section are recovered:

$$V_{u}(x, t) = q_i(x, t) - q_i(x, t)$$ \hspace{2cm} (4-38)

$$\frac{d}{dt} V_{u}(x, t) = \alpha(t) \frac{1}{2\pi} [V_{u} \otimes P_{qq}](x, t).$$ \hspace{2cm} (4-39)

Thus in the leading approximation, all possible non singlet densities evolve with the same kernel $P_{qq}$. 

"Guido Altarelli, Partons in quantum chromodynamics"
The content of the remaining equations for singlet densities is best analyzed by adding up eq. (4-34) for all indices $i$. With the definition:

$$\Sigma(x, t) = \sum_i q_i(x, t) = \sum_{\text{flavours}} [q(x, t) + \bar{q}(x, t)]$$  \hspace{1cm} (4-40)

one finally obtains two evolution equations in two independent densities:

$$\frac{d}{dt} \Sigma(x, t) = \frac{\alpha(t)}{2\pi} \{[\Sigma \otimes P_{qg}](x, t) + [G \otimes 2f P_{qg}](x, t)\} + o(\alpha^2(t))$$  \hspace{1cm} (4-41)

$$\frac{d}{dt} G(x, t) = \frac{\alpha(t)}{2\pi} \{[\Sigma \otimes P_{gq}](x, t) + [G \otimes P_{GG}](x, t)\} + o(\alpha^2(t)).$$  \hspace{1cm} (4-42)

The solution for a quark $q_i$ can then be reconstructed by splitting it into its non singlet component $q_i - (1/2f)\Sigma$ and its singlet component $(1/2f)\Sigma$.

Equivalent equations for the set of moments of quark and gluon densities can be written down in the form of eq. (4-24) where $q_n$ is replaced by a vector and $A_n$ becomes a matrix. For each $n$ the entries of the matrix $A_n$ are related to moments of the $P$ functions as in eq. (4-25).

Note that in order to predict the evolution of densities at the point $x$ one only needs to know the densities at $y \geq x$. On the other hand the construction of each moment implies the knowledge of the densities at all $x$.

4.3. The splitting functions and their physical interpretation. The factorization theorem

We have so far established the important result that in AFT, like QCD, the parton model description of deep inelastic leptoproduction, suitably modified, can be reconciled with the principles of renormalizable field theories. The same formulae as in the naive parton model are found to hold asymptotically provided the naive scaling parton densities are replaced by $Q^2$ dependent effective parton densities with computable logarithmic $Q^2$ dependence and provided equally computable corrections of higher order in $\alpha(Q^2)$ are neglected together with all terms down by integer powers of $m^2/Q^2$ with $m$ some mass or another strong interaction non perturbative energy scale.

The kernels of the evolution equations which govern the $Q^2$ dependence of parton densities in the LLA have been related in a simple way to the leading logarithmic terms of the parton-current cross sections at order $\alpha$. However one crucial point was not properly made evident by that derivation. In order for the parton picture to be completely reproduced and its predictivity not spoiled, the same parton densities, for each given target, must determine all structure functions for any virtual probe (i.e. $\gamma, W^\pm, Z_0$) and other processes as well, for example Drell–Yan processes and so on. It is thus very important to establish that the $P_{BA}$ functions and consequently the parton densities in the LLA only depend on the target and, for each fixed $Q^2$, are insensitive to the nature and the polarization of the probe (vector, axial, V-A and so on).

The universality of the $P_{BA}$ functions is apparent in the RGE approach where it arises because each $\gamma_n^{(i)}$ is the anomalous dimension of a given operator independent of the matrix element and the external probe. Recall in fact eqs. (4-28, 3-56) and note that in the last equation the index $a$, which distinguishes
different structure functions, only appears in the reduced coefficient that enters as an effective charge squared.

In more physical terms a direct proof of the universality of the $P_{BA}$ functions can be given by first introducing a physical interpretation for $P_{BA}$ and then using the latter for a simple calculation of the whole set of kernels without reference to the external current.

Consider the quantities which appear in eqs. (4-17) or (4-30):

$$\Pi_{BA}(x, t) = \delta_{BA} \delta(1 - x) + \frac{\alpha}{2\pi} tP_{BA}(x) + \cdots \quad (4-43)$$

$\Pi_{BA}(x, t)$ can be interpreted as the probability density of finding the parton B in the parton A with a fraction $x$ of the parent longitudinal momentum and $k_\perp \leq Q$. Consequently the splitting functions $P_{BA}(x)$ appear as the lowest order expressions for the variation per unit $t$ of the corresponding probability densities. The diagonal probability densities start with $\delta(1 - x)$ corresponding to nothing happening. At order $\alpha$, terms in $\delta(1 - x)$ are also present for $A = B$, because the probability for A to remain A with the same energy is lowered by the interaction. With the definition given, all terms of order $\alpha$, including $\delta$ function terms, are included in $P_{BA}(x)$. In the derivation of $P_{qq}(x)$ from the cross section at order $\alpha$, contributions proportional to $\delta(1 - x)$ are obtained from virtual diagrams in fig. 4a and from the regularization of the singularities of real diagrams (fig. 4b) near $x = 1$.

It follows that $P_{qq}(x)$ is proportional to a probability density only at $x < 1$, and in particular must be positive definite in this range. At $x = 1$ the dominant term in the expansion for $P_{qq}$ is the lowest order $\delta$ function. The same is true for $P_{Gq}(x)$, while for $B \neq A$ $P_{BA}(x)$ is proportional to a probability density at all $x$ and thus is positive definite.

Following this identification (which is closely related to the approximation in QED based on the equivalent number of photons in an electron [416, 419]), the splitting functions $P_{BA}(x)$ are readily computed from the basic vertices of QCD: $P_{qq}$ and $P_{Gq}$ from the vertex $q \rightarrow q + G$, $P_{qG} = P_{Gq}$ from $G \rightarrow q + \bar{q}$ and finally $P_{GG}$ from $G \rightarrow G + G$, the last vertex being typical of non Abelian gauge theories (the four gluon vertex is of order $g^2$ and cannot contribute in leading order).

The strategy is to first compute the regular part of $P_{BA}(x)$ at $x < 1$. The $\delta$ function terms can then be fixed by the requirements of charge and momentum conservation. The QCD Lagrangian conserves fermion number, flavour and momentum. This implies that, at least asymptotically,

$$\int_0^1 dx \left[ q(x, t) - \bar{q}(x, t) \right] = v_q \quad (4-44)$$

$$\int_0^1 dx \left[ \Sigma(x, t) + G(x, t) \right] = 1 \quad (4-45)$$

where $v_q$ is the valence value for the flavour $q$ in the target, $v_q = 2, 1, 0 \ldots$ for the u, d, s \ldots flavours in the proton, and $\Sigma$ was defined in eq. (4-40). These relations could possibly be corrected by terms down by powers of $\alpha(Q^2)$, at least because the effective parton densities have been defined so far only in the LLA (see later on).
Recalling that the evolution of moments is ruled by moments of the splitting functions (eqs. (4-24, 4-25)) one can cast the constraints in eqs. (4-44, 4-45) in the form:

\[ \int_{0}^{1} dx P_{qq}(x) = 0 \] (4-46)

\[ \int_{0}^{1} dx [P_{qq}(x) + P_{Gq}(x)] = 0 \] (4-47)

\[ \int_{0}^{1} dx [2fP_{qG}(x) + P_{GG}(x)] = 0 \] (4-48)

which correspond to flavour conservation and momentum conservation in q and G splittings respectively.

The constraints in eqs. (4-46 to 48) amount to a considerable simplification also in the derivation of splitting functions from parton-current cross sections because they allow to only restrict to real emission diagrams for the determination of \( P_{qq} \) and \( P_{GG} \). Note in this connection that the diagram in fig. 4-b with the gluon emitted by the final quark superficially seems to contradict the interpretation of \( P_{qq} \) as a correction to the distribution of the initial quark inside the target. This interpretation is only justified if the gluon emission from the initial leg is in some way dominant, because only in this case the gluon emission process can be regarded as a redefinition of the target blob. Actually both diagrams are necessary in a general gauge to ensure the gauge invariance of the result. But by looking more closely one discovers that the second diagram is only necessary in covariant gauges in order to cancel the contributions of the unphysical longitudinal and scalar gluons in the parton like diagram. In other words if one only sums over physical transverse gluons the parton diagram is the only one which leads to log \( Q^2 \) terms. By recalling eq. (2-27) for the gluon propagator in axial gauges one realizes that in fact these gauges automatically project on transverse gluons (physical gauges). Thus by working in physical gauges rather than in covariant gauges one is reduced to consider as dominant diagrams in the LLA those which have a simple partonic interpretation. In particular for computing \( P_{qq} \) at \( x < 1 \) only one diagram is left in axial gauges. We have already observed that this diagram leads to a \( k_\perp \) distribution behaving like \( 1/k_\perp^2 \) and that it is precisely this singularity that contributes the log \( Q^2 \) term to \( \sigma_n \). Thus \( P_{qq} \), which is determined by the coefficient of log \( Q^2 \), is fixed by the residue of the \( 1/k_\perp^2 \) singularity. This residue in turn factorizes into a contribution from the current vertex (the lowest order cross section) that goes into the reduced coefficient, and one from the gluon vertex which is the one relevant for \( P_{qq} \). This qualitatively explains how it is that \( P_{qq} \) and similarly the other kernels, are determined by the vertices of the theory.

By working as for the equivalent photon approximation in QED (see e.g. [67]), one finds that the general form of \( P_{BA}(x) \) at \( x < 1 \) is given by:

\[ P_{BA}(x) = \frac{x(1-x)}{2} \sum \frac{|V_{A\rightarrow BC}|^2}{k_\perp^2} \quad (x < 1) \] (4-49)
where $V_{A\to BC}$ is the invariant vertex for $A \to B + C$ with the coupling constant and the normalization factors $1/\sqrt{2E}$ for each particle having been extracted out. $\Sigma$ refers to the spin and color sums and averages. The factor $1/k^2_\perp$ is cancelled by the vanishing of the numerator at $k^2_\perp = 0$ in all cases of interest. Note that $P_{BA}$ is completely fixed by the vertex $V_{A\to BC}$.

The momenta at the vertex $A \to B + C$ are specified as follows:

$$k_A = (p; p, 0) \quad (4-50)$$

$$k_B = \left( xp + \frac{k^2_\perp}{2xp} ; xp, k_\perp \right) \quad (4-51)$$

$$k_C = \left( (1-x)p + \frac{k^2_\perp}{2(1-x)p} ; (1-x)p, -k_\perp \right). \quad (4-52)$$

Note that the energy is not conserved at the vertex, in the sense of "old" perturbation theory. It is important to observe the symmetry relation:

$$P_{BA}(x) = P_{CA}(1-x) \quad (x < 1). \quad (4-53)$$

The condition $x < 1$ is always specified to recall the existence of additional $\delta$ function terms in the diagonal kernels.

For example in case of $P_{Gq}(x)$ $A$ and $C$ are quarks and $B$ is a gluon and:

$$\sum |V_{q-Gq}|^2 = \frac{1}{2} C_F \text{tr}(k_C \gamma_\mu \hat{k}_A \gamma_\nu) \sum_{\text{pol}} \epsilon^{*\mu} \epsilon^{\nu}. \quad (4-54)$$

where $k_A$ and $k_C$ are given by eqs. (4-50, 4-52), the factor $\frac{1}{2}$ is from the spin average, $C_F$ arises from the color sum and average and the sum over the transverse gluon polarizations leads to

$$\sum_{\text{pol}} \epsilon^{*\mu} \epsilon^{\nu} = \begin{cases} \delta^{\mu_1 \nu_1} - k^2_B k^2_B k^2_B / k^2_B & (\mu, \nu = i, j = 1, 2, 3) \\ 0 & \text{for } \mu \text{ and/or } \nu = 0. \end{cases} \quad (4-55)$$

After some simple algebra one finds:

$$P_{Gq}(x) = C_F \frac{1 + (1-x)^2}{x}. \quad (4-56)$$

By the reflection symmetry in eq. (4-53) one also obtains:

$$P_{qq}(x) = C_F \frac{1 + x^2}{1-x} \quad (x < 1) \quad (4-57)$$

which coincides with the result in eq. (4-16) obtained from a study of the quark-current cross section.

The $1/x$ singularity in $P_{Gq}$ and the corresponding singularity $1/(1-x)$ in $P_{qq}$ arise from the soft gluon bremsstrahlung spectrum and are typical of vector theories. For example in a scalar gluon theory:
\[
\sum |V_{q-Gq}|^2 = \frac{C_F}{2} \text{tr}(k_C k_A) = C_F \frac{k^2}{1-x} \quad (4-58)
\]

so that

\[
P_{Gq}^{\text{SCALAR}}(x) = \frac{1}{2} C_F x \quad (4-59)
\]

and (for the \(\delta\) function see the following discussion)

\[
P_{qq}^{\text{SCALAR}}(x) = \frac{1}{2} C_F [1 - x - \frac{1}{2} \delta(1-x)] \quad (4-60)
\]

In a similar way the result of eq. (4-32) is reproduced from the \(G \rightarrow q + \bar{q}\) vertex:

\[
P_{qG}(x) = \frac{T}{f} [x^2 + (1-x)^2] \quad (4-61)
\]

(Recall the definitions of colour factors in eq. (2-35).) Note the obvious symmetry (in view of eq. (4-53)) under \(x \leftrightarrow (1-x)\) of \(P_{GG}\) and \(P_{qG}\).

The \(\delta\) function terms for the diagonal splitting functions are readily obtained by imposing the constraints in eqs. (4-46 to 4-48), one of them serving as a consistency check. (Otherwise they can be directly computed from the diagrams, see e.g. [276, 15, 16].) The final results are:

\[
P_{qq}(x) = C_F \left[ \frac{1-x^2}{x} + \frac{x}{1-x} + x(1-x) \right] \quad (x < 1) \quad (4-62)
\]

\[
P_{GG}(x) = 2 C_A \left[ \frac{1-x}{x} + \frac{x}{1-x} + x(1-x) \right] + \left( \frac{1}{2} C_A - \frac{2T}{3} \right) \delta(1-x) \quad (4-63)
\]

where in general \([F(x)]_+\) is a distribution defined in terms of any sufficiently regular test function \(\varphi(x)\) by:

\[
\int_0^1 dx \varphi(x) [F(x)]_+ = \int_0^1 dx \left[ \varphi(x) - \varphi(1) \right] F(x) \quad (4-64)
\]

Note that

\[
\int_0^1 dx \left[ F(x) \right]_+ = 0 \quad (4-65)
\]
As an example the non singlet evolution equation eq. (4-39) has to be explicitly handled as follows:

\[ V \otimes P_{qq} = C_F \int x \frac{dz}{z} \left( \frac{1 + z^2}{1 - z} \right) V(x, Q^2) \]

\[ = C_F \int x \frac{dz}{1 + z^2} \left[ \frac{1}{z} V(x, Q^2) - V(x, Q^2) \right] + C_F V(x, Q^2) \left[ \int \frac{dz}{1 - z} \left( \frac{1 + z^2}{1 - z} \right) \right] \]

\[ = C_F \int x \frac{dz}{1 + z^2} \left[ \frac{1}{z} V(x, Q^2) - V(x, Q^2) \right] + C_F V(x, Q^2) \left[ x + \frac{x^2}{2} + 2 \ln(1 - x) \right] \]  

(4-67)

where the last line follows from eq. (4-66) and the last term is present because the lower limit of integration is \( x \) instead of 0 as in eq. (4-65).

We note the following interesting property of splitting functions in the limit \( C_F = C_A = 2T \) \[149\]

\[ P_{qq}(x) + P_{Gq}(x) = 2f P_{qG}(x) + P_{GG}(x) \].  

(4-68)

The validity of this relation is thought to be related to the supersymmetry of the massless QCD Lagrangian in the stated limit when gluons and Weyl fermions both transform according to the regular representation of the group. This interpretation is supported by a recent result which indicates that its validity also extends beyond the LLA provided supersymmetric prescriptions and regularizations are adopted \[30\].

In tables 2, 3 we report a summary of the evolution equations, of the splitting functions and of their moments, which contains all relevant results for the prediction of the \( Q^2 \) behaviour of parton densities in the LLA.

Within leptoproduction equivalent results were originally derived for moments by RGE for the coefficient functions in the light cone expansion by Georgi, Politzer and Gross, Wilczek \[218, 243\] and further studied. The evolution equations for non Abelian theories, their general validity for parton densities also outside leptoproduction and the physical interpretation of the kernels (also following ideas by Kogut and Susskind \[286\]) were obtained by Parisi \[348\], Altarelli, Parisi \[22\]. In the last paper one can find the detailed derivation of the splitting functions from the QCD vertices as reported here (also for polarized partons, as in the following section 4.6). The evolution equations (4-34, 4-36) are often referred to as Altarelli–Parisi equations. In the Abelian case they were independently obtained by Gribov, Lipatov \[238\] and Lipatov \[310\]. Following these papers the non Abelian case was also considered by Dokshitzer \[149\].

Solution of the evolution equations with the kernels collected in table 2 involve the resummation of leading logarithms from diagrams of all orders in perturbation theory. It is important to identify the class of diagrams that contribute the dominant terms in the LLA (and beyond) in view of extending the QCD improved parton model to processes outside the domain of operator expansions and RGE \([107, 189, 238, 310, 90a]\) and more recently: \[249, 149, 311, 198, 275a, 102\]).

In physical gauges the dominant class of diagrams in the LLA are dressed ladders with ordered momenta. The skeleton diagrams are ordinary ladder diagrams (as in fig. 7 which refers to the non singlet case); the absorptive part of the ladder corresponds to the absolute square of the amplitude integrated over phase space. The momenta in the ladder are ordered in the sense that the transverse
Table 2
A summary of the evolution equations in the LLA

### Non singlet

\[
Q^2 \frac{d}{dQ^2} q^{NS} = \frac{\alpha(Q^2)}{2\pi} q^{NS} \otimes P_{qq} \quad \left(\alpha(Q^2) = \frac{1}{b \ln(Q^2/A^2)}; \quad b = \frac{11C_A - 4T}{12\pi}; \quad A \otimes B = \int_y \frac{dy}{y} A(y) B\left(\frac{y}{y'}\right)\right)
\]

### Singlet

\[
\Sigma = \sum_{n=\text{even}} (q + \bar{q})
\]

\[
\begin{align*}
Q^2 \frac{d}{dQ^2} \Sigma &= \frac{\alpha(Q^2)}{2\pi} [\Sigma \otimes P_{qq} + G \otimes 2fP_{qg}] \\
Q^2 \frac{d}{dQ^2} G &= \frac{\alpha(Q^2)}{2\pi} [\Sigma \otimes P_{qg} + G \otimes P_{oo}]
\end{align*}
\]

\[
P_{qq}(x) = C_F \left[ \frac{1 + x^2}{1 - x} + \delta(1 - x) \right] = C_F \left[ \frac{1 + x^2}{1 - x} \right] \rightarrow A^{qg} = C_F \left[ -\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{j=1}^{1} \right] = 2\pi b d^{qg}
\]

\[
P_{qg}(x) = C_F \frac{1 + (1-x)^2}{x} \rightarrow A^{qq}_n = C_F \frac{2 + n + n^2}{n(n+1)} = 2\pi b d^{qq}_n
\]

\[2fP_{qg}(x) = 2T[x^2 + (1-x)^2] \rightarrow A^{qg} = f \frac{2 + n + n^2}{n(n+1)(n+2)} = 2\pi b d^{qg}
\]

### Solutions for moments:

\[
L = \frac{\alpha(\mu^2)}{\alpha(Q^2)} \ln(Q^2/\Lambda^2) - \ln(\mu^2/\Lambda^2)
\]

\[
q^{NS}(Q^2) = q^{NS} L d^{qg}
\]

\[
\begin{align*}
\Sigma_0(Q^2) &= \Sigma_0 \left[ a_n^2 \frac{L d^L - a_n^2 L d^L}{a_n - a_n} + G_{00} \frac{a_n^2 a_n^2}{a_n - a_n} (L d^L - L d^L) \right] \\
G_0(Q^2) &= G_{00} \left[ a_n^2 \frac{L d^L - a_n^2 L d^L}{a_n - a_n} + \Sigma_0 \frac{L d^L - L d^L}{a_n - a_n} \right]
\end{align*}
\]

\[
a_n = \frac{A^{GG}_n - A^{qg}_n \pm \Delta}{2A^{GG}_n} \quad \Delta = \left[ (A^{GG}_n - A^{qg}_n)^2 + 4A^{GG}_n A^{qg}_n \right]^{1/2}
\]

\[
d_n = \frac{1}{4\pi b} [A^{GG}_n + A^{qg}_n \pm \Delta]; \text{ (eigenvalues of } d_n \text{ matrix)}
\]

The momentum of the side particle in the ladder decreases at each step from the current to the target and likewise the longitudinal momentum fraction increases along this same path. The skeleton ladder is then dressed by all vertex and self energy insertions. These virtual corrections provide the "\( + \)" regularization near \( x = 1 \) and replace the fixed by the running coupling. Schematically, for a given moment (for brevity we omit the index \( n \)), the \( h \)-rung ladder gives:
Table 3
Numerical values of the non singlet \( d^{\pi} \) and singlet \( d_s \) exponents and of the coefficients in the general solution for the singlet moments. All quantities are defined as in table 2. Note that

\[
\alpha_s/(\alpha_s - \alpha_s) = 1 + \alpha_s/(\alpha_s - \alpha_s)
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha_s )</th>
<th>( \alpha_s^2 \alpha_s )</th>
<th>( \frac{1}{\alpha_s^2 - \alpha_s} )</th>
<th>( d_s^* )</th>
<th>( d_n )</th>
<th>( d^{\pi}_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.3600</td>
<td>-0.3600</td>
<td>0.6400</td>
<td>0.0000</td>
<td>-0.6173</td>
<td>-0.3951</td>
</tr>
<tr>
<td>3</td>
<td>-0.9270</td>
<td>-0.2468</td>
<td>0.2742</td>
<td>-0.5713</td>
<td>-1.2016</td>
<td>-0.6173</td>
</tr>
<tr>
<td>4</td>
<td>-0.9825</td>
<td>-0.1393</td>
<td>0.1238</td>
<td>-0.7599</td>
<td>-1.6376</td>
<td>-0.7753</td>
</tr>
<tr>
<td>5</td>
<td>-0.9928</td>
<td>-0.0956</td>
<td>0.0744</td>
<td>-0.8912</td>
<td>-1.9536</td>
<td>-0.8988</td>
</tr>
<tr>
<td>6</td>
<td>-0.9963</td>
<td>-0.0723</td>
<td>0.0514</td>
<td>-0.9958</td>
<td>-2.2029</td>
<td>-1.0004</td>
</tr>
<tr>
<td>7</td>
<td>-0.9978</td>
<td>-0.0579</td>
<td>0.0386</td>
<td>-1.0838</td>
<td>-2.4098</td>
<td>-1.0868</td>
</tr>
<tr>
<td>8</td>
<td>-0.9985</td>
<td>-0.0480</td>
<td>0.0305</td>
<td>-1.1599</td>
<td>-2.5873</td>
<td>-1.1620</td>
</tr>
<tr>
<td>9</td>
<td>-0.9990</td>
<td>-0.0409</td>
<td>0.0250</td>
<td>-1.2271</td>
<td>-2.7429</td>
<td>-1.2287</td>
</tr>
<tr>
<td>10</td>
<td>-0.9993</td>
<td>-0.0355</td>
<td>0.0210</td>
<td>-1.2874</td>
<td>-2.8816</td>
<td>-1.2885</td>
</tr>
<tr>
<td>11</td>
<td>-0.9994</td>
<td>-0.0313</td>
<td>0.0181</td>
<td>-1.3419</td>
<td>-3.0068</td>
<td>-1.3429</td>
</tr>
<tr>
<td>12</td>
<td>-0.9996</td>
<td>-0.0279</td>
<td>0.0158</td>
<td>-1.3918</td>
<td>-3.1209</td>
<td>-1.3926</td>
</tr>
</tbody>
</table>

\[
A^k \int \frac{dk_1^2}{k_1^2} \frac{\alpha(k_1^2)}{2\pi} \int \frac{dk_2^2}{k_2^2} \frac{\alpha(k_2^2)}{2\pi} \cdots \int \frac{dk_n^2}{k_n^2} \frac{\alpha(k_n^2)}{2\pi} = \left( \frac{A}{2\pi b} \right)^h \int_t^1 \frac{dt_1}{t_1} \int_t^{t_2} \frac{dt_2}{t_2} \cdots \int_t^{t_{n-1}} \frac{dt_n}{t_n} \approx \frac{1}{h!} \left( \frac{A}{2\pi b} \ln t \right)^h = \frac{1}{h!} (AY)^h
\]

which corresponds to eq. (4-27).

This diagrammatic reconstruction is the first step toward establishing the so-called factorization
Theorem, which is the basis for the QCD version of the parton model. In lepton production we know from the RGE that an exponential factor can be extracted from (moments of) the parton-current cross section and included in a redefinition of (moments of) the parton densities. The evolution equations written in terms of parton densities, with kernels derived with no reference to the probe, strongly suggest that this same factorization also holds for all other processes where, according to the naive parton model, the same parton densities should appear. Examples of such processes are transverse momentum distributions of jets and semiinclusive measurements in lepton production, Drell–Yan processes and so on. The problem is to prove this statement. This is done in the LLA by a diagrammatic analysis showing that to each initial leg exactly the same dressed ladder as above can be associated. More recently the factorization theorem has also been extended to non leading logarithms as well, by showing that the whole exponential factor in eq. (3-59) is in the fact reobtained [27, 160, 309, 171, 330, 247].

The formulation and the physical understanding of the evolution equations have been essential in promoting this line of development toward the reconstruction of the parton model in a theoretically satisfactory context within QCD.

4.4. Explicit solution near $x = 1$

The evolution in the LLA can be explicitly solved near $x = 1$ [238, 239, 149, 95]. Often this is done by going first to moments, solving the corresponding equations in the large $n$ limit (as moments of large order are dominated by the region $x = 1$) and then transforming back. We prefer here to work directly in $x$ space. We restrict our discussion to the valence quark components which are simpler and known to dominate near $x = 1$.

Starting from eqs. (4-67) we have near $x = 1$:

$$
\frac{dV}{dY} = 2C_F \int_x^1 \frac{dz}{1-z} \left[ \frac{1}{z} V\left(\frac{x}{z}, Y\right) - V(x, Y) \right] + C_F V(x, Y) \left[ \frac{3}{2} + 2 \ln(1-x) \right]
$$

(4-70)
where $Y$ was defined in eq. (4-26). In order to solve this equation we make the position:

$$V(x, Y) \approx A(Y) (1 - x)^B(Y).$$  \hspace{1cm} (4-71)

By a simple exercise one finds:

$$\int_x^1 \frac{dz}{1 - z} \left[ \frac{1}{z} \left(1 - \frac{x}{z}\right)^{B(Y)} - (1 - x)^{B(Y)} \right] = \sum_{k=1}^{B(Y)} \frac{1}{k} (1 - x)^{B(Y)}$$

$$= \gamma_E + \psi[B(Y) + 1]$$

where $\psi(x) = (d/dx) \log \Gamma(x)$, $\Gamma(x)$ being the Euler gamma function, and the constant $\gamma_E$ was already met in eq. (3-41). The last expression also holds for non integer $B(Y)$. Eq. (4-70) can then be cast in the form:

$$\frac{1}{A} \frac{dA}{dY} + \left( \frac{dB}{dY} - 2C_F \right) \log(1 - x) + 2C_F [\gamma_E - \frac{3}{2} + \psi(B + 1)] = 0.$$

(4-73)

This implies first that

$$B(Y) = 2C_F Y + c$$

(4-74)

where $c$ is an arbitrary constant and second, from the remaining equation for $A(Y)$, that:

$$V(x, Y) \approx \frac{\text{const}}{\Gamma(2C_F Y + c + 1)} \left(1 - x\right)^{2C_F Y + c}.$$  \hspace{1cm} (4-75)

For the valence content of a given hadron the constant $c$ cannot be determined by the evolution equations alone. On the other hand for a target quark the requirement that in absence of strong interactions (i.e. at $Y = 0$) $V(x, Y) \rightarrow \delta(1 - x)$ leads to $c = -1$. The behaviour of eq. (4-75) is expected to hold in the region $1/(1 - x) \gg 1$ and $(\alpha(Q^2)/\pi) \ln(1/(1 - x)) \ll 1$, this last requirement being important for the LLA to apply (when studying the non leading corrections to the evolution equations in section 5 we shall learn more about neglected terms near $x = 1$). The fact that the leading logarithmic terms are exponentially damped implies that other classes of terms may emerge near $x = 1$, in particular terms neglected because down by powers may eventually dominate the limit near $x = 1$. This is in fact suggested by the connection between structure functions near $x = 1$ and form factors [155, 417, 72, 75] which is preserved in QCD models based on gluon exchange between constituents [85, 322].

Finally we note that in the same approximations of eq. (4-75) the gluon density and the sea densities turn out to vanish with one and two more powers of $(1 - x)$ than valence respectively (apart from logarithms of $(1 - x)$). (Near $x = 0$ the behaviour of structure functions, in particular in the singlet sector, was studied by Dokshitzer [149], Fadin, Kuraev, Lipatov [181], Cabibbo, Petronzio [95]; and more recently by Gribov, Levin, Ryskin [237].)
4.5. Some properties of moments

We consider here the application of the evolution equations to moments of structure functions. For numerical methods of solving the evolution equations in $x$ space, see [90d] (their interpolating formulae for numerical solutions are not to be acritically used) and [430, 353, 211].

For the $n$th moment, defined as in eq. (4-23), of any non singlet quark density we find in the LLA (according to eq. (4-27)):

$$ q_{\text{NS}}^{\gamma}(Q^2) = q_{\text{NS}}^{\gamma}(0) \left( \frac{\alpha_s(t)}{\alpha_s(Q^2)} \right)^{d_n} $$

(4-76)

with

$$ d_n = \frac{A_n}{2\pi b} \quad (4-77) $$

For $n = 1$ one has $d_n^{\text{NS}} = 0$ corresponding to flavour conservation (eq. (4-46)). As shown in table 3, for $n \geq 2$ all $d_n^{\text{NS}}$ are negative and their absolute value increases with $n$. This behaviour of moments implies that the $x$ distribution of all non singlet densities is of constant area but shrinks to smaller $x$ with increasing $Q^2$. At infinite $Q^2$ a $\delta$ function at $x = 0$ is approached.

For the gluon and the singlet quark densities (with the latter, $\Sigma$, defined by eq. (4-40) and measurable from $(F_+ + F_-)/2x$) the moment analysis is a bit more complicated. For each $n$ the system of equations

$$ \frac{d}{dt} \left( \begin{array}{c} \Sigma_n(t) \\ G_n(t) \end{array} \right) = \frac{\alpha(t)}{2\pi} \left( \begin{array}{cc} A_n^{\text{eq}} & A_n^{\text{G}} \\ A_n^{\text{G}} & A_n^{\text{GG}} \end{array} \right) \left( \begin{array}{c} \Sigma_n(t) \\ G_n(t) \end{array} \right) + o(\alpha^2(t)) $$

(4-78)

has two eigenvectors:

$$ O_n^\pm(t) = \Sigma_n(t) + a_n^\pm G_n(t) $$

(4-79)

with diagonal $Q^2$ dependence of the form (4-76) with exponents $d_n^\pm$, related to the eigenvalues of the matrix $A_n$ as in eq. (4-77). The entries of this matrix are the moments of the splitting functions as in eq. (4-25), except that $A_n^{\text{G}}$ is the moment of $2fP_{qG}$. The solution for $\Sigma_n(t)$ and $G_n(t)$ are readily written down in terms of $O_n^\pm(t)$:

$$ \Sigma_n(t) = \frac{1}{a_n^+ - a_n^-} \left[ a_n^+ O_n^+(t) - a_n^- O_n^-(t) \right] $$

$$ G_n(t) = \frac{1}{a_n^+ - a_n^-} \left[ O_n^+(t) - O_n^-(t) \right] $$

(4-80)

The explicit forms of $a_n^\pm$, $d_n^\pm$ and $O_n^\pm(t)$ are shown in table 2 (see also table 3).

Moments with $n = 1$ are not defined in this case because the total number of soft gluons and $q\bar{q}$ pairs is unbound. The moments with $n = 2$, corresponding to momentum fractions, deserve a special mention. (In this connection see also [23, 226].) The sum rule of momentum conservation, eq. (4-45), can be
written down as:

\[ O_2^+(t) = \Sigma_2(t) + G_2(t) = 1. \]  

(4-81)

It implies the existence of one eigenstate with zero eigenvalue. In fact \( a_2^+ = 1, \ d_2^+ = 0 \). Note that the terms proportional to \( C_A \) in \( P_{G2} \) do not contribute to the \( n = 2 \) moment: the splitting of one gluon into two gluons does not change the total momentum fraction carried by gluons. The second eigenvector is readily found to be:

\[ O_2^-(t) = \Sigma_2(t) - \frac{T}{2C_F} G_2(t) \]  

with anomalous exponent

\[ d_2^- = \frac{-4(2C_F + T)}{11C_A - 4T}. \]  

(4-83)

This anomalous exponent being negative, \( O_2^\pm \) vanished at \( Q^2 \to \infty \). Then eq. (4-82) implies a fixed ratio for the asymptotic values of \( \Sigma_2(\infty) \) and \( G_2(\infty) \):

\[ \frac{\Sigma_2(\infty)}{G_2(\infty)} = \frac{T}{2C_F} = \frac{1}{2} \frac{N_d}{N_G} = \frac{3}{16} f \]  

(4-84)

where \( N_q \) and \( N_G \) are the total numbers of quarks and gluons in the theory. By also using eq. (4-81) the asymptotic values can be written down separately:

\[ \Sigma_2(\infty) = \frac{T}{2C_F + T}, \quad G_2(\infty) = \frac{2C_F}{2C_F + T}. \]  

(4-85)

All valence components being extinguished at \( Q^2 \to \infty \), \( \Sigma_2(\infty) \) corresponds entirely to the sea.

It is interesting to observe that the experimental data for the second moment of \( (F_2^+ + F_2^-)/2x \) are above the asymptotic limit (4-85) for \( f = 4 \) and decreasing (fig. 8). This is especially important in view of the relatively large number of gluons in QCD which, according to eq. (4-83), makes the asymptotic limit for \( \Sigma_2 \) particularly small. Fig. 9 shows an indicative picture of the behaviour with \( Q^2 \) of the total momentum fractions carried by quarks and gluons in the nucleon. This trend is well supported by the data (CDHS: ref. [3]). In fig. 10 a well known test of QCD with moments of the structure function \( F_3^{x,\phi} \) is reproduced. To \( F_3 \) quarks and antiquarks contribute with opposite signs, so that its evolution is determined by the non singlet equation. For the log–log plot of any two moments of \( F_3 \) one can predict a straight line with slope fixed with no parameters (in the LLA):

\[ \frac{d \log M_2^\pm(t)}{d \log M_2^\pm} = \frac{d_q^{\pm q}}{d_k^{\pm q}} (1 + o(\alpha(t))). \]

The slope is independent of the value of \( \alpha(t) \) (provided it is sufficiently small) and of the group theory factors in \( P_{qq} \) which drop in the ratio. The linearity of the plot is a test of the adequacy of
Fig. 8. The area under $F_2$ is related to the momentum fractions carried by quarks in the target. The asymptotic values of these fractions can be predicted in QCD. It is important to observe that the data are above the asymptotic prediction (for $f = 4$) and decreasing, which supports the view that present data are indeed in the asymptotic region described by QCD in LLA. This view could not be maintained if, for example, the data were below the asymptotic prediction and decreasing.

Fig. 9. Schematic behaviour of momentum fractions carried by gluons, by valence (here $u + d - ar{u} - ar{d}$), by an SU(3) symmetric sea of light quarks and by the charm quark in the nucleon ($f = 4$).

Fig. 10. In the range of validity of the LLA the $\ln M_n$ versus $\ln M_n$ plot of moments of the structure function $F_1$ is predicted to be a straight line, with slope absolutely predicted in QCD. The value of the slope, which depends on $n,m$ and the vector nature of the interaction, is independent of $A$ and of the gauge group.
perturbation theory and the quantitative value of the slope a test of the vector nature of the gluon and of the consistency with QCD of the observed scaling violations (see also eq. (5-53)) (BEBC/GGM: ref. [79], CDHS: ref. [244]). For a review of experiment and complete references see refs. [379, 153a].

4.6. Polarized parton densities

The formalism of the evolution equations can be directly generalized to parton densities of definite helicity in a polarized target. For an introduction, the naive parton model approximation in this case and the canonical light cone formalism see [186, 256]. For example this is relevant to the description of scaling violations in lepto-production with polarized beam and target. For data see [8] and [63].

Each quark and gluon can be found in one of two helicity states. We denote by $q_{i\pm}(x, t)$ and $G_{\pm}(x, t)$ the quark and gluon densities with helicity $\pm$ in a target of definite polarization. The relation between polarized and unpolarized densities is obviously given by

$$p_{\alpha}(x, t) + p_{\dot\alpha}(x, t) = p_{\alpha}(x, t)$$

(4-86)

The evolution equations for unpolarized targets can be compactly written down as (recall eq. (4-26)):

$$\frac{d}{dY} p_A = \sum_B p_B \otimes P_{AB} + o(\alpha(Q^2))$$

(4-87)

where $p_A$ denotes either a quark or an antiquark of specified flavour or a gluon and $P_{AB}(x)$ are the set of splitting functions ($P_{qq} = \delta_{qq} P_{qq}$ and so on).

In a similar way the evolution equations for polarized densities can be directly written down as:

$$\frac{d}{dY} p_{\alpha} = \sum_B [p_{B_\alpha} \otimes P_{A\alpha\beta} + p_{B_\dot\alpha} \otimes P_{A\dot\alpha\beta}] + o(\alpha(Q^2)).$$

(4-88)

The polarized splitting functions can be related in the usual way to the probabilities of finding a parton $A$ with polarization $a$ in a parton $B$ with polarization $b$, and can be computed from the vertices of QCD [22]. It follows from parity conservation that:

$$P_{A,B}(x) = P_{A,B}(x).$$

(4-89)

Clearly one also has:

$$P_{A,B}(x) + P_{A,B}(x) = P_{A,B}(x) + P_{A,B}(x) = P_{AB}(x).$$

(4-90)

A simple consequence of eq. (4-89) is that the sums $(p_{A_\alpha} + p_{A_\dot\alpha}) = p_A$ and the differences $(p_{A_\alpha} - p_{A_\dot\alpha})$, for any $A$, evolve separately. By using eqs. (4-86, 4-90) the unpolarized evolution equations are immediately recovered for the sums $p_A$. For the differences one finds evolution equations of exactly the same structure but with new kernels:

$$\frac{d}{dY} (p_{A_\alpha} - p_{A_\dot\alpha}) = \sum_B (p_{B_\alpha} - p_{B_\dot\alpha}) \otimes \Delta P_{AB} + o(\alpha(Q^2))$$

(4-91)
\[ \Delta P_{AB}(x) = P_{A,B}(x) - P_{\bar{A},\bar{B}}(x). \] (4-92)

\( \Delta P_{AB} \) is a measure of the tendency of a parton A to remember the polarization of its parent B.

From helicity conservation at the quark gluon vertex it immediately follows that:

\[ \Delta P_{qq}(x) = P_{q,q_+}(x) = P_{qq}(x) \] (4-93)

i.e. the non singlet kernel is the same as in the unpolarized case. By explicit computation one also finds:

\[ \Delta P_{Gq}(x) = C_F \frac{1 - (1 - x)^2}{x} \] (4-94)

\[ \Delta P_{qG}(x) = \frac{T}{f} [x^2 - (1 - x)^2] \] (4-95)

\[ \Delta P_{GG}(x) = C_A \left[ (1 + x^4) \left( \frac{1}{x} - \frac{1}{4(1 - x)^2} \right) - (1 - x)^3 \right] + \delta(1 - x) \frac{11C_A - 4T}{6}. \] (4-96)

Note that all charge moments are well defined in this case: while the total number of gluons and pairs is infinite, the total net helicity is finite. In particular the charge moment of \( \Delta P_{qG} \) is also zero, as is the case for \( \Delta P_{qq} \), so that the net helicity is separately conserved, in leading order, for each flavour of quarks and antiquarks. (Moments of eqs. (4-93 to 4-96) were originally derived by Ahmed, Ross [6, 7], Ito [266], Sasaki [377] from RGE and the light cone operator expansion.)

The evolution equations for parton densities of definite helicities are sufficient for the prediction of scaling violations in leptoproduction on a longitudinally polarized target. However, if the target is transversely polarized additional information is also needed.

To illustrate this point we consider the case of electroproduction with neglect of weak interaction effects. On a spin 1/2 target of momentum \( P \) and helicity \( s \) the hadronic tensor contains as additional antisymmetric part with two new structure functions [186]:

\[ W_{\mu\nu} = 4\pi^2 \frac{E}{M} \int d^4x \ e^{-iqx} \langle P, s | J_\mu(x) J_\nu(0) | P, s \rangle = W_{\mu\nu}^S + iW_{\mu\nu}^A \] (4-97)

\[ iW_{\mu\nu}^A = -ie_{\mu\nu\lambda\rho} q^\lambda \left\{ s^\rho \frac{V_1}{M}(x, Q^2) + [(Pq)_\rho - (qs)_\rho] \frac{V_2}{M^2}(x, Q^2) \right\}. \] (4-98)

\( V_1 \) and \( V_2 \) are related to adimensional structure functions with approximate scaling properties by:

\[ \nu V_1 = G_1 ; \quad \nu^2 V_2 / M = G_2. \] (4-99)

For a longitudinal polarization of the target only \( G_1 \) is relevant asymptotically, while for a transverse polarization both \( G_1 \) and \( G_2 \) are equally important. This can be seen as follows. For a longitudinal polarization

\[ s_t = \pm (P; E, 0)/M \] (4-100)
the only non vanishing components of $W^\Lambda$, in the frame where the $\gamma^*$ and target momenta are given by eqs. (4-1, 4-2), are:

$$W_{21}^\Lambda(s_i) = - W_{12}^\Lambda(s_i) = \pm \frac{1}{M} \left[ G_1 - \left( \frac{2Mx}{Q} \right)^2 G_2 \right] \approx \pm \frac{G_1}{M}. \quad (4-101)$$

This implies that $G_1$ can be measured from the asymptotic value of the asymmetry induced by longitudinally polarized electrons on a longitudinally polarized target. On the other hand, for a transverse polarization of the target, say in the 1 direction:

$$s_\perp = (0; 0, \pm 1, 0) \quad (4-102)$$

the only non vanishing components of $W^\Lambda_{\mu\nu}$ are:

$$W_{0\perp}^\Lambda(s_\perp) = - W_{\perp 0}^\Lambda(s_\perp) = \frac{2x}{Q} (G_1 + G_2). \quad (4-103)$$

Note that in this case the whole tensor is of order $1/Q$ which makes more difficult the experimental determination of $G_2$ from the asymmetries on transversely polarized targets.

By computing the quark-current point like cross section one obtains for the antisymmetric part of the partonic tensor:

$$W_{\mu\nu}^{\Lambda \text{(parton)}} = m e_{\mu\nu\alpha} q^\Lambda w^\sigma \quad (4-104)$$

where $w^\sigma$ and $m$ are the quark polarization and mass. By comparison with eq. (4-98) it is clear that if the target was a free quark then $G_2$ would vanish. On the other hand for a nucleon target, the relation between the target polarization $s$ and the quark polarization $w$ is different in the transverse and longitudinal cases. Precisely in the transverse case (see eq. (4-102)):

$$w_\perp = \pm s_\perp \quad (4-105)$$

while for the longitudinal case (see eq. (4-100)):

$$w_\parallel = \pm \frac{xM}{m} s_\parallel \quad (4-106)$$

where the relation $k_\mu = xP_\mu$ between the quark and the nucleon momentum was taken into account. This difference leads in general to a non trivial relation between the hadronic and partonic tensors:

$$2G_1 = e^2 (q_+ - q_-) \quad (4-107)$$

$$2(G_1 + G_2) = e^2 \frac{m}{Mx} (k_+ - k_-) \quad (4-108)$$

where $e$ is the quark charge and the sum over quarks and antiquarks is omitted for simplicity. $q_\pm$ are the
quark densities of helicity ± in a nucleon of positive helicity whose difference evolves in $Q^2$ according to eqs. (4-91), and $k_\pm$ are the quark densities with ± transverse polarization in a nucleon with positive transverse polarization. By making the target to coincide with a free quark, then $M = m$, $q_- = k_- = 0$ and $q_+ = k_+ = \delta(1 - x)$ so that one again obtains $G_2 = 0$.

In summary, relatively larger asymmetries are connected with longitudinally polarized targets. These asymmetries are described by the structure function $G_1$. $G_1$ can be expressed in terms of quark densities of definite helicity. The theory of scaling violations for $G_1$ can be simply described in parton language in the massless theory. In leading order in $\alpha(Q^2)$ the polarized evolution equations (4-91) with the kernels specified by eqs. (4-93 to 4-96) contain all the necessary information. The conservation law for the first moment of $P_{\text{qu}}$, eq. (4-46), amounts to asymptotically preserving the Bjorken sum rule [70]. The subleading corrections of order $\alpha(Q^2)$ to this sum rule have also been computed (together with more general non leading corrections to $G_1$) [283].

On the other hand the structure function $G_2$ can only be reached experimentally by measuring the small asymmetries on a transversely polarized target (which asymptotically vanish as $1/Q$ as seen from eq. (4-103)). The theory of scaling violations for $G_2$ is more involved than for $G_1$ because $G_1 + G_2$ has no parton analogue for massless partons. This is clear from eq. (4-108) where transversely polarized quarks appear: a notion with no sense in the massless theory. Thus the original treatment [6, 7, 266, 377] of the scaling violations for $G_2$ by light cone expansion in the massless theory cannot be considered as physically satisfactory [21]. Recently a more complete treatment was given, by considering the quark mass term as an additional interaction [31].

5. Leptoproduction beyond the leading logarithmic approximation

In this section we consider the corrections to the LLA in leptoproduction. We have seen that the naive parton model results are formally reproduced in the LLA in terms of $Q^2$ dependent effective parton densities which satisfy evolution equations with known kernels. Corrections of order $\alpha(Q^2)$ to the parton formulae and of order $\alpha^2(Q^2)$ to the evolution equations have been neglected in section 4 where a physical picture of scaling violations was developed. On the other hand the general structure of the expansion in $\alpha(Q^2)$ for moments of structure functions, as derived from RGE, was discussed in section 3. We start the present discussion by making the connection between the two methods more precise. Then we consider in succession the non leading corrections to the parton formulae and to the evolution equations.

For a compact description of all independent structure functions in leptoproduction it is convenient to introduce the notation:

$$\mathcal{F}_a = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \equiv (2F_1, F_2/x, F_3).$$

(5-1)

In order to simplify the writing we temporarily restrict the discussion to the non singlet projections of $\mathcal{F}_a$. We start from eq. (4-22 or 4-35) which we write in the form:

$$\mathcal{F}_a(x, Q^2) = \int \frac{dy}{y} q(y, Q^2) e^2 \left[ \delta \left( \frac{x}{y} - 1 \right) + \frac{\alpha(Q^2)}{2\pi} f^a \left( \frac{x}{y} \right) + \cdots \right] = q \otimes e^2 \left[ \delta + \frac{\alpha(Q^2)}{2\pi} f^a + \cdots \right]$$

(5-2)
with \( q(x, t) \) obeying the evolution equations (4-34, 4-36) (as often done, here too the sum over \( q \) and \( \bar{q} \) flavours was omitted). In the formalism of section 3 the moments of the structure functions \( \mathcal{F}_a \) are given by eq. (3-56). The correspondence between the two formulations might be put as follows \([61, 284]\):

\[
\int_0^1 dx x^{n-1} q(x, t) = Q_n(\alpha) \exp \int_\alpha^\infty \frac{\gamma_n(\alpha) d\alpha}{\beta(\alpha)} \tag{5-3}
\]

\[
\int_0^1 dx x^{n-1} e^2 \left[ \delta(1 - x) + \frac{\alpha(t)}{2\pi} f^a(x) + \cdots \right] = C_n[a(\alpha(t))]. \tag{5-4}
\]

However it has already been stressed that beyond the LLA neither \( C_n(\alpha) \) nor \( \gamma_n(\alpha) \) are separately well defined. Correspondingly we have seen that the set of \( f^a(x) \) is not unambiguous and in fact these non leading terms depend on the regularization adopted for infrared singularities and on the renormalization prescription (through virtual diagrams). This is an indication that the effective parton densities have been so far completely specified only within the accuracy of the LLA. In the naive parton model one thought to have an intrinsic, measurable definition of parton densities. But in going from naive to QCD improved parton densities the original definition was modified by inclusion of a universal (convolution) factor, extracted from the parton-current cross section and containing the leading logarithmic series of terms with all related mass singularities. Beyond the LLA the definition becomes ambiguous because one could clearly also include an additional universal non leading logarithmic factor without altering the results of section 4. A change in the definition of \( q(x, t) \) and \( G(x, t) \) beyond the LLA corresponds to a change of the reduced coefficients \( C_n(\alpha) \), of the anomalous dimension functions \( \gamma_n(\alpha) \), to a variation of the non leading corrections to individual processes and also to the evolution equations for parton densities.

A precise definition of parton densities beyond the LLA can be fixed in many different ways. Depending on the definition adopted the charge and momentum sum rules (eqs. (4-44, 4-45)) can be either preserved to all orders in the running coupling or receive computable corrections. One possibility is to exactly specify the computation method and the renormalization prescription and then use eq. (5-3) for defining \( q(x, t) \). A second possibility, which we prefer, is to link the definition of parton densities beyond the LLA to some suitable processes acting as standard “partonometers”.

According to this idea quark densities which exactly satisfy the valence sum rules eqs. (4-44), i.e. without corrections of order \( \alpha(Q^2) \), can be defined from the structure functions \( \mathcal{F}_2 \) for all possible beams on a given target \([15, 16, 297]\). That is, one defines quark densities by setting to all orders in \( \alpha(t) \):

\[
\mathcal{F}_2(x, Q^2) = \sum_i e_i^2 q_i(x, Q^2) \tag{5-5}
\]

or equivalently

\[
\int_0^1 dx x^{n-1} \sum_i e_i^2 q_i(x, t) = C_2[a(\alpha(t))] Q_n(\alpha) \exp \int_\alpha^\infty \frac{\gamma_n(\alpha) d\alpha}{\beta(\alpha)} \tag{5-6}
\]

where \( e_i^2 \) are the appropriate naive parton model couplings squared. Clearly the r.h.s. is now
independent of prescriptions. $F_2$ is given a special role in that it is connected with the Adler sum rule [4]. This sum rule which follows from the algebra of charges is exact and not only asymptotically true. In particular when all powers of $m^2/Q^2$ are neglected it can be written in the form (to all orders in $\alpha(Q^2)$):

$$\int_0^1 \frac{dx}{x} \left[ F_2^{\nu T}(x, Q^2) - F_2^{\nu T}(x, Q^2) \right] = A_0$$

(5-7)

where $T$ is the target and $A_0$ is a fixed number depending on $T$ and the structure of weak couplings. Note that $(F_2^{\nu T} - F_2^{\nu T})/x$ is a combination of differences $\Sigma_i C_i(n_i - \bar{n}_i)$ between quarks and antiquarks of the same flavour, with $\Sigma_i C_i = 0$. Thus by defining all quark densities from $F_2$ we automatically ensure that the valence sum rules eqs. (4-44) are preserved from corrections of order $\alpha(Q^2)$.

With the definition of $q_i(x, t)$ from $F_2$ the LLA formulae, which for each term in the current, are schematically given by

$$\mathcal{F}_1 = \mathcal{F}_2 \propto q + \bar{q} ; \quad \mathcal{F}_3 \propto -q + \bar{q}$$

(5-8)

in next to leading accuracy are modified into:

$$\mathcal{F}_2 \propto q + \bar{q}$$

(5-9)

$$\mathcal{F}_1 \propto (q + \bar{q}) \otimes \left[ \delta + \frac{\alpha(t)}{2\pi} (f^1_q - f^1_{\bar{q}}) \right] + 2 \frac{\alpha(t)}{2\pi} G \otimes (f^3_{\bar{q}} - f^3_q) + o(\alpha^2(t))$$

(5-10)

$$\mathcal{F}_3 \propto (-q + \bar{q}) \otimes \left[ \delta + \frac{\alpha(t)}{2\pi} (f^3_q - f^3_{\bar{q}}) \right] + o(\alpha^2(t)).$$

(5-11)

The non singlet restriction has been dropped here. The differences $(f_{q,G}^a - f_{q,G}^d)$ are completely well defined (see for example the discussion following eq. (3-73)). This can easily be understood because the leading logarithmic series cancels in the differences ($\mathcal{F}_a - \mathcal{F}_2$) and so does their ambiguous fall-out on the $f_{q,G}(x)$ terms. An explicit computation leads to [144, 55, 15, 369]:

$$f^2_q(x) - f^2_{\bar{q}}(x) \equiv f^2_q(x) = C_F \cdot 2x$$

(5-12)

$$f^3_{\bar{q}}(x) - f^3_q(x) \equiv f^3_{\bar{q}}(x) = \frac{T}{f} \cdot 4x (1 - x)$$

(5-13)

$$f^3_q(x) - f^3_{\bar{q}}(x) \equiv f^3_{\bar{q}}(x) = C_F (1 + x).$$

(5-14)

Note that only the real diagrams in figs. 4b and 6 contribute to eqs. (5-12 to 5-14). We see that the correction terms to the naive parton model formulae for the structure functions in leptoproduction are extremely well behaved at all $x$ and even near $x = 0$ or 1.

Of particular importance is the expression of the longitudinal structure function [99, 432, 336, 425, 258, 100, 20]

$$F_L(x, Q^2) = F_2(x, Q^2) - 2x F_1(x, Q^2).$$

(5-15)
One immediately obtains from eqs. (5.12, 5.13) with $N = 3, T = f/2$:

$$F_L(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} x^2 \int_0^1 \frac{dy}{y^3} \left\{ \frac{8}{9} F_2(y, Q^2) + 2 \left( \sum_{i=1}^{2f} e_i^2 \right) \left( 1 - \frac{x}{y} \right) y G(y, Q^2) \right\} + o(\alpha^2(Q^2))$$  \hspace{1cm} (5.16)

$$\sigma_L(Q^2) = \int_0^1 dx F_L(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \left\{ \frac{8}{9} \int_0^1 dx F_2(x, Q^2) + \frac{1}{6} \left( \sum_{i=1}^{2f} e_i^2 \right) \int_0^1 dx x G(x, Q^2) \right\} + o(\alpha^2(Q^2))$$  \hspace{1cm} (5.17)

where $(\sum_{i=1}^{2f} e_i^2)$ is the sum of all coefficients of $q$ and $\bar{q}$ in the naive parton model expression for $F_2/\alpha$ (with $f = 4$ it is 20/9 in electroproduction and 8 in $\nu$ and $\bar{\nu}$ scattering from charged currents). It would be of great importance to experimentally verify these expressions in detail. The present data only allow a very vague consistency statement (for a complete list of references to the data see [379]). Note that a measure of $F_L$ would allow a determination of the shape of the gluon density which otherwise in totally inclusive leptoproduction is to be extracted from the observed scaling violations of $F_2$ [62].

An important application of eqs. (5.12 to 5.14) is the prediction of preasymptotic corrections to sum rules from the algebra of current densities. An example is the Gross-Llewellyn-Smith sum rule [242] for which one finds:

$$\int_0^1 dx \left[ F_{3}^{T}(x, Q^2) + F_{3}^{\bar{T}}(x, Q^2) \right] = L_0 \left[ 1 - \frac{\alpha(Q^2)}{\pi} + o(\alpha^2(Q^2)) \right]$$  \hspace{1cm} (5.18)

where $L_0$ is the predicted asymptotic value.

Another important phenomenological consequence has to do with the problem of a quantitative determination from the data of the small sea densities [15]. Typically the sea densities are measured from quantities that in the naive parton model are proportional to antiquarks or to strange quarks in the nucleon (see for example [3]). An example is the right-handed cross section from V-A currents. When non leading corrections of order $\alpha(Q^2)$ are included the naive parton model formulae are modified according to eqs. (5.9 to 5.14) and terms of order $\alpha(Q^2)$ times $q$ and/or $G$ parton densities are to be added to the sea term which is the only one present in the LLA. The effect of these per se small correction terms is not completely negligible at moderate $Q^2$ because the leading term, being proportional to the sea, is particularly small in these cases.

We now consider the non leading corrections to the evolution equations. For each structure function the convolution structure of the equations is kept in higher order but the kernels start depending on the particular structure function which is considered and on the definition adopted for parton densities. This is seen for the non singlet case by going back to eq. (3.56) and writing it in the form:

$$\frac{d}{dt} M_n^\alpha(t, \alpha) = \beta[\alpha(t)] \frac{d}{d\alpha(t)} M_n^\alpha(t, \alpha) \rightarrow \gamma_n[\alpha(t)] + \beta[\alpha(t)] \left[ \frac{d}{d\alpha(t)} [\gamma_n[\alpha(t)]] \right] C_n^{-1} [\alpha(t)]$$

$$\rightarrow \{ \alpha(t) \gamma_n^{(1)} + \alpha^2(t) [\gamma_n^{(2)} - bc_n^{(1)}] + \cdots \} M_n^\alpha(t, \alpha)$$  \hspace{1cm} (5.19)
where eqs. (3-9, 3-26, 3-57, 3-58) were used. Since products in moment space are equivalent to convolutions in density space, one has:

\[
\frac{d}{dt} \mathcal{F}_a(x, t) = \int_0^1 \frac{dy}{y} \mathcal{F}_a(y, t) \mathcal{P}_a \left( \frac{x}{y}, \alpha(t) \right) = \mathcal{F}_a \otimes \mathcal{P}_a
\]

(5-20)

with

\[
\int_0^1 dx x^{-1} \mathcal{P}_a(x, \alpha) = \alpha \gamma_n^{(1)} + \alpha^2 [\gamma_n^{(2)} - b C_n^{(1)\alpha}] + \cdots
\]

(5-21)

Consider quark densities defined in terms of \( \mathcal{F}_2 \) from eq. (5-5). Then the terms of order \( \alpha^2(Q^2) \) in the non singlet evolution equations are completely determined. For an extension to the singlet sector a definition of the gluon density beyond the LLA is also necessary. This has not been necessary for the next to leading corrections to the parton formulae because gluons start interacting with electroweak currents only at order \( \alpha(Q^2) \). Thus shifting the gluon density by terms of order \( \alpha(Q^2) \) only affects structure functions at order \( \alpha^2(Q^2) \). On the other hand the gluon density is already present in the evolution equations in the LLA and consequently the singlet kernels are sensitive at order \( \alpha^2(Q^2) \) to the definition of the gluon density. We are not interested in specifying a particular definition for the gluon density because all explicit examples will be restricted to the non singlet sector. We denote by \( \mathcal{P} \) the general kernels for the parton densities in a given definition; in particular in the non singlet sector we shall always take:

\[
\mathcal{P}(x, \alpha) = \mathcal{P}_2(x, \alpha) \quad \text{(non singlet)}.
\]

(5-22)

Quite in general we can then write for the evolution of parton densities:

\[
\frac{d}{dt} q_i = \sum_{j=1}^f q_i \otimes \mathcal{P}_{q_i q_j} + \sum_{j=1}^f \bar{q}_j \otimes \mathcal{P}_{q_i q_j} + G \otimes \mathcal{P}_{q_i G}
\]

(5-23)

\[
\frac{d}{dt} G = \sum_{j=1}^f q_j \otimes \mathcal{P}_{G_{q_j}} + \sum_{j=1}^f \bar{q}_j \otimes \mathcal{P}_{G_{q_j}} + G \otimes \mathcal{P}_{G G}.
\]

(5-24)

By charge conjugation we have:

\[
\mathcal{P}_{q_i q_j} = \mathcal{P}_{q_j q_i}; \quad \mathcal{P}_{q_i q_j} = \mathcal{P}_{q_j q_i}
\]

\[
\mathcal{P}_{q_i G} = \mathcal{P}_{G q_i}; \quad \mathcal{P}_{G_{q_i}} = \mathcal{P}_{G q_i}.
\]

(5-25)

The equivalence of all flavours in the massless theory implies that in the quark sector one only needs to distinguish flavour conserving kernels from flavour non conserving kernels:

\[
\mathcal{P}_{q_i G} = \mathcal{P}_{q G}; \quad \mathcal{P}_{G_{q_i}} = \mathcal{P}_{G q}
\]

(5-26)
\[ P_{qq} = \begin{cases} D: \text{diagonal} & P_{qq}^D \\ ND: \text{non diagonal} & P_{qq}^{ND} \end{cases} \]

For example \( P_{uu} = P_{qq}^D, P_{du} = P_{qq}^{ND} \). Note that \( P_{qq}^{ND} \) all start at order \( \alpha^2 \).

The structure of the gluon equation is not changed with respect to eq. (4-42). We therefore concentrate on the quark sector. From eqs. (5-23, 5-26 to 5-28) we obtain:

\[
\frac{d}{dt} q_i = q_i \otimes Q_{qq} + \bar{q}_i \otimes Q_{qq} + \Sigma_q \otimes P_{qq}^{ND} + \Sigma_q \otimes P_{qq}^{ND} + G \otimes P_{qG} \tag{5-29}
\]

where

\[
Q = P_{qq}^D - P_{qq}^{ND} \tag{5-30}
\]

and \( \Sigma_q, \Sigma_{\bar{q}} \) are the sums of all quark and all antiquark densities respectively so that (see eq. (4-40)):

\[
\Sigma = 
\]

As first observed by Ross and Sachrajda [373] \( P_{qq}^D \) and \( P_{qq}^{ND} \) are different already at order \( \alpha^2(Q^2) \). For example \( P_{uu} \neq P_{\bar{u}u} \). The difference at this order is due to the presence in the final state of two identical quarks in the \( uu \) case but not in the \( du \) case, as illustrated in fig. 11. On the other hand at order \( \alpha^2 \) the equality holds:

\[
P_{qq}^{ND} = P_{qq}^{ND} \quad \text{(at order \( \alpha^2 \)).} \tag{5-32}
\]

For example \( P_{du} = P_{\bar{d}u} \). Eq. (5-32) is true to this order because all flavour changing transitions arise through one gluon exchange, as in fig. 11. We shall limit the following discussion to the next to leading approximation where eq. (5-32) is valid. Then eq. (5-28) becomes:

\[
\frac{d}{dt} q_i = q_i \otimes Q_{qq} + \bar{q}_i \otimes Q_{qq} + \Sigma \otimes P_{qq}^{ND} + G \otimes P_{qG} \tag{5-33}
\]

By taking differences in order to restrict to the non singlet sector one finds:

\[
\frac{d}{dt} (q_i - q_m) = (q_i - q_m) \otimes Q_{qq} + (\bar{q}_i - \bar{q}_m) \otimes Q_{qq} \tag{5-34}
\]

Fig. 11. Lowest order diagrams for the probability of finding a \( \bar{u} \) (or \( \bar{d} \)) antiquark in a quark \( u \).
and similarly for a difference of antiquarks:

$$\frac{d}{dt}(q_l - q_m) = (q_l - q_m) \otimes Q_{qq} + (q_l - q_m) \otimes Q_{qq}$$

or of a quark and an antiquark:

$$\frac{d}{dt}(q_l - q_m) = (q_l - q_m) \otimes Q_{qq} + (q_l - q_m) \otimes Q_{qq}.$$  

(5-36)

Note, for example, that eq. (5.35) implies that the difference $(\bar{u} - \bar{d})$ cannot be 0 at all $Q^2$ in a proton [373].

The previous results show the existence of two independent non singlet kernels at order $\alpha^2$. In particular one predicts a different evolution for, say, $(\mathbb{F}_2^p - \mathbb{F}_2^N) = \frac{1}{4}(u + \bar{u} - d - \bar{d})$ and $(\mathbb{F}_2^p - \mathbb{F}_2^p) = 2(u - \bar{u} - d + \bar{d} - s + \bar{s} + c - \bar{c})$ where these expressions are valid (for $f = 4$) because of our definition (eq. (5.5)) of quark densities. In fact the two evolution equations are:

$$\frac{d}{dt}(\mathbb{F}_2^p - \mathbb{F}_2^N) = (\mathbb{F}_2^p - \mathbb{F}_2^N) \otimes (Q_{qq} + Q_{qq})$$

$$\frac{d}{dt}(\mathbb{F}_2^p - \mathbb{F}_2^p) = (\mathbb{F}_2^p - \mathbb{F}_2^p) \otimes (Q_{qq} - Q_{qq}).$$

(5-37)

The validity of the Adler sum rule implies that:

$$\int_0^1 dx [Q_{qq}(x, \alpha) - Q_{qq}(x, \alpha)] = 0$$

(5-38)

while the corresponding moment of $(Q_{qq} + Q_{qq})$ is not expected to vanish, so that the charge moment of $(\mathbb{F}_2^p - \mathbb{F}_2^N)$ starts depending on $Q^2$ at order $\alpha^2$.

$(Q_{qq} \pm Q_{qq})$ are the non singlet kernels for the structure functions $F_2$. The non singlet kernels of other structure functions can easily be found by using eqs. (5-9 to 5-11 and 5-19 to 5-21). As examples consider $[\mathbb{F}_3^p \pm \mathbb{F}_3^p]$. In the naive parton model or in the LLA in QCD one has $[\mathbb{F}_3^p \pm \mathbb{F}_3^p] = 2[-u \pm \bar{u} + d + \bar{d} + s + \bar{s} - c \pm \bar{c}]$. It then follows that the corresponding evolution equations are:

$$\frac{d}{dt}[\mathbb{F}_3^p \pm \mathbb{F}_3^p] = [\mathbb{F}_3^p \pm \mathbb{F}_3^p] \otimes \left[(Q_{qq} \mp Q_{qq}) + \alpha^2(t) \frac{b}{2\pi} (f_3^2 - f_3^2)\right]$$

(5-39)

where the difference $(f_3^2 - f_3^2)$ was given in eq. (5-14).

The explicit form of the non singlet kernels in next to leading accuracy was first computed in moment form by Floratos, Ross, Sachrajda [193]. The moment expression was considerably simplified (see later eq. (5-49)) by Gonzales-Arroyo, Lopez, Yndurain [229]. Independent recomputations have recently been done by Curci, Furmanski, Petronzio [135] and Floratos, Lacaze, Kounnas [191] leading to the following compact form in density space:
\[ Q_{qq}(x, \alpha) = \frac{\alpha}{2\pi} C_F \left( \frac{1+x^2}{1-x} \right) + \left( \frac{\alpha}{2\pi} \right)^2 \left\{ C_F \left[ -2 \left( \frac{1+x^2}{1-x} \right) \ln x \ln(1-x) - 5(1-x) - \left( \frac{3}{1-x} + 2x \right) \ln x ight. \\
- \frac{2}{3} \left( 1 + x \right) \ln^2 x \right. \\
+ \frac{2}{3} C_F C_A \left[ \left( \frac{1+x^2}{1-x} \right) \left( \ln^2 x - \frac{11}{3} \ln \frac{1-x}{x^2} + \frac{367}{16} - \frac{\pi^2}{3} \right) \\
+ 2 \left( 1 + x \right) \ln x + \frac{61}{12} - \frac{215}{12} \ln x \right] + \frac{2}{3} C_F T \left[ \left( \frac{1+x^2}{1-x} \right) \left( \ln \frac{1-x}{x^2} - \frac{29}{12} + \frac{13}{4} \right) \ln x \right] \right\}, \]
\[ + \delta(1-x) \int_0^1 dx \ Q_{qq}(x, \alpha) \quad [\alpha = \overline{MS}] \] (5-40)

\[ Q_{qq}(x, \alpha) = \left( \frac{\alpha}{2\pi} \right)^2 \left( C_F - \frac{1}{3} C_A \right) C_F \left[ 2 \left( 1 + x \right) \ln x + 4(1-x) \right. \\
\left. + \frac{1+x^2}{1-x} \left( \ln^2 x - 4 \ln x \ln(1+x) - 4 \ln x \right) \right] \] (5-41)

with [308]:

\[ \text{Li}_2(x) = - \int_0^x dy \ln(1-y). \] (5-42)

In the case of QCD:

\[ \int_0^1 dx \ Q_{qq}(x, \alpha) = -0.1598(a/2\pi)^2 + \cdots \] (5-43)

Note that \( Q_{qq} \) is regular near \( x = 1 \) so that no "+" redefinition, eq. (4-65), was needed in this case as opposite to that of \( Q_{qq} \). The expression of \( Q_{qq} \) is completely independent of prescriptions once the definition of the coupling constant is specified, which explains the explicit mention to the modified minimal subtraction in eq. (5-40). We recall from eq. (3-64) that a change in the definition of the coupling induces a variation of the second order term proportional to the lowest order result \( P_{qq}(x) \).

In the limit \( x \to 1 \) the dominant terms in \( Q_{qq} \) are given by:

\[ Q_{qq}(x, \alpha) \xrightarrow[x \to 1]{} \left( \frac{\alpha}{2\pi} \right)^2 P_{qq}(x) - \left( \frac{\alpha}{2\pi} \right)^2 2\pi b C_F \frac{1+x^2}{1-x} \ln(1-x) + \cdots \] (5-44)

while the behaviour of \( Q_{qq} \) is very damped near \( x = 1 \):

\[ Q_{qq}(x, \alpha) \xrightarrow[x \to 1]{} \left( \frac{\alpha}{2\pi} \right)^2 C_F \left( C_F - \frac{1}{3} C_A \right) \left( 1-x \right)^5 \frac{1}{10} + \cdots \] (5-45)

The behaviour of \( Q_{qq} \) is equivalent to \( \alpha \log n + \alpha^2 a (\log n)^2 \) \((a \text{ being a constant})\) for moments at large \( n \). It
is interesting to observe that the most singular term near \( x = 1 \) at order \( \alpha^2 \) in \( Q_{q\bar{q}} \) can be reabsorbed into the leading term of order \( \alpha \) by a change of scale \( Q^2 \rightarrow Q^2(1 - x) \). In fact for \( \alpha(Q^2) \) given by eq. (3-31) one has:

\[
\alpha[Q^2(1 - x)] = \alpha(Q^2)[1 - b\alpha(Q^2) \ln(1 - x) + \cdots].
\] (5-46)

The physical origin of this rescaling is easily traced back by recalling eqs. (4-13, 4-14) for the parton-current cross section at order \( \alpha \). The log \( Q^2 \) is actually a log \( (k_1^2)_{\text{max}} \) where \( k_1 \) is the transverse momentum of the produced parton and its maximum value is \( k_1^2 \sim Q^2(1 - x) \) near \( x = 1 \). This phase space effect is present for each rung of the ladder and can be controlled to all orders. In fact the region \( (1 - x) \sim 0 \) corresponds to the soft gluon region (recall that \( (1 - x) \) is the energy lost by the quark). One can therefore take advantage of the techniques of soft gluon resummation [74, 177, 429]. As a result, by diagrammatic [238, 149, 292] or by coherent state methods [232, 136, 90e, 140, 337, 231] for the non singlet moments one obtains the exponentiation (to be compared with eq. (4-27)):

\[
q_n(Q^2) = q_n^0 \exp \left[ \int_0^1 dx (x^{n-1} - 1) C_F \frac{1 + x}{1 - x} \int_{\mu^2}^{Q^2(1 - x)} \frac{dk_1^2 \alpha(k_1^2)}{k_1^2} \right].
\] (5-47)

Equivalently the corresponding evolution equation can be written, at leading accuracy in the expression for \( \alpha(Q^2) \), in the form [26]:

\[
\frac{d}{dt} q(x, Q^2) = \int_x^1 \frac{dy}{y} q(y, Q^2) \left\{ P_{q\bar{q}} \left( \frac{x}{y} \right) \frac{\alpha[Q^2(1 - x/y)]}{2\pi} \right\}.
\] (5-48)

This result is interesting as a nice illustration of a source of non uniformity near \( x = 1 \). Arguments have been given to prove that to all orders this phase space effect provides the leading sub-dominant corrections near \( x = 1 \) [115, 116]. Directly from eqs. (4-13, 5-40) it is seen that the phase space effect is instead not dominant by itself near \( x = 0 \). Phenomenological applications of eq. (5-48) have been recently discussed [53b, 56]. Its importance is in practice severely limited by ignorance of power behaved scaling violations (mass effects [334, 48, 219, 144], see however [174, 52], higher twists [230, 358, 341] and so on).

A relatively simple analytic form for the even (odd) moments of \( Q_{q\bar{q}} \pm Q_{q\bar{q}} \) \( (Q_{q\bar{q}} - Q_{q\bar{q}}) \) has been given. In the following expressions the \( \pm \) alternative on the l.h.s. should correspond to even/odd \( n \) on the r.h.s. respectively. This explains how it is that two distinct non singlet kernels arise from a unique non singlet leading twist operator with definite anomalous dimension: the light cone formalism only predicts even (odd) moments of crossing even (odd) structure functions. \( Q_{q\bar{q}} \pm Q_{q\bar{q}} \) are obtained as separate inversions from even/odd moments [373]. It is clear on the other hand that the formulae in \( x \) space eqs. (5-40, 5-41) also allow to derive odd (even) moments of crossing even (odd) structure functions. With this in mind the above mentioned expressions for moments are given by Gonzalez-Arroyo, Lopez, Yndurain [229] in the form:
\[
\int_0^1 dx x^{n-1} [Q_{qq}(x, \alpha) \pm Q_{qq}(x, \alpha)]_{\text{order } \alpha^2} \equiv \alpha^2 [\gamma_n^{(2)} - b C_n^{(1)}]
\]

\[
= -\left(\frac{\alpha}{2\pi}\right)^2 \left[ C_F (C_F - \frac{3}{2} C_A) \left\{ 2 S_1(n) \frac{2n + 1}{n(n+1)^2} + 2 \left( 2 S_1(n) - \frac{1}{n(n+1)} \right) \left( S_2(n) - S_2(n) \frac{n}{2} \right) \right\} + 8 \hat{S}(n) + 3 S_2(n) \left( \frac{3}{8} - S_3(n) \frac{n}{2} \right) - \frac{3n^3 + n^2 - 1}{n^3(n+1)^3} - 2(-1)^n \frac{2n^2 + 2n + 1}{n^3(n+1)^3} \right] + C_F T \left[ -\frac{20}{9} S_1(n) + \frac{4}{3} S_2(n) + \frac{1}{6} + 2 \frac{11n^2 + 5n - 3}{9n^2(n+1)^2} \right] + C_F C_A \left[ S_1(n) \left( \frac{67}{9} + \frac{2n + 1}{n^2(n+1)^2} \right) - 2 S_1(n) S_2(n) + S_2(n) \left( -\frac{13}{6} + \frac{1}{n(n+1)} \right) \right] - \frac{43}{48} \frac{151n^4 + 263n^3 + 97n^2 + 3n + 9}{18n^3(n+1)^3}
\]

\[
+ \frac{11 C_A}{6} - 4 T \left[ S_3(n) + \frac{5}{3} S_1(n) - S_2(n) - \frac{S_1(n)}{n(n+1)} + \frac{3}{2n} + \frac{2}{n+1} + \frac{1}{n^2} - \frac{9}{2} \right] \] \quad [\alpha \equiv \overline{\text{MS}}]

where:

\[
S_k(n) = \sum_{j=1}^n \frac{1}{j^k}
\]

\[
S_k(n) \frac{n}{2} + \frac{1 + (-1)^n}{n} S_k(n) - \frac{n}{2} S_k(n) \frac{n-1}{2}
\]

\[
\hat{S}(n) = \sum_{j=1}^n \frac{(-1)^j}{j} S_1(j)
\]

In table 4 values in the \overline{\text{MS}} prescription of \gamma_n^{(2)} - b C_n^{(1)} and of the parameters \epsilon_n^{(a)} defined in eqs. (3-66 to 3-68) are reported separately for each structure function \((a = 1, 2, 3)\) and for both crossing even and odd cases. In particular the difference between moments of \(Q_{qq} + Q_{qq} \) and \(Q_{qq} - Q_{qq}\) is always small and rapidly goes to zero with \(n\) (a consequence of eq. (5-45)) thus a posteriori justifying the neglect of this difference which is often found in the literature. This numerical evaluation of non leading corrections for all leptoproduction structure functions shows that, for reasonable values of \(n\), the convergence of the perturbative expansion appears satisfactory.

When fitting \(A_{\overline{\text{MS}}}^{(a)}\) without including \(\epsilon_n^{(a)}\) a different value should be found for each \(n\) (and \(a\)) provided that the sensitivity of the experiment is adequate and \(Q^2\) large enough that all other preasymptotic effects can be neglected. By including \(\epsilon_n^{(a)}\) this dependence should accordingly be substantially reduced as the remaining effect is displaced to next order. When comparing the fits in the LLA with the fits including next to leading corrections the corresponding accuracy in the expression of the running coupling must be kept by going from the LLA form in eq. (3-28) to the improved formula in eq. (3-31). As already mentioned, it must be kept in mind that the definition of \(A_{\overline{\text{MS}}}^{(a)}\) is changed in going from eq. (3-28) to eq. (3-31) as is clear from eqs. (3-29, 3-32) (thus the label \overline{\text{MS}} is not really sufficient to specify which \(A\) one is referring to). As a consequence when repeating the fit in terms of the improved
Table 4
Numerical values in the $\overline{\text{MS}}$ prescription of $h_n^a - \gamma^{(2)}_n - bC_n^{1,\text{ew}}$ and $e_n^a = (\pi\beta_0)\left|bC_n^{1,\text{ew}} - \gamma^{(2)}_n + b'\gamma^{(1)}_n\right|$ (see eqs. (3.66), (5.19) of text) for crossing even and crossing odd non singlet moments of structure functions. $a$ labels the structure function ($2F_1$, $3F_2$ or $F_3$). The difference between crossing even and crossing odd values goes rapidly to zero with increasing $n$ and becomes irrelevant.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h^1$ EVEN</th>
<th>$h^1$ ODD</th>
<th>$e^1$ EVEN</th>
<th>$e^1$ ODD</th>
<th>$h^2$ EVEN</th>
<th>$h^2$ ODD</th>
<th>$e^2$ EVEN</th>
<th>$e^2$ ODD</th>
<th>$h^3$ EVEN</th>
<th>$h^3$ ODD</th>
<th>$e^3$ EVEN</th>
<th>$e^3$ ODD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1326</td>
<td>0.1407</td>
<td>-0.6283</td>
<td>-0.6667</td>
<td>-0.810 · 10$^{-2}$</td>
<td>0.000</td>
<td>0.3836 · 10$^{-1}$</td>
<td>0.000</td>
<td>0.2030</td>
<td>0.2111</td>
<td>-0.9616</td>
<td>-1.000</td>
</tr>
<tr>
<td>2</td>
<td>-0.1556</td>
<td>-0.1552</td>
<td>0.8022 · 10$^{-1}$</td>
<td>0.7821 · 10$^{-1}$</td>
<td>-0.2494</td>
<td>-0.2490</td>
<td>0.5247</td>
<td>0.5227</td>
<td>-0.1322</td>
<td>-0.1318</td>
<td>-0.3089 · 10$^{-1}$</td>
<td>-0.3290 · 10$^{-1}$</td>
</tr>
<tr>
<td>3</td>
<td>-0.4188</td>
<td>0.9576</td>
<td>0.9573</td>
<td>-0.4892</td>
<td>-0.4891</td>
<td>1.291</td>
<td>-0.4071</td>
<td>0.9020</td>
<td>0.9017</td>
<td>1.729</td>
<td>2.466</td>
<td>3.129</td>
</tr>
<tr>
<td>4</td>
<td>-0.6443</td>
<td>1.763</td>
<td>-0.7006</td>
<td>-0.7005</td>
<td>2.029</td>
<td>-0.6372</td>
<td>1.729</td>
<td>2.466</td>
<td>3.129</td>
<td>3.733</td>
<td>4.288</td>
<td>4.802</td>
</tr>
<tr>
<td>5</td>
<td>-0.8407</td>
<td>2.488</td>
<td>-0.8876</td>
<td>2.710</td>
<td>-0.8360</td>
<td>2.466</td>
<td>3.129</td>
<td>3.733</td>
<td>4.288</td>
<td>4.802</td>
<td>5.281</td>
<td>5.731</td>
</tr>
</tbody>
</table>
formulae a finite rescaling of $\Lambda_{MS}$ (independent of $n$ and $a$) is to be expected, which is needed to keep
the value of the coupling at the reference mass fixed. This reference mass is in fact either in the same
energy region than the given experiment or at some other finite scale. In view of this fact it would
be better to express the results in terms of $\alpha(\mu^2)$ directly rather than in terms of $\Lambda$.

By comparing different moments the dependence on the definition of the coupling can be made to
drop away [50, 346, 357], as in eq. (3-69) or similarly for the non leading corrections to the slope parameters
in the log $M_n^2$ vs. log $M_k^2$ plot of fig. 10 (recall eq. (3-66)):

$$
\frac{d \ln M_n^2(t)}{d \ln M_k^2(t)} \left( \frac{d}{d t} \ln M_n^2(t) \right) = \frac{d_{\alpha}^{\text{eq}}}{d_{\alpha}^{\text{eq}}} \left\{ 1 + \left( \frac{e_3^2}{d_{\alpha}^{\text{eq}}} - \frac{e_2^2}{d_{\alpha}^{\text{eq}}} \right) \frac{\alpha(t)}{\pi} + \ldots \right\}.
$$

(5-53)

In these prescription independent formulations of non leading corrections the size of the latter can
still be altered by choosing a different scale, say $h_n Q^2$, for each moment. For example we have seen in
eqs. (5-47, 5-48) that the non leading corrections to lepton production structure functions suggest that
$Q^2(1 - x)$ is a more appropriate scale near $x = 1$. In moment language this precisely means a different
scale for each moment. Thus the pattern of non leading corrections can be used to infer the appropriate
scale choices to each process ([39], see also [370]).

For a determination of the singlet kernels a definition of the gluon density beyond leading accuracy is
also to be fixed, as already discussed. In this respect, it is important to observe that the singlet
equations, being non homogeneous, require for their use in non leading accuracy a correspondingly
precise knowledge of the quark and gluon densities. But the gluon density can only be measured at
present from scaling violations. Thus in principle the improved singlet evolution equations provide us
with the means for a more precise measure of the gluon density. Although very important from the
theoretical point of view, in practice the usefulness to this purpose of the singlet evolution equations in
next to leading accuracy is limited by the quality of the data and the uncertainties of the size of other
(power behaved) preasymptotic corrections.

Note that at non leading accuracy the quark–quark entry of the singlet matrix of evolution kernels
does not coincide with either of the non singlet kernels. In fact by eq. (5-33) one readily finds (when eq. (5-32) is valid):

$$
\frac{d \Sigma}{d t} = \Sigma \otimes \left[ Q_{\text{qq}} + Q_{\text{qq}} + 2 f P_{\text{qq}}^{\text{ND}} \otimes \left] + 2 f G \otimes P_{\text{qG}} \right. \right.
$$

(5-54)

In the literature the singlet kernels in next to leading accuracy are computed from a definition of the
gluon density, in the spirit of eqs. (5-3; 5-4), based on $\overline{\text{MS}}$ (for both the coupling and the operators).
This definition preserves the energy momentum sum rule. We do not give here the explicit expressions
of the kernels. For this we refer the reader to [209, 210] where the list is most complete and clear. These
independent calculations appear to have corrected some marginal errors in the otherwise extremely
valuable previous computations ([194], see also [228, 254, 255, 192]). This statement is based on a
recent paper [30] which shows that the supersymmetric relation eq. (4-68) is still valid to non leading
accuracy provided that one adopts a set of supersymmetric prescriptions and the existing discrepancy is
solved in favour of Furmanski, Petronzio.

We conclude the description of the QCD theory of scaling violations by some remarks on the
comparison with experiment [379, 153a].

The existence of scaling violations in the deep inelastic region is definitely established. Beyond a few
GeV$^2$ for $Q^2$ and $W^2$ the scaling violations are small and when they are neglected evident signatures for the validity of the naive parton model emerge. The success of the naive parton model predictions in relating different structure functions from different beams and targets is uncontroversial and impressive. The formulation of this model appears as one of the main achievements of particle physics in the last two decades. The success of the naive parton model is by itself a strong argument in favour of QCD if one believes that the general framework of renormalizable field theories is adequate for the strong interactions as well as for the electro-weak interactions. Moreover, the pattern of the observed $Q^2$ dependence in all experiments is perfectly compatible with the QCD predictions (figs. 12, 13). This result is also of great significance because of the wealth of data by now accumulated. There are obvious difficulties in precisely matching the sets of data from different experiments. Thus it is neither simple nor perhaps sensible to make an overall fit of the existing data. However, within each experiment, from

![Graph showing $x F_3$ as a function of $Q^2$](image)

Fig. 12. Data on $x F_3$ at different values of $x$ as functions of $Q^2$ from the CDHS collaboration for $Q^2 > 2$ GeV$^2$, $W^2 > 11$ GeV$^2$. From this experiment the quoted value of $\lambda_{GB}$ is $\lambda_{GB} = 0.2 \pm 0.15$ GeV (courtesy of Dr. H. Wahl).
Fig. 13. Data on $F_2$ at different values of $x$ as functions of $Q^2$ from the CDHS collaboration. At $x > 0.4$ the evolution of $F_2$ is taken as pure non-singlet (courtesy of dr. H. Wahl).

relatively low energies as in the SLAC–MIT experiments [365, 76], up to the highest energies, both with charged [29, 38, 77a] and neutral lepton beams [408], a description of scaling violations in terms of QCD logarithms alone appears in all cases possible with values of $\Lambda_{\overline{MS}}$ of a few hundred MeV. The really delicate problem is the separation of the QCD scaling violations from the $Q^2$ dependence induced by mass effects, higher twist corrections, threshold phenomena and so on. The analysis, currently still in progress, of high energy, high statistics experiments is especially focused on this problem. Perhaps a trend of decreasing $\Lambda_{\overline{MS}}$ is observed from low energy to high energy experiments and seems to indicate that some part of the measured scaling violations is in fact due, as is reasonable, to non QCD effects. After a tentative separation of these terms the value of $\Lambda_{\overline{MS}}$ deduced from high energy experiments is typically $\Lambda_{\overline{MS}} \sim 0.1–0.3$ GeV. (For an up to date summary see, for example, [169].) Note, however, that at
high $Q^2$, a small error on $\alpha(Q^2)$ implies a large error on $\Lambda$, so that, as already stated, it would be better to present the data in terms of $\alpha(Q^2)$ at some value of the energy scale.

6. The photon structure functions

Deep inelastic scattering on a real photon target can be experimentally measured in $e^+e^-$ annihilation as shown schematically in fig. 14. One can ensure a sufficiently large value of $-Q^2$ on one side by fixing the electron angle (tagging), while most of the photons on the other side, where the electron is not tagged, are quasi real. The first data on deep inelastic scattering on a photon target have appeared recently [see e.g. 407].

The interest of deep inelastic scattering on a photon target stems from the fact, first discovered by Witten [425], that the photon structure functions can be absolutely predicted in the asymptotic limit, and yet the result does not coincide with the free field prediction [277, 409, 6].

The photon structure functions $F_2^\gamma, F_3^\gamma/x$ are determined as usual once the quark (and gluon) densities in the photon $q_\gamma$ (and $G_\gamma$) are known. In the LLA these densities can be determined as follows. One starts by observing that the evolution equations are modified in this case by the presence of another parton species: the photon itself [146, 339]. They read:

$$Q^2 \frac{d}{dQ^2} q_\gamma(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} [q_\gamma \otimes P_{qq} + G_\gamma \otimes P_{qG}] + \frac{\alpha_{em}}{2\pi} e_\gamma^2 \gamma \otimes P_{q\gamma} + \cdots \quad (6-1)$$

$$Q^2 \frac{d}{dQ^2} G_\gamma(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} [\Sigma_\gamma \otimes P_{Gq} + G_\gamma \otimes P_{GG}] + \cdots \quad (6-2)$$

The last term in eq. (6-1), proportional to the photon density in a photon $\gamma^\prime$, is of order $\alpha_{em} \approx 1/137$ and cannot be neglected in this case because $q_\gamma^\prime$ and $G_\gamma^\prime$ are themselves of order $\alpha_{em}$. We shall always work in first order in $\alpha_{em}$. The photon density starts at zeroth order in $\alpha_{em}$ and we can make the identification:

$$\gamma^\prime(x, Q^2) = \delta(1-x). \quad (6-3)$$

In fact a factor of $\alpha_{em}$ is already brought in by the probability $(\alpha_{em}/2\pi)e_\gamma^2 P_{q\gamma}$ of finding a quark in the photon. $P_{q\gamma}$ and $P_{qG}$ only differ by a colour factor ($T/f$ is replaced by $\text{tr} (1) = N$, the number of colour replicas for each quark):

$$P_{q\gamma}(x) = N[x^2 + (1-x)^2]. \quad (6-4)$$

Fig. 14. Two photon process in $e^+e^-$ annihilation.
In eq. (6-2) $\Sigma^\gamma$ is defined as usual (see eq. (4-40)) and there is no term in $P_{G\gamma}$ because there is no vertex in the theory directly connecting a photon with a gluon.

In the unphysical limit of completely neglecting the strong interactions one would obtain from eqs. (6-1, 6-3) the naive parton model result:

$$q\gamma(x, Q^2)_{\text{naive}} = \frac{\alpha_{em}}{2\pi} e_1^\gamma N[x^2 + (1 - x)^2] \ln(Q^2/\Lambda_\gamma^2) + \cdots \tag{6-5}$$

where the dots stand for non logarithmic terms which can be asymptotically neglected. The energy $\Lambda_\gamma$ is a convenient scale that, on a purely conventional basis, can be set equal to the coupling constant scale: $\Lambda_\gamma \sim \Lambda$. Eq. (6-5) corresponds to the absorptive part of the box diagram contribution to the $\gamma^* \gamma \to \gamma^* \gamma$ forward amplitude (fig. 15a).

In order to proceed in studying the effect of strong interactions we separate as usual the non singlet and the singlet quark evolution equations:

$$Q^2 \frac{d}{dQ^2} (q\gamma)_{\text{NS}} = \frac{\alpha(Q^2)}{2\pi} (q\gamma)_{\text{NS}} \otimes P_{qq} + \frac{\alpha_{em}}{2\pi} (e_1^\gamma)_{\text{NS}} P_{q\gamma} + \cdots \tag{6-6}$$

$$Q^2 \frac{d}{dQ^2} \Sigma^\gamma = \frac{\alpha(Q^2)}{2\pi} [\Sigma^\gamma \otimes P_{qq} + G^\gamma \otimes 2fP_{q\gamma}] + \frac{\alpha_{em}}{2\pi} 2f(e_2^\gamma)P_{q\gamma} + \cdots \tag{6-7}$$

where:

$$(q\gamma)_{\text{NS}} = q\gamma - \frac{1}{2f} \Sigma^\gamma \tag{6-8}$$

$$\langle e^k \rangle = \frac{1}{f} \sum_{i=1}^f e_i^k \tag{6-9}$$

$$(e_1^\gamma)_{\text{NS}} = e_1^\gamma - \langle e^2 \rangle. \tag{6-10}$$

We first consider the non singlet equation which contains all the interesting features in simplest form. By dropping all inessential labels and going to moments, the relevant equation is of the form:

$$Q^2 \frac{d}{dQ^2} q_n(Q^2) = \alpha(Q^2) q_n(Q^2) b d_n^{uu} + \alpha_{em} e^2 b d_n^{\gamma\gamma} \tag{6-11}$$

Fig. 15. A sequence of ladder diagrams (with gluons as rungs) for deep inelastic scattering on a photon target. Labeled with "a" the box diagram corresponding to the naive parton model.
where \( d_n \) are the logarithmic exponents defined through eqs. (4-25, 4-27, 4-77). The general solution of this inhomogeneous equation is given by the general solution of the associated homogeneous equation plus a particular solution of the complete equation. By recalling eq. (3-28) for \( a(0^2) \) one obtains:

\[
q_n(Q^2) = q_n^0 \left[ \ln \frac{Q^2}{A^2} \right]^{d_n^a} + \frac{\alpha_{em} e^2 d_n^{\gamma\gamma}}{(1 - d_n^{\gamma\gamma})} \frac{1}{\alpha(Q^2)}.
\]

(6-12)

The particular solution is given by the second term and it would exactly coincide with the naive parton result (see eq. (6-5)) if not for the factor \((1 - d_n^{\gamma\gamma})^{-1}\). This rescaling factor is smaller than 1 for \( n > 1 \), because \( d_n^{\gamma\gamma} < 0 \) in this range (table 3). The particular solution is asymptotically dominant over the first term which contains the whole hadronic structure of the photon (for example the vector meson dominance contributions). Thus the asymptotic behaviour of \( q_n(Q^2) \) is an absolute prediction.

The way the asymptotic solution is built up in perturbation theory is elucidated by a diagrammatic analysis in terms of dressed ladders in physical gauges [312] (fig. 15). The ladder structure (the dressing replaces \( a \) with \( a(Q^2) \)) leads to the recursive structure (compare with eq. (4-69)):

\[
q_n(Q^2) = \alpha_{em} e^2 b d_n^{\gamma\gamma} \left[ \ln \frac{Q^2}{A^2} + \int \frac{dQ_1^2}{Q_1^2} \alpha(Q_1^2) b d_n^{\gamma\gamma} \ln \frac{Q_1^2}{A^2} \right.
\]

\[
+ \int \frac{dQ_1^2}{Q_1^2} \alpha(Q_1^2) b d_n^{\gamma\gamma} \int \frac{dQ_2^2}{Q_2^2} \alpha(Q_2^2) b d_n^{\gamma\gamma} \ln \frac{Q_2^2}{A^2} + \cdots \bigg] .
\]

(6-13)

All orders in the recursion contribute with a linear term in \( \log(Q^2/A^2) \) because \( \alpha(Q^2) b \log(Q^2/A^2) \approx 1 \):

\[
q_n(Q^2) = \alpha_{em} e^2 b d_n^{\gamma\gamma} \ln \frac{Q^2}{A^2} \left[ 1 + d_n^{\gamma\gamma} + (d_n^{\gamma\gamma})^2 + \cdots \right] = \frac{\alpha_{em} e^2 b d_n^{\gamma\gamma} \ln(Q^2/A^2)}{(1 - d_n^{\gamma\gamma})}.
\]

(6-14)

Note that the free field limit corresponds to \( b \rightarrow \infty \), i.e. \( d_n^{\gamma\gamma} = 0 \) and \( b d_n^{\gamma\gamma} \) finite.

It is straightforward to also solve the singlet equations by following quite similar lines. The complete asymptotic solutions can then be written as:

\[
(q^\gamma)_n = \frac{\alpha_{em} e^2 d_n^{\gamma\gamma}}{\alpha(Q^2)} \left[ \frac{e^2 - \langle e^2 \rangle}{1 - d_n^{\gamma\gamma}} + \frac{(e^2)(1 - d_n^{\gamma\gamma})}{(1 - d_n^{\gamma\gamma})(1 - d_n^{\gamma\gamma})} \right]
\]

(6-15)

\[
(G^\gamma)_n = \frac{\alpha_{em} e^2 d_n^{\gamma\gamma} 2 f(e^2) d_n^{\gamma\gamma}}{\alpha(Q^2)(1 - d_n^{\gamma\gamma})(1 - d_n^{\gamma\gamma})}.
\]

(6-16)

\( d_n^{\gamma\gamma} \) are the eigenvalues of the singlet logarithmic exponent matrix as given in tables 2, 3.

Note that

\[
\frac{(\Sigma^\gamma)_n}{(G^\gamma)_n} = \frac{1 - d_n^{\gamma\gamma}}{d_n^{\gamma\gamma}}.
\]

(6-17)

In particular for \( n = 2 \) the ratio of the momentum fractions carried by quarks and gluons in a photon
turns out to be independent of the number of quark flavours \( f \) (and of \( Q^2 \)) [199]:

\[
\frac{\langle \Sigma^\gamma \rangle_{n=2}}{(G^\gamma)_{n=2}} = \frac{11C_A}{8C_F} = \frac{99}{32}.
\] (6-18)

It is now easy to reconstruct the structure functions. From \( F_2^\gamma(x, Q^2) = 2xF_1^\gamma(x, Q^2) = \sum_{i=1}^{2f} e_i^2 q_i(x, Q^2) \), by taking moments, we immediately find (recall eq. (6-9)):

\[
(2F_1^\gamma)_n = \left( \frac{F_2^\gamma}{x} \right)_n = \frac{\alpha_{em}2f d_n^{\gamma} \gamma}{\alpha(Q^2)} \left[ (e_4) - (e^2)^2 + (e_2)^2(1 - d_n^{G}) \right] - \frac{8\alpha(Q^2)}{[1 - d_n^{G}]} \frac{(e_2)^2(1 - d_n^{G})}{(1 - d_1^{G})(1 - d_2^{G})}.
\] (6-19)

Note that we did not include a factor \( (2f) \) in \( d_n^{\gamma} \) as we had done in \( d_n^{G} \). With respect to the naive parton model result the effect of QCD produces a pronounced depletion of the structure function at large \( x \) because of the gluon radiation that softens the quark distributions.

The contribution of \( q_7 \) and \( G_7 \) to the longitudinal structure function of the photon \( F_1^\gamma \), defined as in eq. (5-15), is directly obtained by eq. (5-16). In addition there is also a contribution from the photon density, eq. (6-3), which can immediately be obtained from the gluon term by a simple coupling and colour factor rescaling. One finally has:

\[
F_1^\gamma(x) = \frac{\alpha_{em}2f}{2\pi} 24f(e_4)x^2(1 - x) + x^2 \int \frac{du}{u^3} \left[ \frac{8\alpha(Q^2)}{3 \pi} F_2^\gamma(u, Q^2) + 4f(e_2) \left( \frac{\alpha(Q^2)}{2\pi} uG^\gamma(u, Q^2) \right) \left( 1 - \frac{x}{u} \right) \right].
\] (6-20)

Note that the second term is independent of \( Q^2 \) as is the first one because both \( F_2^\gamma \) and \( G^\gamma \) are proportional to \( 1/\alpha(Q^2) \). Thus \( F_1^\gamma \) scales in the LLA. In terms of moments:

\[
\int_0^1 dx x^{n-2} F_1^\gamma(x) = 2f \frac{\alpha_{em}}{2\pi} \left\{ \frac{12(e_4)}{(n+1)(n+2)} + d_n^{qG} \left[ \frac{8}{3(n+1)} \frac{(e_4) - (e^2)^2}{1 - d_n^{G}} 
+ (e_2)^2(1 - d_n^{G}) \left( 1 - d_n^{G} \right) \right] \right\}.
\] (6-21)

The next to leading corrections to \( F_2^\gamma \) and \( 2F_1^\gamma \) are also absolutely computable for \( n > 2 \). In fact at subleading accuracy the moments with \( n \geq 2 \) are of the form:

\[
\left( \frac{F_2^\gamma}{x} \right)_n = \frac{E_n}{\alpha(Q^2) + H_n^2} (n > 2)
\] (6-22)

where \( E_n \) can be read from eq. (6-19) and \( \alpha(Q^2) \) is to be taken here in the improved form eq. (3-31). The important point is that the hadronic components do not enter in the next to leading corrections \( H_n^{n^2} \) because all the logarithmic exponents \( d_n^{qG}, d_n^G \) are negative for \( n > 2 \). The moment with \( n = 2 \) is exceptional:
Here $E_2$ is computable while $H_2$ is affected by the unknown matrix element of the hadronic energy momentum tensor (whose anomalous exponent $d_2$ vanishes) between photon states. The values of $H_2$ for $n > 2$ and of $E_2$ were computed by Bardeen, Buras [54] (see also [157]). For $n > 2$ the non leading corrections (in the MS prescription for $\alpha$) turn out to be sizeable at practical values of $Q^2$ and their sign is in the direction of increasing the difference with the naive parton model. (For an analysis of the final state see [84].) The behaviour at $x = 1$ has been studied by Chase [108]. The case that the target photon is also virtual was studied by Uematsu, Walsh [403].

7. The Sudakov form factor of partons

The behaviour of the quark form factor in the LLA and beyond is a very important and delicate problem in QCD. Its importance arises from the fact that this is the simplest problem involving the resummation of a series of doubly logarithmic terms. A complete understanding of this problem would allow the control of many similar doubly logarithmic effects of great relevance in phenomenology. Some of them will be discussed in the following sections. Thus an intense research activity on this area is currently under way.

Consider the one loop calculation in fig. 4a for the quark form factor. We take an on shell massless quark and regularize both infrared and ultraviolet singularities by dimensional regularization. It is convenient to work in the Landau gauge where the quark self energy diagrams vanish and the vertex diagrams of fig. 4a leads by itself to the gauge invariant result. One obtains for the coefficient of $e\gamma_\mu$ (see for example [16]):

$$
\Gamma(q^2) = 1 + \frac{\alpha}{4\pi} C_F \left( \frac{4\pi\mu^2}{-q^2} \right)^{\epsilon} \frac{1}{\Gamma(1-2\epsilon)} \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + o(\epsilon) \right],
$$

where $\epsilon$ was defined in eq. (3-38) and $q$ is the momentum of the virtual photon $\gamma^*$. The double pole in $\epsilon$ is of infrared origin. At fixed $\epsilon$ and large $Q^2 = |q^2|$ it corresponds to:

$$
\Gamma\left( \frac{Q^2}{\mu^2} \right) = 1 - \frac{\alpha}{2\pi} C_F \ln^2(Q^2/\mu^2).
$$

The presence of these infrared doubly logarithmic terms is a well known feature [74, 177, 429]. At order $n$ in $\alpha$ they are of the form $(\alpha^2)^n$. They cancel when one adds the real emission diagrams in order to construct some measurable quantity (all charges being necessarily accompanied by soft radiation). In our discussion of structure functions we never met these terms directly because we adroitly avoided the point $x = 1$. It is however instructive to mention that if one extends the calculation in the same regularization to the real diagrams then one obtains for the structure function $F_2$ of the quark:

$$
F_2(x, Q^2)_{\text{real}} = \alpha \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon} \frac{1}{\Gamma(1-2\epsilon)} \left\{ \frac{2}{\epsilon^2} \delta(1-x) - \frac{1}{\epsilon} \frac{1+x^2}{(1-x)} + \frac{3}{2\epsilon} \delta(1-x) 
\right.
\left. + \left[ (1+x^2) \left( \frac{\ln(1-x)}{1-x} \right) - \frac{3}{2(1-x)} \frac{1+x^2}{1-x} \ln x + 3 + 2\delta(1-x) \right] \right\}.
$$

(7-3)
One easily checks the cancellation of the double poles after multiplication of the square of eq. (7-1) by 
\( \delta(1-x) \) and also that the total residue of the single poles is indeed \( P_{qq}(x) \).

The contribution to \( \Gamma(Q^2) \) of the leading double logarithmic series of terms \((\alpha t^2)^n\) was proven long ago to exponentiate in QED [396]

\[
\Gamma\left(\frac{Q^2}{\mu^2}\right) = \exp\left[ -\frac{\alpha}{4\pi} \ln^2 \frac{Q^2}{\mu^2} \right] \quad \text{(QED, mass shell).} \tag{7-4}
\]

The corresponding equation for QCD is simply obtained by inserting a colour factor:

\[
\Gamma\left(\frac{Q^2}{\mu^2}\right) = \exp\left[ -\frac{\alpha}{4\pi} C_F \ln^2 \frac{Q^2}{\mu^2} \right] \quad \text{(QCD, mass shell).} \tag{7-5}
\]

This extension is not trivial because of the gluon self couplings which are absent in the Abelian case. In QCD the exponentiation has been explicitly verified to 3 loops [131, 101] and proven to all orders by Frenkel and Taylor [200] and by Kinoshita and Ukawa [281] following an analysis by Korthals-Altes and de Rafael [295], and Coquereaux and de Rafael [130]. Similar results also hold for other form factors of partons, for example the form factor of the gluon.

The physical meaning of eqs. (7-4, 7-5) appears to be that the probability amplitude for a (either Abelian or non Abelian) charge to be deflected with high momentum transfer without radiating soft quanta is exponentially small (at fixed infrared cut-off).

The detailed form of the exponent is dependent on the infrared regularization procedure. It is well known that for an off shell quark the exponent differs by a factor of 2. For example:

\[
\Gamma\left(\frac{Q^2}{p^2}\right) = \exp\left[ -\frac{\alpha}{2\pi} \ln^2 \frac{Q^2}{p^2} \right] \quad \text{(QED, off shell).} \tag{7-6}
\]

Up to this point we have been working in the leading doubly logarithmic approximation and in particular with a fixed coupling. We now consider the problem, very relevant in QCD, of how to make the coupling to run. In this case this has to do with non leading terms because, according to eq. (3-28), \((\alpha - \alpha(t)) t^2 = \beta a^2 t^3\), which is non leading with respect to terms of order \(a^2 t^4\). On the other hand the control of non leading terms is important because of the exponential damping of the leading sequence of terms.

This problem has been studied by Mueller [331] (see also [125]). If one translates into QCD the results obtained for QED with negative \(\alpha\), a non unitary but asymptotically free theory, then one obtains:

\[
\Gamma(Q^2) = \exp\left[ -C_F \int \frac{dk^2}{k^2} \frac{\alpha(k^2)}{2\pi} \ln \frac{Q^2}{k^2} \right] \quad \text{(QCD, on shell).} \tag{7-7}
\]

For \(\alpha(k^2)\) in the LLA as given by eq. (3-28) the above formula was proven in non Abelian theories by Kinoshita and Ukawa [281]. Within the stated accuracy, they in fact proved the validity for the form factor of a modified RGE proposed by Korthals-Altes and de Rafael [295] with a \(t\) dependent anomalous dimension term. This modified RGE leads to eq. (7-7) as a solution. For fixed \(\alpha\) eq. (7-5) is
reobtained. For running $\alpha(k^2)$ the asymptotic exponent is computed by observing that:

$$\int_{k^2}^{Q^2} \frac{dk^2}{k^2} \alpha(k^2) \ln \frac{Q^2}{k^2} = \frac{1}{2\pi b} \int \frac{d\ln k^2}{\ln k^2} \left( \ln \frac{Q^2}{A^2} - \ln \frac{k^2}{A^2} \right) \sim \frac{1}{2\pi b} \ln \frac{Q^2}{A^2} \ln \frac{Q^2}{A^2} + \cdots$$

This implies by using eq. (3-54) for $b$:

$$\Gamma(Q^2) = \text{const.} \exp\left[ -\frac{6C_F}{11C_A - 4T} \ln \frac{Q^2}{A^2} \ln \frac{Q^2}{A^2} + \cdots \right]$$

(QCD, on shell).

This is thought to be the correct asymptotic behaviour when all terms down by logarithms are included and all terms down by powers are neglected. However, strictly speaking, it cannot be excluded that terms that order by order are down by powers finally dominate the asymptotic behaviour.

8. Jets in leptoproduction and their transverse momentum

A summary was presented in sections 4 and 5 of the well established and detailed predictions of QCD on scaling violations for structure functions in deep inelastic leptoproduction. It has become increasingly clear in the last few years that a study of the final state, even limited to some appropriately chosen bulk properties, is an important cross check providing distinct additional signatures which are crucial in order to experimentally establish the QCD mechanism for scaling violations. This is of special importance because, from a study of scaling violations in a restricted energy domain, it is difficult to experimentally disprove the view that most or even the whole of the $Q^2$ dependence may originate from mass effects, higher twist operators, thresholds and other power violations of the naive parton model. On the other hand it is simple to realize that the same mechanism that produces the scaling violations in QCD necessarily implies the occurrence of multijet events and of an approximately linear rise of the average transverse component of the total momentum of each parton fragmentation jet. A clear cut experimental evidence for these “hard” parton effects (already obtained in other domains of deep inelastic phenomena, mainly in $e^+e^-$ annihilation and Drell–Yan processes) is a severe blow against all “soft” mechanisms for scaling violations.

We first describe in qualitative terms the difference between the structure of the final state in the naive parton model and according to QCD.

In leptoproduction a well defined reference line is provided by the virtual $\gamma^*$ line in the laboratory frame. All transverse momenta are defined here with respect to this line. In the naive parton model all deep inelastic events in leptoproduction are expected to appear as collinear within an angular accuracy $p_{\perp}/E$, where $p_{\perp}$ is an “intrinsic” ($Q^2$ independent) transverse momentum of the partons in the target and/or of the hadrons in the parton jets. The collinear events consist of two jets, the target fragmentation jet and the struck parton jet, at opposite sides of the rapidity interval, smoothly connected by a soft hadron plateau. Correspondingly the longitudinal cross section $\sigma_z$, eqs. (5-15 to 5-17), vanishes as a power of $m^2/Q^2$ or $p_{\perp}/Q^2$. This follows because in the frame (4-1) where $\gamma^*$ carries no energy the hit parton simply reverses its momentum. In the massless limit the quark helicity is conserved in the interaction with $\gamma^*$; in the collinear
limit, this implies a spin flip for the hit parton and a unit of spin component for $\gamma^*$. Thus $\sigma_L$, which corresponds to a $\gamma^*$ of zero helicity, can only arise from mass and acollinearity effects.

In QCD this picture is invalidated because of the probability of order $\alpha(Q^2)$ of emitting, for example, a gluon of large $k_\perp \sim Q$ from the quark line. This produces a hard tail in the $k_\perp$ distribution of the final parton. We have already seen that this high $k_\perp$ tail behaves as $1/k_\perp^2$ (eq. (4-14)), and extends up to $k_\perp^2 = \frac{1}{4} W_p^2$ where $W_p^2$ is the total energy squared of the parton-current system in its center of mass (eq. (4.34-13)). This small (because of order $\alpha(Q^2)$) but long tail is responsible for scaling violations, for a contribution of order $\alpha(Q^2)$ to $\sigma_L$ and for a linear rise with energy of the average $k_\perp$ of partons:

$$Q^2 \frac{d\sigma_T}{dQ^2} = Q^2 \frac{d}{dQ^2} \int \frac{dk_\perp^2}{k_\perp^2} \alpha(k_\perp^2) \approx \alpha(Q^2) + \cdots$$

(8-1)

$$\frac{\sigma_T}{\sigma_L} = \int \frac{dk_\perp^2}{k_\perp^2} \alpha(k_\perp^2) \frac{k_\perp^2}{Q^2} = \alpha(Q^2) + \cdots$$

(8-2)

$$\langle k_\perp^2 \rangle = \int \frac{dk_\perp^2}{k_\perp^2} \alpha(k_\perp^2) k_\perp^2 = \alpha(Q^2) \cdot Q^2 + \cdots$$

(8-3)

In the last equation for $\langle k_\perp^2 \rangle$ the dots indicate terms which are constant in $Q^2$, apart from logarithms, and cannot be computed in perturbation theory. In fact these terms are affected by the non-perturbative intrinsic $p_\perp$ distribution. On the other hand the slope of the linearly rising term is determined by values of $k_\perp \sim Q$ on the perturbative tail and is computable. Thus the quantity of interest in QCD is not the absolute value of $\langle k_\perp^2 \rangle$ but rather its rate of change. In studying the variation of $\langle k_\perp^2 \rangle$ the intrinsic $p_\perp$ can be forgotten because it does not depend on $Q^2$. We have seen that the important hard contributions arise from a ladder of emissions ordered in $k_\perp$. The intrinsic transverse momentum has to do with the $p_\perp$ distribution in the target as seen at the bottom of the ladder at a fixed reference scale $\mu$ where the non-perturbative domain sets in.

The occurrence of a hard parton interaction at order $\alpha(k_\perp^2)$ produces, when $k_\perp \sim Q$, a three jet event: the target fragmentation jet along the $\gamma^*$ direction and two parton jets with nearly opposite large transverse momentum [132]. The three jet axes lie approximately on a plane, so that the increase of $k_\perp$ should be limited to the component in the event plane $(k_\perp)_{in}$ as opposed to the $(k_\perp)_{out}$ component which only starts increasing at order $\alpha^2(Q^2)$. Note that the parton plane does not coincide with the lepton plane. The definitions of in and out with respect to these two planes should not be confused. The angular correlations between the two planes have been studied in the asymptotic limit, although the effects of hadronization and of the intrinsic $p_\perp$ tend to wash out the QCD effects at present energies [221, 325, 117, 323, 293, 326]. When $k_\perp$ is not sufficiently large the two parton jets are not resolved and the QCD effect appears as a "fattening" of a single jet in the current fragmentation region. On the other hand if the two parton jets are resolved then, to a first approximation, each of them should look as in a normal two jet event (at a somewhat lower energy). This is because the fattening of the jets of a three jet event only occurs at order $\alpha^2(Q^2)$, together with the occurrence of four jet events, and so on.

The asymptotic distributions of suitably defined transverse momentum variables (or other "jettiness" parameters) for a hadronic target can be obtained by an application of the factorization theorem. This
Guido Allarelli, Partons in quantum chromodynamics

Theorem instructs us to take convolutions of the corresponding parton distributions with the effective $Q^2$ dependent parton densities in the target. In the LLA it is sufficient to study the $k_\perp$ distributions of the two basic parton processes at order $\alpha$ given by (4-9) and (4-29). The results are summarized in the following.

We define the variables:

$$z = \frac{Q^2}{2(kq)}, \quad u = \frac{(kp_\perp)}{(kq)} \quad 0 \leq z, u \leq 1$$

(8-4)

where $k$ is the incoming parton four momentum and $p_\perp$ is the four momentum of one of the two final partons which we imagine to observe. By interchange of the two final partons $u \leftrightarrow (1-u)$. When the intrinsic momentum of the initial parton is neglected $k$ is indeed along the reference $\gamma^*$ line. In terms of $z$ and $u$ the absolute value of the transverse momentum of each of the final partons is given by:

$$k_\perp(z, u) = Q \sqrt{\frac{1-z}{z}} u(1-u).$$

(8-5)

For each given polarization of $\gamma^*$ (L, RH, LH for longitudinal, right-handed, left-handed respectively) and for each specified nature of the current ($V, V-A$, mixture of $V$ and $A$) one can write down the corresponding angular distributions. For example consider a general $V$ and $A$ coupling of $\gamma^*$ to a quark current of the form $\gamma_\mu (eV + e_A \gamma_5)$. In electroproduction $e_A = 0$, $e_V = e$ (the quark charge) and for a $V-A$ weak charged current $e_V = -e_A = 1$. Also we denote the angular distributions on a parton target with the same symbols (see eq. (5-1)) as for the corresponding structure functions $F_a$ (but now in terms of two variables):

$$F_a(z, u) = \frac{d\sigma}{du}(z, u)$$

(8-6)

with

$$F_a(z) = \int_0^1 du F_a(z, u).$$

(8-7)

For an initial quark or antiquark, as in (4-9), we have (for $z, u < 1$) [190, 20, 321]:

$$F_1^a(z, u) = \frac{d\sigma}{du}(z, u) = (e^2 + e_A^2) C_F \frac{\alpha}{2\pi} \left[ \frac{1 + (1 - u - z)^2}{(1-z)(1-u)} + 2zu \right]$$

(8-8)

$$F_2^a(z, u) = \frac{d\sigma}{du}(z, u) = (e^2 + e_A^2) C_F \frac{\alpha}{2\pi} 4zu$$

(8-9)

$$F_3^a(z, u) = -\frac{d\sigma}{du}(z, u) = 2e_V e_A C_F \frac{\alpha}{2\pi} \left[ \frac{1 + (1 - u - z)^2}{(1-z)(1-u)} + 2zu - 2(1 - z)(1 - u) \right].$$

(8-10)

Eq. (8-8) corresponds to eq. (4-15). Similarly eqs. (8-7, 8-9) lead to eq. (5-12). Note that for a $V-A$ current:
\[ F_{RH}^q(z, u) = F_{LH}^q(z, u) = C_F \frac{\alpha}{2\pi} 4(1-z)(1-u) \quad (8-11) \]

where

\[ F_{LH}^{RH} = F_1 + F_3. \quad (8-12) \]

These cross sections are regular at \( u = 1 \) because, for a V - A current, the RH (LH) cross section on a quark (antiquark) starts at order \( \alpha \), so that the occurrence of logarithms in the integrated cross sections is delayed to order \( \alpha^2 \), in the same way as for \( F_L \).

Similarly for an initial gluon (\( 0 < u < 1 \)):

\[ F_1^g(z, u) = (e_\gamma^2 + e_\alpha^2) \frac{\alpha}{2\pi} \frac{1}{2} \left[ z^2 + (1-z)^2 \right] \left[ \frac{1}{u} + \frac{1}{1-u} - 2 \right] \quad (8-13) \]

\[ F_1^g(z, u) = (e_\gamma^2 + e_\alpha^2) \frac{\alpha}{2\pi} 4z(1-z). \quad (8-14) \]

All previous angular distributions have been computed from real diagrams (figs. 4b, 6) and no contributions at \( z = 1 \) were reported, as well as no regularization at \( u = 0, 1 \). We are not interested in these details because they would lead to vanishing contributions to the quantities of interest when convoluted with powers of \( k_\perp \), given in eq. (8-5). For the same reason we did not show \( F_3^g(z, u) \) which is antisymmetric in \( u \leftrightarrow (1-u) \) (the totally inclusive structure function \( F_3^g(z) \) vanishes).

From the angular distributions in eqs. (8-8 to 8-14) one can obtain the moments of the parton \( k_\perp \) distributions:

\[ \frac{\alpha}{2\pi} c(e_\gamma, e_\alpha) K^m_a(z) = \int_0^1 \left( \frac{k_\perp(z, u)}{Q} \right)^m \mathcal{F}_a(z, u) \, du \quad (8-15) \]

where \( c(e_\gamma, e_\alpha) \) are the appropriate couplings which we prefer to factorize together with \( \alpha/2\pi \).

By taking averages over the effective parton distributions in the target one finally obtains:

\[ \langle k_\perp^m \rangle = \frac{\alpha(Q^2)}{2\pi} W^m f_m(x, y, Q^2) + \cdots \quad (8-16) \]

with the slopes \( f_m \) given by:

\[ f_m(x, y, Q^2) = \left( \frac{x}{1-x} \right)^{m/2} \int_0^1 \frac{dz}{z} \left[ 2F_1(z, Q^2) \left[ \left( 1-y + \frac{y^2}{2} \right) K_{1}^{m,a} \left( \frac{x}{z} \right) + (1-y) K_{1}^{m,a} \left( \frac{x}{z} \right) \right] + \sum_{\text{pairs}} (e^2 + e^2) G(z, Q^2) \left[ \left( 1-y + \frac{y^2}{2} \right) \right. \right. \]

\[ \times \left. K_{1}^{m,a} \left( \frac{x}{z} \right) + (1-y) K_{1}^{m,a} \left( \frac{x}{z} \right) \right] + o(\alpha(Q^2)). \quad (8-17) \]
The above equation applies to a process with cross section proportional to the quantity \( \sigma(x, y, Q^2) \) given by:

\[
\sigma(x, y, Q^2) = \left(1 - y + \frac{y^2}{2}\right) 2x F_1(x, Q^2) + (1 - y) F_2(x, Q^2) + y \left(1 - \frac{y}{2}\right) x F_3(x, Q^2).
\]

(8-18)

In eqs. (8-17, 8-18) \( y = (E - E')/E \) with \( E, E' \) being the initial and final lepton energies in the lab. frame. The sum \( \Sigma_{\text{partons}}(e^2 + e^2_\gamma) \) is 10/9 and 4 in electroproduction and \( \nu \) (or \( \bar{\nu} \)) scattering from charged currents respectively (for \( f = 4 \)). In eq. (8-16) the energy scale

\[
W = Q \sqrt{\frac{1-x}{x}}
\]

(8-19)

(the total invariant mass of the hadronic system) was adopted because it turns out that with this choice the slopes \( f_n(x, y, Q^2) \) show little \( x \) dependence for \( x \approx 0.2 \). In the LLA we of course could as well replace \( Q^2 \) with \( W^2 \) in the argument of the running coupling constant.

In the idealized case of a clear cut three jet event one could measure \( k_\perp \), the parton transverse momentum, by identifying it with the total transverse momentum of each of the two large angle jets in the current fragmentation region, namely with the transverse component (with respect to the \( \gamma^* \) reference line) of the sum of the momenta of all hadrons in one jet:

\[
k_\perp = \langle k_\perp \rangle_{\text{parton}} = \sum_{\text{H in one jet}} \langle k_\perp \rangle_{\text{Hadron}}.
\]

(8-20)

We stress again that we are not considering here how fat the jet is, but how much its backbone, i.e. the parton momentum, is acollinear. According to eq. (8-20) \( \langle k_\perp \rangle \) is to be identified with \( \langle \Sigma_{\text{H}}(k_\perp)_{\text{H}} \rangle \). The total \( k_\perp \) of a whole jet has the important advantage of being independent of the details of the hadronization mechanism, in particular of the fragmentation functions (because of the momentum conservation sum rule eq. (9-24)). This is obviously not the case for the average \( k_\perp \) of an individual hadron of a given type, for example a charged pion, which is so commonly considered (see for example [324]). In a qualitative way it is clear that the rise with energy of the average \( k_\perp \) of the parton (with respect to the \( \gamma^* \) line) necessarily implies a related increase of the average transverse momentum of each individual hadron in its jet. However for a quantitative analysis the total transverse momentum of the parton jet is far simpler, being independent of the fragmentation process. Also, by first adding up all transverse momentum vectors in one parton jet and then taking powers of result, one obtains a quantity which is free of collinear (and soft) mass singularities.

However the case of an ideal three jet event where it is simple to disentangle the two parton jets does not provide a sufficiently general measurement procedure. Such a procedure must reduce to the previous prescription for a clear cut three jet event (always in the limit of negligible intrinsic transverse momentum). It must also be insensitive to mass singularities not only at order \( \alpha \) but in higher orders as well. Many practical definitions are possible (see for example [81]). The differences among them only influence the asymptotic limit at order \( \alpha^2(Q^2) \). One possible example is the following. Consider a variable plane through the \( \gamma^* \) reference line. Add the transverse momenta of all hadrons on one side of the plane. By rotating the plane around the \( \gamma^* \) line maximize the result and call the corresponding maximum the \( k_\perp \) of the parton jet.

More on the techniques developed for final state analysis will be discussed later in the case of \( e^+e^- \).
annihilation. Most of the methods and the variables of use in that case can directly be borrowed for leptoproduction as well [367, 390]. For related work see also [253, 328].

Recent data provide evidence that a tail of events at large $k_\perp$ is developing which expands in $Q^2$ and/or $W^2$. (For example [39, 47, 141].)

We have so far considered the region of very large transverse momenta $k_\perp \sim Q$. When $Q$ is sufficiently large one can consider a problem with two scales of momenta:

$$\mu^2 \ll k_\perp^2 \ll Q^2.$$  \hspace{1cm} (8-21)

In this regime terms of order $\log(Q^2/k^2_\perp)$ can no more be neglected with respect to terms of order $\log(k^2_\perp/\mu^2)$ as it was true for $k_\perp \sim Q$. We shall see that this leads us to consider a problem of resummation for a sequence of doubly logarithmic terms in close relation with the Sudakov exponentiation considered in section 7. This problem was first formulated and discussed by Dokshitzer, Dyakonov and Troyan [150]. The solution that they proposed was modified by Parisi and Petronzio [352] who established the direct connection of this problem with the Sudakov form factor. Explicit calculations to two loops have confirmed their solution ([315], see also [270, 176]).

The origin of the double logarithms can be understood as follows. Consider the quantity:

$$\Sigma(k^2_\perp, Q^2) = \frac{1}{\sigma(Q^2)} \int \frac{d\sigma}{dp^2_\perp} dp^2_\perp$$ \hspace{1cm} (8-22)

where $\sigma(Q^2)$ is the total cross section. $\Sigma$ is the generating function (by differentiation with respect to $k^2_\perp$) of $(1/\sigma) d\sigma/dk^2_\perp$. We write down the expression for $\Sigma(k^2_\perp, Q^2)$ to order $\alpha(k^2_\perp)$, for a parton quark target, in the form of 1 minus the probability of hard radiation:

$$\Sigma(k^2_\perp, Q^2) = 1 - \int_{k^2_\perp}^{Q^2} \frac{dp^2_\perp}{p^2_\perp} \frac{\alpha(p^2_\perp)}{2\pi} \int_0^{1-p^2_\perp/Q^2} dx P_{q\bar{q}}(x) + \cdots$$ \hspace{1cm} (8-23)

the upper limit of integration in $x$ arises because of the condition eq. (4-13) on the maximum allowed value of $p^2_\perp$ at fixed $x$ (we dropped a factor of 4 because $Q^2$ or $Q^2/4$ is the same in the LLA):

$$p^2_\perp \leq Q^2 \frac{1-x}{x}, \text{ i.e. } x \leq \frac{1}{1+p^2_\perp/Q^2} \sim 1 - \frac{p^2_\perp}{Q^2}.$$ \hspace{1cm} (8-24)

For the sake of this argument one can take the expression of $P_{q\bar{q}}(x)$ at $x < 1$. The double logarithms are produced from the soft singularity of $P_{q\bar{q}}(x)$ near $x = 1$ (which we know is typical of vector gluons):

$$\Sigma(k^2_\perp, Q^2) = 1 - \int_{k^2_\perp}^{Q^2} \frac{dp^2_\perp}{p^2_\perp} \frac{\alpha(p^2_\perp)}{2\pi} \int_0^{1-p^2_\perp/Q^2} dx C_F \left( \frac{1+x^2}{1-x} \right) + \cdots$$

$$= 1 - \int_{k^2_\perp}^{Q^2} \frac{dp^2_\perp}{p^2_\perp} \frac{\alpha(p^2_\perp)}{2\pi} \left[ 2C_F \ln \frac{Q^2}{p^2_\perp} + \cdots \right].$$ \hspace{1cm} (8-25)
Consider first the case of QED with $\alpha$ fixed and $C_F = 1$. Then:

$$\Sigma(k_\perp^2, Q^2) = 1 - \frac{\alpha}{2\pi} \ln^2 \frac{Q^2}{k_\perp^2} + \cdots$$  \hspace{1cm} (QED) \hspace{1cm} (8-26)

which corresponds to:

$$\left(\frac{1}{\sigma} \frac{d\sigma}{dk_\perp^2}\right)_{\alpha k_\perp^2 > 0} = \frac{\alpha}{\pi} \frac{1}{k_\perp^2} \ln \frac{Q^2}{k_\perp^2} + \cdots = \alpha \nu(k_\perp) + \cdots$$ \hspace{1cm} (8-27)

In the region $(\alpha/\pi) \log^2(Q^2/k_\perp^2) \ll 1$ the leading expression for $\Sigma$ or $(1/\sigma) d\sigma/dk_\perp^2$ is obtained by taking double logarithms into account to all orders. The double logarithms arise from the emission of photons (gluons) with momenta much smaller than $Q^2$. In this soft region the cross section for production of $n$ photons (gluons) factorizes into independent emissions [178, 233, 232, 90e]:

$$\frac{1}{\sigma} d\sigma_n = \frac{\alpha^n}{n!} d^2 k_\perp^1 \cdot d^2 k_\perp^2 \cdots d^2 k_\perp^n \nu(k_\perp^1) \frac{\nu(k_\perp^n)}{\pi} \cdots$$ \hspace{1cm} (8-28)

In the leading bilogarithmic approximation this corresponds to the exponential result:

$$\Sigma(k_\perp^2, Q^2) \approx \exp \left[ -\frac{\alpha}{2\pi} \ln^2 \frac{Q^2}{k_\perp^2} \right] \approx \Gamma^2 \left( \frac{Q^2}{k_\perp^2} \right)$$ \hspace{1cm} (QED) \hspace{1cm} (8-29)

where we made explicit the connection with the square of the on shell Sudakov form factor in eq. (7-4). Equivalently:

$$\frac{1}{\sigma} d\sigma = \frac{\alpha}{\pi} \frac{1}{k_\perp^2} \ln \frac{Q^2}{k_\perp^2} \exp \left[ -\frac{\alpha}{2\pi} \ln^2 \frac{Q^2}{k_\perp^2} \right]$$ \hspace{1cm} (QED). \hspace{1cm} (8-30)

More in general one can write:

$$\frac{1}{\sigma} \frac{d\sigma}{d^2 k_\perp} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \prod_{i=1}^{n} d^2 k_\perp^i \frac{\nu(k_\perp^i)}{\pi} \delta^{(2)} \left( k_\perp - \sum_{i} k_\perp^i \right)$$ \hspace{1cm} (QED). \hspace{1cm} (8-31)

The $\delta$ function term can be cast in a convenient form by the familiar impact parameter formalism:

$$\delta^{(2)} \left( k_\perp - \sum_{i} k_\perp^i \right) = \frac{1}{(2\pi)^2} \int d^2 b \exp \left\{ -ib \left( k_\perp - \sum_{i} k_\perp^i \right) \right\}.$$ \hspace{1cm} (8-32)

One then obtains the eikonal result [196, 111, 1, 46]:

$$\frac{1}{\sigma} \frac{d\sigma}{d^2 k_\perp} = \frac{1}{(2\pi)^2} \int d^2 b \exp(-ib \cdot k_\perp) \exp \left\{ \frac{\alpha}{\pi} \int d^2 p_\perp \nu(p_\perp) \left[ \exp(ibp_\perp) - 1 \right] \right\}$$ \hspace{1cm} (QED). \hspace{1cm} (8-33)

The factors $(\exp(ibp_\perp) - 1)$ arise from interpreting $\nu(p_\perp)$ as a distribution of the "+" type (near $p_\perp = 0$) [24].
The extension to the non Abelian case is done by following the same discussion as for the Sudakov form factor. One obtains:

$$
\Sigma(k^2, Q^2) = \exp \left[ -C_F \int_{k^2}^{Q^2} \frac{dp_\perp^2}{p^2} \alpha(p_\perp^2) \ln \frac{Q^2}{p^2} \right] = I^2(Q^2, k^2) 
$$

(8-34)

where the square of the Sudakov form factor is again reproduced (see eq. (7-7)). Alternatively one could exponentiate the whole lowest order differential distribution:

$$
\Sigma(k^2, Q^2) = \exp \left[ - \int_{k^2}^{Q^2} \frac{dp_\perp^2}{p^2} \frac{1}{\alpha} \frac{d\sigma}{dp_\perp^2} \right]. 
$$

(8-35)

This is an equivalent procedure at the leading doubly logarithmic approximation with the additional advantage of being automatically in agreement with the calculation at order \( \alpha(Q^2) \) for \( k_\perp \sim Q \).

By inserting the LLA for the running coupling one obtains the asymptotic expansion:

$$
\Sigma(k^2, Q^2) \approx \exp \left[ - \frac{4G_F}{11C_A - 4T} \left( \ln \frac{Q^2}{\Lambda^2} \ln \frac{\ln(Q^2/\Lambda^2)}{\ln(k^2/\Lambda^2)} - \ln \frac{Q^2}{k^2} \right) \right] \quad \text{(QCD)}.
$$

(8-36)

In analogy with eq. (8-33) one finds [139]

$$
\frac{1}{\sigma} \frac{d\sigma}{d^2k_\perp} = \frac{1}{(2\pi)^2} \int d^2b \exp(-ibk_\perp) \exp \left[ \frac{1}{\pi} \int \frac{dp_\perp^2}{p^2} (\exp(ibp_\perp) - 1) \frac{\alpha(p_\perp^2)}{\pi} C_F \ln \frac{Q^2}{p^2} \right] \quad \text{(QCD)} 
$$

(8-37)

(in this connection see also [118, 126, 127]). It was pointed out by Parisi and Petronzio [352] that previous approach can be extended down to \( k_\perp = 0 \) provided that \( Q^2 \) is large enough. Physically this means that at sufficiently large \( Q^2 \) the dominant contribution to the cross section at \( k_\perp = 0 \) arises from the cancellation of several hard emissions each with large transverse momentum.

The generating function \( \Sigma \) for the actual hadronic process can be obtained by folding the partonic \( \Sigma \) with the effective parton densities as usual. The only new point is that the effective scale will be \( k^2 \perp \) in this case rather than \( Q^2 \) because what matters is the maximum allowed transverse momentum, as we have seen many times by now.

9. \( e^+e^- \) annihilation

9.1. The total hadronic cross section

\( e^+e^- \) annihilation at high energy is the simplest and most fundamental deep inelastic process. In the one photon approximation, the total hadronic cross section at asymptotic energies is related to the Fourier transform of the vacuum expectation value of the product (or the commutator) of two electromagnetic currents in the massless theory (when weak effects are disregarded):

$$
\sigma_{e^+e^- \rightarrow \pi} = \frac{4\pi\alpha^2_{em}}{3Q^2} II(Q^2)
$$

(9-1)
where $\Pi(Q^2)$ is defined by:

$$T_{\mu\nu} = \int dx \, e^{i x\cdot p} \langle 0 | [J_\mu(x), J_\nu(0)] | 0 \rangle = \frac{1}{6\pi} ( - g_{\mu\nu} Q^2 + q_\mu q_\nu ) \Pi(Q^2). \quad (9-2)$$

For $e^+ e^- \rightarrow \mu^+ \mu^-$, $\Pi(Q^2) = 1$ and the corresponding cross section is given by:

$$\sigma_{e^+ e^- \rightarrow \mu^+ \mu^-} \equiv \sigma_0 = \frac{4\pi \alpha^2}{3 Q^2}. \quad (9-3)$$

Similarly the cross section into a spinless point-like pair with unit charge would be $\frac{1}{4} \sigma_0$.

The hadronic component of $\Pi(Q^2)$ is dominated by the c-number term in the operator expansion of the current commutator near the light cone. The corresponding coefficient obeys the simplest renormalization group equation (3-6) with no anomalous dimension term. This is because the current is conserved and no anomalous dimension is associated to a c-number. According to the general discussion in section 3.1, eq. (3-11), the solution can be cast in the form:

$$\sigma_{e^+ e^- \rightarrow \text{hadrons}} = \sigma_{\text{TOT}} = \sigma_0 R[\alpha(t)] \quad (9-4)$$

where, in the massless theory, $R[\alpha(t)]$ is obtained from $R(t, \alpha)$ by setting $R[\alpha(t)] = R[0, \alpha(t)]$.

This is certainly the correct asymptotic limit for $Q^2$ large and spacelike. An analytic continuation along a circle of large radius would then be necessary to reach the vicinity of the physically interesting region at large timelike values of $Q^2$ \cite{364, 329, 53, 380}. However order by order in perturbation theory the expansion corresponding to the asymptotic limit in eq. (9-4) appears to be verified at all timelike $Q^2$ sufficiently large in comparison with the masses of the quarks in the final state. In particular the asymptotic limit is also valid below threshold for production of still heavier quarks and their effect near threshold can be reabsorbed in a change of the running coupling, in agreement with the decoupling theorem \cite{33}. Thus diagrammatic methods can lead to more powerful results than the elegant and far simpler light cone analysis.

In fixed point theories $R[\alpha(Q^2)]$ tends to a constant (scaling) value. But only in asymptotically free theories this value coincides with the free field value (naive parton model):

$$R[\alpha(Q^2)] \rightarrow R(0) = N \sum_{i=1}^{f} e_i^2_{\text{QCD}} = 3 \sum_{i=1}^{f} e_i^2, \quad (9-5)$$

$e_i$ being as usual the quark charges.

The experimental value of $R$ beyond the c and b quark thresholds up to the highest explored energy is in beautiful agreement (fig. 16) with the naive parton prediction within the experimental accuracy (of about 15% due mainly to systematic errors) (see for example \cite{422}). This experiment provides a spectacular support for parton physics and also represents the most direct experimental evidence for the existence of three colours of quarks.

The QCD corrections to the naive parton limit turn out to be too small to be appreciated with the present experimental accuracy. They are obtained by computing the expansion in $\alpha$ of the total cross section for $e^+ e^- \rightarrow \gamma^* \rightarrow X$ where X is a partonic final state made up of quarks, antiquarks and gluons. The structure of the first few terms in the perturbative expansion for $R$ is found to be:
The important feature is the absence of the leading logarithmic series. This corresponds to the vanishing of the anomalous dimension term in the RGE and can be viewed as a consequence of the known theorems on the cancellation of mass singularities [279, 280, 306]. According to these theorems mass singularities are related to the presence of degenerate initial and/or final states in the massless limit (i.e. states with equal four momentum: typically the state of one massless parton and the state obtained from its splitting into two collinear massless partons). Mass singularities disappear if all degenerate initial or final states are added up. In leptoproduction after summing over all final states in the computation of the various total cross sections, one is still left with the mass singularities associated with the initial parton line. This explains the factorizability of these mass singularities which can be reabsorbed in the redefinition of the initial parton density (precisely this method of proving the factorization theorem was used in [27]). For $e^+e^-$ total cross section the initial state is the vacuum and consequently no mass singularities are left after adding up all possible final states. At order $\alpha$ this cancellation is obtained after adding up the cross section for $e^+e^- \rightarrow q\bar{q}$, including virtual corrections, and the cross section for $e^+e^- \rightarrow q\bar{q}G$ (fig. 17). The presence in higher orders of terms in $t$ like $t^2$ is connected with the definition of the coupling constant which itself introduces some mass singularities. In fact all these terms can be eliminated by setting $t = 0$, so that $\alpha(\mu^2)$ becomes $\alpha(Q^2)$. With this natural choice:

$$R[t, \alpha] = R(0) \left(1 + \alpha c^{(1)} + \alpha^2 c^{(2)} + \alpha^2 t c^{(2)} + \cdots\right).$$

(9-6)

$$R = R(0) \left[1 + c^{(1)} \alpha(Q^2) + c^{(2)} \alpha^2(Q^2) + \cdots\right].$$

(9-7)
The coefficient $c^{(1)}$ is known since a long time (in QED: [271] and in QCD: [35, 431]):

$$c^{(1)} = \frac{3}{4} C_F / \pi.$$  \hfill (9-8)

Recently the second order coefficient $c^{(2)}$ has been independently computed by three different collaborations [112, 148, 105]. In the MS definition of $\alpha$ the result is given by:

$$c^{(2)} = \frac{C_F}{(4\pi)^2} \left[ \frac{123}{2} C_A - \frac{3}{2} C_F - 11 f - 48 \pi \beta (3) b \right] \quad (\alpha = \overline{\text{MS}})$$ \hfill (9-9)

where $\beta (3) = 1.2021 \ldots$ and $b$ is given in eq. (3-54). Thus in QCD (recall eq. (9-5) for $R(0)$):

$$R = R(0) \left[ 1 + \frac{\alpha(Q^2)}{\pi} + (1.986 - 0.115 f) \left( \frac{\alpha(Q^2)}{\pi} \right)^2 + \cdots \right] \quad (\alpha = \overline{\text{MS}}).$$ \hfill (9-10)

As is seen, the known portion of the expansion appears as well behaved. The asymptotic limit is approached from above but, as already mentioned, the size of the expected corrections to the naive parton model prediction is below the present experimental accuracy.

Note that the MS prescription was defined in a quite abstract way, with no direct reference to any physical process. The fact that the expansions for $R$ and for the leptoproduction moments with reasonable $n$ are found to be well behaved in this prescription provides some physical justification for the present definition.

9.2. Scaling violations for fragmentation functions

We now consider the process of one hadron inclusive production in $e^+e^-$ annihilation, where $H$ is a specified hadron with four momentum $p$. This process is the twin in the timelike region of leptoproduction on the target $\bar{H}$: $\gamma^*(Q^2) + H \rightarrow X$ compared to $\gamma^*(Q^2) \rightarrow H + X$. For simplicity we shall leave aside all weak interaction effects which do not introduce essential modifications in the QCD aspects of interest to us here.

The center of mass of $\gamma^*$ is the natural frame of reference in $e^+e^-$ annihilation:

$$q = (Q; 0).$$ \hfill (9-11)

In this frame the energy of $H$ can be described as a fraction of the beam energy $Q/2$:

$$P = (zQ/2; p), \quad 0 \leq z \leq 1.$$ \hfill (9-12)

The invariant definition of $z$ is given by:

$$z = 2(Pq)/Q^2.$$ \hfill (9-13)

The structure functions $F_{1,2}^H(z, Q^2)$ can be introduced by formal analogy with leptoproduction (as in that case the label $H$ will often be dropped). A quantitative definition can be phrased in terms of the differential cross section in $z$ and $\cos \theta$, $\theta$ being the center of mass angle between the hadron and the
beam directions, after the azimuthal integration is performed:

\[ \frac{d\sigma}{dz \, d \cos \theta} = \frac{1}{2} \sigma_0 z \left[ 2 \bar{F}_t(z, Q^2) + \frac{z}{2} \sin^2 \theta \, \bar{F}_s(z, Q^2) \right] \]  

(9-14)

where \( \sigma_0 \) is defined in eq. (9-3).

Alternatively one can introduce transverse and longitudinal structure functions by:

\[ \bar{F}_t(z, Q^2) = 2 \bar{F}_1(z, Q^2) \]  

(9-15)

\[ \bar{F}_L(z, Q^2) = 2 \bar{F}_1(z, Q^2) + z \bar{F}_2(z, Q^2) . \]  

(9-16)

After integration over \( \theta \) the energy distribution is obtained in the form:

\[ \frac{d\sigma}{dz} = \sigma_0 z \left[ 3 \bar{F}_1(z, Q^2) + \frac{z}{2} \bar{F}_2(z, Q^2) \right] = \sigma_0 z \left[ \bar{F}_t(z, Q^2) + \frac{1}{2} \bar{F}_L(z, Q^2) \right] . \]  

(9-17)

As in all inclusive processes the general relations hold:

\[ \int d^3 p \, \frac{d\sigma^H}{d^3 p} = \langle n^H \rangle \sigma_{\text{TOT}} \]  

(9-18)

\[ \sum_H \int d^3 p \, I_3^H \frac{d\sigma^H}{d^3 p} = I_3 \sigma_{\text{TOT}} \]  

(9-19)

\[ \sum_H \int d^3 p \, P^\mu \frac{d\sigma^H}{d^3 p} = q^\mu \sigma_{\text{TOT}} \]  

(9-20)

where \( I_3^H \) and \( I_3 \) are the third components of isospin for \( H \) and the initial state respectively (\( I_3 = 0 \) in \( ee^- \)), \( \langle n^H \rangle \) is the average multiplicity of \( H \), and \( q^\mu \) is the total incoming four momentum.

In the naive parton model with spin \( 1/2 \) quarks one has:

\[ \bar{F}_L^H(z) = 0 \]  

(9-21)

\[ z \bar{F}_t^H(z) = 3 \sum_{i=1}^f e_i^2 [D_{0q_i}^H(z) + D_{0\bar{q}_i}^H(z)] . \]  

(9-22)

The factor of 3 is from colour. In this limit the angular distribution in eq. (9-14) reduces to \( 1 + \cos^2 \theta \), an experimentally well established signature for the approximate validity of the parton dynamics at high energies. \( D_{0q_i}^H(z) \) is the fragmentation (or decay) function, namely the number density of \( H \) in the jet of the parton \( p \). This identification is confirmed by eqs. (9-18 to 9-22), which within the accuracy of the naive parton model, lead to:
\[ \sum_{H} \int dz I_{3}^{H} D_{0p}^{H}(z) = I_{3}^{p} \]  
\[ (9-23) \]

\[ \sum_{H} \int dz z D_{0p}^{H}(z) = 1 \]  
\[ (9-24) \]

for each parton \( p \), with \( I_{3}^{p} \) being the isospin component of \( p \). Eq. (9-23) is a particular example of a non singlet charge conservation sum rule. Eq. (9-24) is the momentum conservation sum rule in the jet of parton \( p \).

The parton model description of one hadron inclusive \( e^{+}e^{-} \) annihilation is sketched in fig. 18. The virtual photon \( \gamma^{*} \) produces a hard parton \( p \) with four momentum \( k \), its energy being a fraction \( y \) of the beam energy:

\[ k = (yQ/2; k) \]  
\[ (9-25) \]

Then independently of the other partons, the produced parton \( p \) fragments into hadrons. In fact no hard interaction can take place between two produced partons because their separation in rapidity becomes too wide at large enough energy. For the same reason one can add the probabilities rather than the amplitudes of finding \( H \) in the jets of two different partons. The neglected interferences vanish as powers when \( Q^{2} \to \infty \). The necessary reshuffling of colour and electric charge is attributed to wee parton exchanges in the central region, common to several parton jets, and is assumed to preserve the asymptotic structure implied by the hard interactions.

In the limit of massless partons and negligible intrinsic transverse momentum of the fragments the relation between the four momentum of \( H \) and that of the parent parton is given by \( P = z/y k \) (see eqs. (9-12, 9-25)). The cross section for producing \( H \) with fraction \( z \) of the beam energy is obtained as a convolution of the cross section for producing a parton with energy fraction \( y \) times the density of hadrons \( H \) in the parton \( p \) with fraction \( z/y \) the parton momentum:

\[ z\hat{\sigma}_{a}^{H}(z, Q^{2}) = \int \frac{dy}{y} \sum_{i} \sigma_{a}^{\gamma^{*}\rightarrow p}(y, Q^{2}) D_{0p}^{H}(\frac{z}{y}) \]  
\[ (9-26) \]

where \( \sigma_{a}^{\gamma^{*}\rightarrow p} \) is a rescaled (adimensional) cross section for production of \( p \) and, in analogy with eq. (5-1), we have set:

\[ \hat{\sigma}_{a}^{H} = (2\hat{F}_{1}^{H} - z\hat{F}_{2}^{H}) \]  
\[ (9-27) \]

Fig. 18. \( e^{+}e^{-} \to \gamma^{*} \to H + \) all in the parton model.
In the naive version of the parton model the additional assumption of identifying $\sigma_{u}^{*-p}$ with the point-like cross sections is made:

$$\sigma_{\text{free}}^{*-u} = 3e_{i}^{2} \delta(1-z).$$  \hfill (9-28)

As was the case in leptoproduction, the above parton description is kept in QCD but the cross section for parton production cannot be restricted to the free field value and is to be computed to all orders in $\alpha$.

It is instructive to consider the cross section to order $\alpha$ in detail, as computed from the diagrams in fig. 17. To this order we have to consider the one parton inclusive cross sections for $q$ and $\bar{q}$ production and also for $G$ production:

$$\sigma_{u}^{*-u} = \sigma_{u}^{*-q} = \sigma_{\text{free}}^{*-u} + \sigma_{aa}^{*-u} + \cdots$$  \hfill (9-29)

$$\sigma_{aa}^{*-G} = \sigma_{aa}^{*-G} + \cdots$$  \hfill (9-30)

where by $\sigma_{aa}$ we denote the terms of order $\alpha$ in the corresponding cross section. It is easy to realize that the cancellation of mass singularities valid for the total cross section is no more true for $\sigma_{aa}^{*-q,G}$. This is because the sum over degenerate final states is no more complete in this case. For example the cross section for production of a quark with fraction $y$ does not include the collinear state of a quark with fraction $y(1-\eta)$ and a gluon with fraction $y\eta$.

Thus we expect the logarithmic terms to reappear in $\sigma_{aa}^{*-q,G}$:

$$\sigma_{aa}^{*-q} = 3e_{i}^{2} \left[ \delta(1-z) + \frac{\alpha}{2\pi} \left( tP_{qq}(z) + \tilde{f}_{q}^{a}(z) \right) \right] + \cdots$$  \hfill (9-31)

$$\sigma_{aa}^{*-G} = 3e_{i}^{2} \frac{\alpha}{2\pi} \left[ 2tP_{Gq}(z) + \tilde{f}_{G}^{a}(z) \right] + \cdots$$  \hfill (9-32)

In the previous equations we anticipated the result that the coefficients of the leading logarithmic terms are the same splitting functions as in leptoproduction. This is a simple consequence of crossing and can be checked by the following simple calculation.

Consider the cross sections $\sigma_{aa}^{*-G}$ and $\sigma_{aa}^{*-q}$, the last one at $y < 1$, so that the only diagrams to contribute are those in fig. 17b with real gluon emission. We denote by $x_{p}$ the fraction of beam energy carried by the parton $p$ in the center of mass frame:

$$E_{p} = x_{p}Q/2 \quad (p = q, \bar{q}, G).$$  \hfill (9-33)

For a final state of three partons it is obviously true that

$$x_{u} + x_{q} + x_{G} = 2.$$  \hfill (9-34)

We also note the relations:

$$s = (q - k_{q})^{2} = Q^{2}(1-x_{q})$$  \hfill (9-35)

$$t = (q - k_{q})^{2} = Q^{2}(1-x_{q})$$  \hfill (9-36)
\[ u = (q - k_G)^2 = Q^2(1 - x_G). \] (9-37)

An easy calculation leads to the following well known formula for the total cross section of the parton process \( e^+e^- \rightarrow q\bar{q}G \) at order \( \alpha \) [165]:

\[ \sigma = \sigma_{\text{TOT}} C_F \frac{\alpha}{2\pi} \left[ \frac{s + t + 2uQ^2}{st} \right] \] (9-38)

where \( \sigma_{\text{TOT}} \) is given by eq. (9-4) with \( \alpha = 0 \). By eqs. (9-35 to 9-37) this cross section can be written in the equivalent forms

\[
\frac{1}{\sigma_{\text{TOT}}} \frac{d\sigma}{dx_q dx_{\bar{q}}} = C_F \frac{\alpha}{2\pi} \frac{x_q^2 + x_{\bar{q}}^2}{(1 - x_q)(1 - x_{\bar{q}})}
\] (9-39)

\[
\frac{1}{\sigma_{\text{TOT}}} \frac{d\sigma}{dx_q dx_G} = C_F \frac{\alpha}{2\pi} \frac{x_q^2 + (2 - x_q - x_G)^2}{(1 - x_q)(x_q + x_G - 1)}.
\] (9-40)

This last equation also applies with \( q \leftrightarrow \bar{q} \). The soft infrared singularities at \( x_q, x_{\bar{q}} = 1 \) make the integrated \( q\bar{q}G \) partial cross section not defined by itself. This singularities would be cancelled, as for example in the total cross section, by adding the contribution of order \( \alpha \) to the \( q\bar{q} \) cross section from the interference with the point-like term of the virtual gluon exchange diagrams. As in the case of leptonproduction for a rigorous approach one should introduce a regularization, add both real and virtual contributions to the moments of the structure functions, observe the cancellation of the infrared regulator and read out the coefficients of \( t \) (see for example [17]). However, in this case also, for a determination of the coefficients of \( t \) it is sufficient to simply look at the residues of the soft singularities, as is fully confirmed by the complete calculation.

In order to obtain \( \sigma_{\alpha}^{x \rightarrow q} \) at \( x_q < 1 \) from eq. (9-39) one must integrate over all possible values of \( x_q \). By observing that:

\[
\int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 2) f(x_1, x_2) = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 f(x_1, x_2)
\] (9-41)

one formally obtains (the logarithmic terms are independent of the index \( a \)):

\[
\sigma_{\alpha}^{x \rightarrow q}(x_q) = C_F \frac{\alpha}{2\pi} \int_{1-x_q}^1 dx_q \frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})} = C_F \frac{\alpha}{2\pi} \frac{1 + x_q^2}{1 - x_q} t + \cdots
\]

\[
= \frac{\alpha}{2\pi} P_{qq}(x_q)t + \cdots \quad (x_q < 1)
\] (9-42)

where the logarithmic divergence of the integral at \( x_q = 1 \) has been reinterpreted as a factor of \( t \). The \( \delta \) function term at \( x_q = 1 \) can be reconstructed either by the full computation or by imposing the asymptotic validity of the isospin conservation sum rule in eq. (9-23).
Similarly for $\sigma_\alpha^{\gamma^* \rightarrow G} $ one obtains:

$$
\sigma_\alpha^{\gamma^* \rightarrow G}(x_G) = C_F \frac{\alpha}{2\pi} \int_{1-x_G}^{1} dx_q \frac{x_q^2 + (2 - x_q - x_G)^2}{(1-x_G)(x_q + x_G - 1)}
$$

$$
= 2C_F \frac{\alpha}{2\pi} \frac{1 + (1-x_G)^2}{x_G} t + \cdots = 2C_F \frac{\alpha}{2\pi} P_{\alpha q}(x_G) t + \cdots
$$

(9-43)

The factor of 2 arises in this case from the separate and symmetric contributions from the poles at $x_q = 1$ and $x_q = 1 - x_G$ (the gluon can be emitted by the quark or by the antiquark).

Thus, to order $\alpha$, we obtained the result (recall eq. (9-26)):

$$
z \tilde{S}_\alpha(z, Q^2) = 3e^2 \int_{1-x_G}^{1} \frac{dy}{y} \left\{ \left[ \delta(1-y) + \frac{\alpha}{2\pi} (tP_{\alpha q}(y) + \tilde{f}_G^a(y)) \right] \left[ D_{\alpha q}(\frac{z}{y}) + D_{\alpha q}(\frac{z}{y}) \right] 
+ \frac{\alpha}{2\pi} \left[ t 2P_{\alpha q}(y) + \tilde{f}_G^a(y) \right] D_{\alpha q}(\frac{z}{y}) \right\} + \cdots
$$

(9-44)

where the sum over flavours was omitted for simplicity. These expressions show a complete analogy with the case of leptoproduction and this parallelism is kept in higher orders as well. As was done in that case for quark densities, one introduces here effective $Q^2$ dependent fragmentation functions by multiplication by a factor, extracted from the cross sections, and including the leading logarithmic series of terms in all orders and the related mass singularities. In the LLA the naive parton model formulae are recovered in terms of these scale dependent decay functions when terms down by powers of the running coupling are neglected:

$$
z \tilde{S}_\alpha(z, Q^2) = 3e^2 \left[ D_q(z, Q^2) + D_q(z, Q^2) \right] + o(\alpha(Q^2)).
$$

(9-45)

The fragmentation functions obey the evolution equations [221, 402, 345]:

$$
\frac{d}{dt} D_q(z, t) = \frac{\alpha(t)}{2\pi} \left[ P_{qq} \otimes D_q + P_{\alpha q} \otimes D_G \right] + o(\alpha^2(t))
$$

(9-46)

$$
\frac{d}{dt} D_G(z, t) = \frac{\alpha(t)}{2\pi} \left[ P_{qG} \otimes \sum_{i=1}^{l} (D_q + D_q) + P_{GG} \otimes D_G \right] + o(\alpha^2(t))
$$

(9-47)

where the equation for $D_G$ can be derived along similar lines as for $D_q$.

The only difference with respect to the evolution equations for quark densities is in the transposition $P_{qG} \leftrightarrow P_{\alpha q}$ (fig. 19). The evolution equations for fragmentation functions are also conveniently handled by separating the singlet and non singlet sectors:

$$
\frac{d}{dt} D_{NS} = \frac{\alpha(t)}{2\pi} D_{NS} \otimes P_{qq} + o(\alpha^2(t))
$$

(9-48)
Fig. 19. Schematic illustration of the evolution equation for the quark fragmentation function in the LLA.

\[
\frac{d}{dt} D_S = \frac{\alpha(t)}{2\pi} [D_S \otimes P_{qq} + D_G \otimes 2fP_{Gq}] + o(\alpha^2(t)) \tag{9-49}
\]

\[
\frac{d}{dt} D_G = \frac{\alpha(t)}{2\pi} [D_S \otimes P_{qG} + D_G \otimes P_{GG}] + o(\alpha^2(t)) \tag{9-50}
\]

where \( D_{NS} \) stands for any difference of the form \( D_{q_i} - D_{q_j} \) and we set:

\[
D_S = \sum_{i=1}^f (D_{q_i} + D_{\bar{q}_i}). \tag{9-51}
\]

Note that in the singlet sector the kernel matrices for quark densities and for fragmentation functions are related as follows:

\[
\begin{pmatrix}
  P_{qq} & 2fP_{qG} \\
  P_{Gq} & P_{GG}
\end{pmatrix}_{\text{spacelike}} \leftrightarrow \begin{pmatrix}
  P_{qq} & 2fP_{Gq} \\
  P_{qG} & P_{GG}
\end{pmatrix}_{\text{timelike}} \tag{9-52}
\]

The eigenvalues of the anomalous dimension matrix are clearly the same.

The non singlet charge sum rules as in eq. (9-23) are protected in the LLA from scaling violations by the vanishing of the first moment of \( P_{qq} \). The momentum sum rules eqs. (9-24) are also compatible with the evolution equations in the LLA, in the sense that if valid for each parton \( p \) at a given \( t \), their validity at \( t + dt \) is guaranteed by eqs. (4-47, 4-48), i.e. the same equations that express momentum conservation at the basic QCD vertices.

It is interesting to recall a formal correspondence between leptoproduction and annihilation, called the analytic continuation rule ([238, 310, 149], see also [154]). This is based on properties of diagrams under crossing. The world “formal” means that, first it is a property of tree diagrams not necessarily respected by the regularization method and the renormalization prescription. Second, it is of no direct physical significance because it cannot be extended to non perturbative quantities like quark densities and decay functions. The analytic continuation rule is a statement about the parton-current cross sections in the spacelike and timelike regions which, when all colour factors are removed, can be stated as:

\[
\eta_a \sigma_a (1/x, t) = \tilde{\sigma}_a (x, t) \tag{9-53}
\]

where \( \sigma_a \) and \( \tilde{\sigma}_a \) are the cross sections in leptoproduction and annihilation respectively and \( \eta_a \) is a phase. In particular for the leading logarithmic terms one has, when colour factors are removed:

\[
[\eta_{AB} x P_{AB}(1/x)]_{\text{spacelike}} = [P_{AB}(x)]_{\text{timelike}}; \quad \eta_{AB} = (-1)^{2s_A + 2s_B + 1} \tag{9-54}
\]
where \( s_A, s_B \) are the parton spins. For example \( P_{qq} \) is invariant:

\[
P_{qq} \sim \frac{1 + x^2}{1 - x} \rightarrow x \left[ \frac{1 + (1/x)^2}{1 - 1/x} \right] = \frac{1 + x^2}{1 - x}.
\]

(9-55)

On the other hand:

\[
P_{qG} \sim x^2 + (1 - x)^2 \rightarrow x \left[ \left( \frac{1}{x} \right)^2 + \left( 1 - \frac{1}{x} \right)^2 \right] = \frac{1 + (1 - x)^2}{x} \sim P_{Gq}
\]

(9-56)

and so on. The identity of the diagonal kernels and the interchange of the off diagonal ones in the evolution equations for the spacelike and timelike regions is thus reproduced by this rule. In the following we shall see that the analytic continuation rule for kernels also holds beyond the leading order with appropriate definitions and modifications.

For parton densities the theory of scaling violations is best proven by light cone operator expansion and RGE for the operator coefficients. Even in this case some assumptions have to be made on the non perturbative domain. Effects of confinement, as hadronization or colour and charge reshuffling are assumed not to upset the hierarchy of asymptotic terms implied by different light cone singularities. Without knowing the dynamics of confinement the results on the \( Q^2 \) behaviour of structure functions can be completely proven only for a world of quarks and gluons. On the other hand, we have seen that, even in a world of unconfined partons, the absolute values of the structure functions at a given \( Q^2 \) cannot be computed in perturbation theory as these quantities depend on \( \alpha(k^2) \) with \( k^2 \) in the whole domain \( k^2 \leq Q^2 \). It is however natural to extend to hadrons the results on scaling violations by trading the non computable parton densities in a parton with the even less computable parton densities in a hadron. For annihilation and fragmentation functions the situation is quite similar. In a world of quarks and gluons one can reproduce, by a refined diagrammatic analysis, all results on the \( Q^2 \) dependence of the decay functions for a parton in a parton jet. This program of extending the factorization theorem in the timelike region has been developed by many authors (Dokshitzer [149]; Amati, Petronzio, Veneziano [27]; Efremov, Radyushkin [160]; Libby, Sterman [309]; Stirling [392]; Ellis, Georgi, Machacek, Politzer, Ross [171]; Konishi, Ukawa, Veneziano [292]). The most complete parallelism with the operator approach in the spacelike region of \( Q^2 \) has been accomplished with the cut vertex formalism in the timelike region developed by Mueller [330] (see also Gupta and Mueller [247]). The cut vertices in the timelike region are in one to one correspondence with the operators in the light cone expansion. To each cut vertex an “anomalous dimension” function \( \bar{\gamma}(\alpha) \), the same for all processes, can be associated. A given process is related to the cut vertices by process dependent coefficient functions \( \bar{c}(t, \alpha) \). The moments of one hadron inclusive structure functions are shown to have formally identical expressions in terms of coefficient functions and anomalous dimensions of cut vertices as moments of structure functions in leptoproduction. For example for the non singlet structure functions in annihilation one has:

\[
\bar{M}_n^a(t) = \tilde{C}_n^a[\alpha(t)] \exp \left[ \int_{\alpha}^{\alpha(t)} \frac{\bar{\gamma}_n(\alpha)}{\beta(\alpha)} \, d\alpha \right] \bar{M}_0^a.
\]

(9-57)

The validity of this formalism for the actual structure functions of one hadron inclusive production in \( e^+e^- \) annihilation depends to some extent on the non perturbative dynamics. But, as mentioned above, this is the case also in leptoproduction and the theoretical status of the QCD improved parton model is by now quite comparable in the spacelike and timelike regions.
The calculation of the first non leading corrections to the structure functions in the timelike region and to their evolution equations has been recently completed.

Independent of all definitions of decay functions beyond the LLA the expression of the longitudinal structure function, defined in eq. (9-16), is given by (for $N = 3$):

$$z F_L(z, t) = \frac{3\alpha(t)}{2\pi} \frac{8}{3} \int \frac{dy}{y} \frac{\sum_{i=1}^{F}(\frac{z}{y}, t) + D_\eta(\frac{z}{y}, t)}{2 \left( \sum_{i=1}^{F} e_i^2 \right) D_\sigma(\frac{z}{y}, t) \frac{2(1-y)}{y}}. \tag{9-58}$$

By recalling eqs. (5-15, 5-16) for $F_L$ in lepton production we see that the kernels which appear in the last equation for $F_L$ are related by the analytic continuation rule, eq. (9-53), to their analogue in the spacelike region. The factor of $8/3$ in the gluon kernel is due to crossing of colour and spin factors.

It is interesting to observe that from eqs. (9-58, 9-24) it follows that:

$$\sum_i \int_0^1 dz \left( \frac{\alpha(t)}{\pi} + \ldots \right) \frac{\sigma_0}{\sum_{i=1}^{F}(\frac{z}{y}, t)} = 3u_0 \left( \frac{\alpha(t)}{\pi} + \ldots \right) \tag{9-59}$$

On the other hand the general eq. (9-20) can be written in the present case as (recall eq. (9-4)):

$$\sum_i \int_0^1 dz \left[ \frac{\alpha(t)}{\pi} + \ldots \right] \frac{\sigma_0}{\sum_{i=1}^{F}(\frac{z}{y}, t)} \left( 1 + \frac{\alpha(t)}{\pi} + \ldots \right). \tag{9-60}$$

Thus, at order $\alpha(t)$, the whole correction to $R$ in eq. (9-8), is seen to arise from the longitudinal part. This means that if one defines quark fragmentation functions beyond the leading accuracy through $F_T$, i.e. by enforcing eq. (9-22) with no corrections of order $\alpha(t)$, then the momentum sum rule eq. (9-24) (for $p = q, \bar{q}$) is automatically satisfied with no corrections of order $\alpha(t)$. On the other hand this definition of quark fragmentation functions leads to computable corrections of order $\alpha(t)$ to the charge sum rule eq. (9-23). Both the momentum and the charge sum rules are instead preserved to all orders if the quark decay functions are defined by:

$$\frac{d\sigma^H_T}{dz} = \frac{\sigma_T \sigma_T[\alpha(t)]}{R(0)} \frac{3}{\sum_{i=1}^{F}(\frac{z}{y}, t) + D_\eta(\frac{z}{y}, t)} \tag{9-61}$$

with $\sigma_T \sigma_T[\alpha(t)]$ given by eqs. (9-4, 9-7). We prefer the definition of eq. (9-22) which emphasizes that a fraction of order $\alpha(t)$ of events is due to three jets. Moreover $F_T$ corresponds to an angular distribution $(1 + \cos^2 \theta)$ which is in fact appropriate for $qq$ production. This choice is however only of aesthetical importance. Physical predictions are independent of such conventions. These definitions are relevant for the calculation of non leading corrections to more complicated processes involving fragmentation functions (lepton production with one hadron detected in the final state, $e^+e^-$ with two hadrons detected in different jets and so on [375, 17, 42]).

The non leading corrections to the evolution equations have been computed both for the non singlet and singlet sectors [135, 191, 192, 209, 274, 275, 333a]. The relevant kernels are quantitatively different than in the spacelike case, although the two sets of kernels can be related by a suitably modified version of the analytic continuation rule. We consider in the following the non singlet case.
Consider for example the non singlet structure functions \((Z\bar{Z})_{K^+K^0}\) where \(\bar{\mathcal{F}}_a\) are defined in eq. (9-27). These are crossing even structure functions, as always the case in annihilation via one photon. The corresponding evolution equations can be written as:

\[
\frac{d}{dt} (z\bar{\mathcal{F}}_a)_{K^+K^0} = (z\bar{\mathcal{F}}_a)_{K^+K^0} \otimes \bar{\mathcal{P}}_a .
\]

One can relate \(\bar{\mathcal{P}}_a\) to \(\mathcal{P}_a\), the kernel which governs the corresponding evolution equation for \(\mathcal{F}e^p-e^N\) in leptoproduction (for \(a = 2\), \(\mathcal{P}_a\) is precisely \(Q_{qq} + Q_{qq}\) as in eq. (5-37)). To this purpose it is convenient to split \(\mathcal{P}_a\) and \(\bar{\mathcal{P}}_a\) into a regular part and a \(\delta\) function term at \(x = 1\):

\[
\bar{\mathcal{P}}_a = \mathcal{P}_a^{\text{reg}} + \mathcal{P}_a^{\text{sing}} \delta(1-x) .
\]

Then one finds that the regular parts satisfy the analytic continuation rule:

\[
-x\mathcal{P}_a^{\text{reg}} \left( \frac{1}{x}, \alpha \right) = \bar{\mathcal{P}}_a^{\text{reg}}(x, \alpha) .
\]

For the sake of this continuation all \(\log(1-x)\) in eqs. (5-40, 5-41) are to be taken as \(\log|1-x|\). Instead for the singular terms one finds:

\[
\mathcal{P}_a^{\text{sing}}(\alpha) = \mathcal{P}_a^{\text{sing}}(\alpha) - b \frac{\alpha^2(t)}{2\pi} C_F \pi^2 .
\]

The meaning of this last term is simple. The analytic continuation rule holds provided it is extended to both the variables \(Q^2\) and \(x\): the change from negative to positive \(q^2\) is in fact at the origin of the additional term in eq. (9-65). As this issue is of importance also for Drell–Yan processes (section 11) we consider it in detail.

We recall that the moments of \(\mathcal{P}_a\) correspond to \(\alpha^2[\gamma^{(a)}_n - bC^{(1)a}_n]\) (see eqs. (5-19, 5-21)). The additional term proportional to \(\pi^2\) in eq. (9-65) arises from the reduced coefficient \(C^{(1)a}_n\), and in fact is proportional to \(b\). Precisely it is obtained from the virtual diagram contribution in eq. (7-1) (which is only present at \(x = 1\) in the structure functions). This contribution is formally identical both in the spacelike and the timelike regions when expressed in terms of \(q^2\) as in eq. (7-1). However in terms of \(Q^2 = |q^2|\), we have \(-Q^2 = q^2\) for \(\mathcal{F}_a\) and \(Q^2 = q^2\) for \(\bar{\mathcal{F}}_a\). Because of the double pole \(1/e^2\), the real part of the relative factor \((-1)^e = 1 + i\pi\varepsilon - \frac{1}{3}\pi^2\epsilon^2 + \cdots\) leads to a difference between \(\mathcal{F}_a\) and \(\bar{\mathcal{F}}_a\) given by:

\[
(\bar{\mathcal{F}}_a - \mathcal{F}_a)_{\text{virtual}} = \frac{\alpha}{2\pi} C_F \pi^2 \delta(1-x)
\]

which exactly corresponds to eq. (9-65). We see that the \(\pi^2\) term is associated with \(\log^2 q^2\) in the vertex correction. The leading term \(\log^2 Q^2\) is cancelled by the real diagrams (recall eq. (7-3) and the related discussion) but the effect of the change of sign of \(q^2\) is felt by the non leading terms.

We recall from eq. (9-55) that in leading order \(P_{qq}\) is form invariant under the analytic continuation rule and this fact implies the equality in the LLA of non singlet scaling violations in the spacelike and timelike regions. On the other hand in non leading accuracy, although the analytic continuation rule still holds for the regular parts (eq. (9-64)), however \(\mathcal{P}_a\) are no more form invariant. For example by
recalling the definition of $e_a^n$ in eq. (3-66) for moments of $\mathcal{F}_a$ and $z\mathcal{F}_a$ one obtains \[135\]:

$$
\left(\bar{e}_a^n\right)^{K'^0-K^0} - \left(\bar{e}_a^n\right)^{P^+P^-N^0} = \frac{1}{2} \left[ 2S_a(n) - \frac{\pi^2}{3} - \frac{2n+1}{n^2(n+1)^2} \right] \left[ 3C_F - \frac{C_F^2}{\pi b} \left( -2S_a(n) + \frac{3}{2} + \frac{1}{n(n+1)} \right) \right] - \frac{1}{4} C_F \left( \frac{2n+1}{n(n+1)} - 2\pi^2 \right)
\] (9.67)

where $S_a(n)$ was defined in eq. (5-50). By a numerical evaluation one realizes that this difference is rather large so that the next to the leading corrections to moments of structure functions in the timelike region are substantially more conspicuous (for $n > 1$) than in the spacelike region.

There is one interesting point which is worth mentioning about the singlet sector. In leading order $P_{GG}$ and $P_{Gq}$ both show a simple pole near $x = 0$ which corresponds to a pole near $n = 1$ in the moments. For example $\bar{\gamma}_n^{GG}(\alpha) = (\alpha/2\pi)A_n^{GG} \sim \alpha C_A/\pi(n-1)$ (see table 2). From the explicit calculations in two loops the leading triple pole at $n = 1$ in the timelike region was found to be consistent with the branch point structure for $\bar{\gamma}_n^{GG}(\alpha)$ given by:

$$
\bar{\gamma}_n^{GG}(\alpha) = \frac{1}{2} \left[ - (n-1) + \sqrt{(n-1)^2 + 8\alpha C_A/\pi} \right].
\] (9.68)

Recently this conjecture was further supported by a three loop calculation of the leading term near $n = 1$ [333]. The removal of the pole singularity makes it possible to evaluate the $Q^2$ behaviour of the average multiplicity $\bar{n}(Q^2)$ of gluons, which is proportional to the moment with $n = 1$ of the corresponding fragmentation function. For a gluon jet eq. (9.68) leads to:

$$
\bar{n}(Q^2) \propto \exp \int_0^{\alpha(Q^2)} \frac{\bar{\gamma}_n^{GG}(\alpha)}{\beta(\alpha)} = \exp \sqrt{\frac{2C_A}{\pi b} \ln \frac{Q^2}{\Lambda^2}}.
\] (9.69)

This remarkable result was first obtained, with a slight difference in the numerical factor at exponent, by Bassetto, Ciafaloni, Marchesini [60], Amati, Bassetto, Ciafaloni, Marchesini, Veneziano [26]; see also Furmanski, Petronzio, Pokorski [212].

The gluon component of singlet fragmentation functions is expected to be dominant near $x = 0$. Furthermore since all $Q^2$ dependence arises from the partonic cascade in the jet the prediction in eq. (9.69) should presumably be relevant for hadrons as well. This matter is currently under study and it may be that more will be learnt on this issue soon.

9.3. Annihilation into real photons

If the hadron $H$ is replaced by a real photon one is led to consider $D_q^\gamma$ and $D_G^\gamma$: the fragmentation functions of quarks and gluons into a real photon. The theory of the photon structure functions described in section 6 can be applied to this case almost verbatim. We only quote the obvious modifications [312, 339].

The evolution equation for $D_q^\gamma$ in the LLA can be written as:

$$
Q^2 \frac{d}{dQ^2} D_q^\gamma = \frac{\alpha(Q^2)}{2\pi} \left[ D_q^\gamma \otimes P_{q\gamma} + D_G^\gamma \otimes P_{q\gamma} \right] - \frac{\alpha_{em}(Q^2)}{2\pi} e_q^2 D_q^\gamma \otimes P_{\gamma q}
\] (9.70)
while the corresponding equation for \( D_\gamma \) is unaltered and coincides with eq. (9-47). In eq. (9-70) we can replace \( D_{\gamma} \), the number density of photons in a photon, by \( \delta(1-x) \) as we are working in lowest order in \( \alpha_{em} \). \( P_{\gamma q} \) is simply obtained from \( P_{Gq} \) by removing the colour factor \( C_F \): 

\[
P_{\gamma q}(x) = \frac{1 + (1-x)^2}{x^2}.
\] 

(9-71)

By the same procedure and with the same notations as in section 6 the asymptotic results in the LLA for \( (D_{\gamma q})_n \) and \( (D_{\gamma})_n \) (i.e. the \( n \)th moments of \( D_{\gamma q} \) and \( D_\gamma \)) are found to be (in analogy with eqs. (6-15, 6-16)):

\[
(D_{\gamma q})_n = \frac{\alpha_{em} d_{\gamma q} n}{\alpha(Q^2)} \left[ \langle e_i^2 \rangle - \frac{(e^q_1 + \langle e^q_\cdots e^q_n \rangle)}{1 - d_n^{Gq}} \right].
\]

(9-72)

\[
(D_{\gamma})_n = \frac{\alpha_{em} d_{\gamma q} n^{Gq} d_n^{Gq}}{\alpha(Q^2)(1 - d_n^q)(1 - d_n^q)}.
\]

(9-73)

(Recall from table 2 that a factor of 2\( f \) is included in \( d_n^{Gq} \).) In the same approximation the structure functions are given by:

\[
z\bar{F}_{\gamma}(z, Q^2) = 3 \sum_{n=1}^{L} e_n^2 [D_{\gamma q}(z, Q^2) + D_{\gamma}(z, Q^2)].
\]

(9-74)

Similarly for \( \bar{F}_\gamma \) one only needs to go back to eq. (9-58) and just add the pointlike photon contribution which is obtained by a simple rescaling of the gluon term. With \( N = 3 \) we have:

\[
z\bar{F}_\gamma(z) = \frac{\alpha_{em}}{2\pi} 24 f(e^q_1) \frac{1-z}{z} + \frac{3 \alpha(Q^2)}{2\pi} \frac{1}{3} \int \frac{dy}{y} \left\{ \sum_{n=1}^{L} e_n^2 \left[ D_{\gamma q}(\frac{z}{y}, Q^2) + D_{\gamma}(\frac{z}{y}, Q^2) \right] + 2 f(e^q_1) D_{\gamma}(\frac{z}{y}, Q^2) \right\}.
\]

(9-75)

Note that \( \bar{F}_\gamma \) is independent of \( Q^2 \) in the LLA because the factor of \( \alpha(Q^2) \) which we see in eq. (9-73) is compensated by its inverse which is included in the fragmentation functions as given by eqs. (9-72, 9-73).

9.4. Jets

In this section we study those bulk properties of the final state which do not depend on the identification of particular hadronic channels. Experiments on \( e^+e^- \) annihilation at high energy offer the best opportunity of systematically testing the distinct signatures predicted by QCD for the structure of the final state, averaged over a large number of events. However many aspects of this analysis are not specific of \( e^+e^- \) annihilation and can be extended to other processes as well, as already seen in section 8 for leptoproduction.
Typical of asymptotic freedom is the hierarchy of configurations which emerges from the smallness of \( \alpha(Q^2) \) at high energy. When all corrections of order \( \alpha(Q^2) \) are neglected one recovers the naive parton model prediction for the final state: almost collinear events with two back to back jets with limited intrinsic transverse momentum and an angular distribution as \((1 + \cos^2 \theta)\) with respect to the beam direction. At order \( \alpha(Q^2) \) a tail of events with large transverse momentum, \( k_\perp \sim Q \), is generated. Three jet events appear to this order. The skeleton of a three jet event is a three hard parton event, the third parton being a gluon emitted by a q or \( \bar{q} \) line. Events are therefore approximately planar. The average transverse momentum in the event plane \( \langle k_\perp \rangle_{\text{in}} \) is predicted to increase linearly with \( Q \) (apart from logarithms), while \( \langle k_\perp \rangle_{\text{out}} \) is still fixed in this approximation. Similarly the most energetic jet, called the slim jet, should look as a jet of a two jet event (at scaled down energy) and correspondingly \( \langle k_\perp \rangle_{\text{slim}} \) is fixed, while \( \langle k_\perp \rangle_{\text{int}} \) increases with \( Q \). At order \( \alpha^2(Q^2) \) a hard perturbative non planar component starts to build up and some fraction of four jet events is predicted to show up: both \( \langle k_\perp \rangle_{\text{out}} \) and \( \langle k_\perp \rangle_{\text{slim}} \) start increasing.

The topological signatures just described in a qualitative way are quite well supported by the available data (see for example [422] and figs. 20, 21). All together these experiments provide the best evidence for QCD in addition and complementary to the measurements on scaling violations in leptoproduction, on \( R \) in \( e^+e^- \) and on Drell—Yan processes.

In addition to these topological signatures other quantitative tests of more specific aspects of QCD dynamics are possible. The aim is to distinguish the gluon jet from the quark jets, to establish the spin-one nature of the gluon, to compare the computed matrix element with experiment, to measure \( \alpha(Q^2) \) and so on. These tasks are currently being actively pursued and a number of important results have been already obtained. This part of the program is complicated not only by problems of uncertainties and model dependence in the analysis, associated with the non perturbative effects of hadronization, but also by theoretical problems connected with the discovery, by explicit computations, of large non leading terms in the perturbative expansions of some of the quantities of interest.

Fig. 20. Distributions of the mean transverse momentum squared for event for charged particles (in the event plane) at c.m energies of 12, 27.4—31.6, 33—36.6 GeV. The development with energy of the high \( p_T^2 \) tail is well described by QCD (plus a model of hadronization).

Fig. 21. Energy flow diagram for events with \( T < 0.8 \) and \( Q_\perp > 0.1 \) compared with QCD (plus a model of hadronization), a \( q\bar{q} \) model with average intrinsic \( p_\perp \) of 500 MeV and a mixed phase space and \( q\bar{q} \) model.
It is clear that precise definitions must be given of the intuitive terms used in the introduction, like jet, \(n\)-jet event, line of the event, plane of the event and so on. It is also important to identify those quantities which are best suited for both a perturbative calculation and a comparison with experiment. On this point it is crucial to observe that, since we aim at an inclusive description without fragmentation functions or the like, there is no place where mass singularities can be reabsorbed. Thus a necessary condition for such an inclusive quantity to be computable in perturbation theory is that it should be infrared safe, i.e. free of soft and collinear mass singularities in the massless theory. In fact only those quantities which do not depend on the details of some infrared cut off mechanism can possibly be asymptotically insensitive to the long wavelength part of the problem related to the non perturbative physics of hadronization and confinement (see for example refs. [388, 217, 349, 400]).

We start by considering the notion of jet and of \(n\)-jet event. We recall that one cannot separate the contributions to the total hadronic cross section at order \(\alpha(Q^2)\) of the \(q\bar{q}\) and \(q\bar{q}G\) final states. The positive term from \(q\bar{q}G\), whose differential cross section is given in eq. (9-38 or 9-39), obviously leads to a singular contribution to the total cross section. This singularity is cancelled by adding the interference term which contributes at order \(\alpha(Q^2)\) to the \(q\bar{q}\) final state. This type of situation persists to all orders and consequently it makes no sense to think of the total cross section as a sum of \(n\)-parton \((n\text{-jet})\) partial cross sections.

A definition of jets in terms of energy flow inside given cones was considered in the literature, which, although of limited use in practice, is however instructive under many respects [388]. One defines, for example, a two jet event as one where a fraction of energy larger than \(1 - \epsilon\) is contained inside two opposite half cones of semiaperture \(\delta\), with \(\epsilon\) and \(\delta\) small. From an explicit calculation at order \(\alpha(Q^2)\) one finds (see also [389, 69, 410]):

\[
F(\epsilon, \delta) = \frac{\sigma_{2\text{jet}}(\epsilon, \delta)}{\sigma_{\text{TOT}}} = 1 - \frac{\alpha(Q^2)}{\pi} C_F (4 \ln 2 \epsilon + 3) \ln \delta + \frac{\pi^2}{3} - \frac{7}{4} + \alpha(\epsilon, \delta) + \cdots
\]  

(9-76)

The logarithmic terms, related to soft and collinear singularities, are easily seen by the following derivation to be universal properties of all \(q\) or \(q\bar{q}\) jets, independent of the production processes [208, 338].

Consider a quark produced with energy \(E\). In the LLA one can argue in terms of probabilities. The produced quark can (a) do nothing with probability \(P_0\); (b) produce by splitting a soft quantum with energy \(\leq \epsilon E\), with probability \(P_\epsilon\); (c) emit a quasi collinear pair with aperture angle between the two partons less than \(2\delta\), with probability \(P_\delta\); (d) finally it can split into a hard non collinear pair with probability \(P_{\text{hard}}\). Conservation of probabilities implies \(P_0 + P_\epsilon + P_\delta + P_{\text{hard}} = 1\). The probability of finding an energy \((1 - \epsilon)\cdot E\) inside a semicone of half aperture \(\delta\) around the initial quark direction is given by \(P_0 + P_\epsilon + P_\delta\). It is simpler to compute \((1 - P_{\text{hard}})\) which, with logarithmic accuracy, is given by:

\[
[1 - P_{\text{hard}}(\epsilon, \delta)]_\alpha = 1 - \int_\epsilon^{1 - \epsilon} dx P_{\alpha q}(x) \int \frac{dk^2}{2\pi} \frac{\alpha(k^2)}{k^2} 
\]

\[
= 1 - C_F \frac{\alpha(E^2)}{2\pi} (4 \ln \epsilon + 3) \ln \delta + \cdots = f_q(\epsilon, \delta).
\]  

(9-77)

The upper limit of \(1 - \epsilon\), which arises from the requirement that the final quark be hard, is not crucial in
this case. The fraction of two jet events as given in eq. (9-76) is immediately obtained as $1 - 2P_{\text{hard}}(2\epsilon, \delta)$, because there are two jets and the event energy is twice the jet energy.

The same reasoning can also be applied to a gluon jet [385, 163, 387, 173]:

$$[1 - P_{\text{hard}}(\epsilon, \delta)]_G = 1 - \int_{\epsilon}^{1-\epsilon} \! \! dx \frac{1}{2}[P_{G\text{G}}(x) + 2f_{P\text{qG}}(x)] \int_{E^2/\delta^2} \! \! \frac{d^2 k^2}{k^2} \frac{\alpha(k^2)}{2\pi}$$

$$= 1 - \frac{\alpha(E^2)}{2\pi} \left[ 4CA \ln \epsilon + \frac{11C_A - 4T}{3} \right] \ln \delta + \cdots = f_G(\epsilon, \delta). \quad (9-78)$$

The factor of 1/2 in front of $(P_{G\text{G}} + 2f_{P\text{qG}})$ avoids an obvious double counting. Note that a gluon jet appears fatter than a quark jet, a fact that can also be seen otherwise.

The exponentiation of the leading infrared singularities can be invoked to resum to all orders the string of terms started in eqs. (9-77, 9-78) [136, 139, 386]:

$$f_q(\epsilon, \delta) \approx \exp \left\{ C_F (2 \ln \epsilon + \frac{3}{2}) \int_{E^2/\delta^2} \! \! \frac{d^2 k^2}{k^2} \frac{\alpha(k^2)}{2\pi} \right\} \quad (9-79)$$

$$f_G(\epsilon, \delta) \approx \exp \left\{ \left[ 2CA \ln \epsilon + \frac{11C_A - 4T}{6} \right] \int_{E^2/\delta^2} \! \! \frac{d^2 k^2}{k^2} \frac{\alpha(k^2)}{2\pi} \right\}. \quad (9-80)$$

Actually it is only at the doubly logarithmic level that one is guaranteed that no other comparable terms are left out. Therefore the validity of these formulae is only assumed in the region

$$\frac{\alpha(E^2)}{\pi} \ll \frac{\alpha(E^2)}{\pi} \ln \frac{1}{\epsilon} \sim \frac{\alpha(E^2)}{\pi} \ln \frac{1}{\delta} \ll \frac{\alpha(E^2)}{\pi} \ln \epsilon \ln \delta \ll 1.$$

At present energies there is not enough phase space for these conditions to hold and at the same time to require that $\delta, \epsilon \geq \langle p_\perp \rangle_{\text{intrinsic}}/E$ as is necessary to avoid the details of the hadronization process.

The evolution of a hard parton (jet) can be followed at the parton level by describing the successive hard parton branchings. The rules of this "jet calculus" in the LLA and beyond can be formulated [291, 292, 29, 275, 100a]. Interesting results are derived for ratios of multiplicities of quarks and gluons in a parton jet, longitudinal and transverse momentum distributions, two parton correlations and so on. Similarly, interesting results are obtained on the probability of formation of colour singlet parton clusters [28, 59, 60].

Rather than insisting on a particular definition of a jet, suitable variables can be introduced for a quantitative description of final state topologies. These variables are for example useful to define the axis or the plane of the event and consequently longitudinal and transverse momentum distributions. All such variables should be linear in energy and/or momentum in order to meet the necessary condition of infrared stability: configurations differing by the splitting of a parton into two collinear partons should contribute with the same coefficient in order not to spoil the cancellation of collinear singularities and moreover the emission of an additional soft parton should lead to a vanishingly small increment.
Thrust and spherocity are two alternative infrared safe “jettiness” measures for a quantitative parametrization of the continuous range from the topology of a sphere to that of a collinear (ideal) two jet event. Thrust is defined as [182]:

\[
T = \max_e \frac{\sum_i |p_i \cdot e|}{\sum_i |p_i|}
\]  

(9-81)

where \(\sum_i\) is a sum over all visible particles in one event and the unit vector \(e\) is varied until the maximum result is found. The direction identified by the maximal vector \(e\) is called the thrust axis. One has:

\[
\frac{1}{2} \leq T \leq 1 \quad T = \begin{cases} \frac{1}{2}: & \text{sphere} \\ 1: & \text{line} \end{cases}
\]  

(9-82)

Similarly spherocity [217], which is less used in practice, is defined as:

\[
S = \left(\frac{4}{\pi}\right)^2 \min \left(\frac{\sum_i |p_i \cdot e|}{\sum_i |p_i|}\right)^2
\]  

(9-83)

where \(p_{\perp}\) is the transverse momentum with respect to the minimum direction (spherocity axis). The bounds are in this case:

\[
0 \leq S \leq 1 \quad S = \begin{cases} 1: & \text{sphere} \\ 0: & \text{line} \end{cases}
\]  

(9-84)

Spherocity is to be contrasted to the non infrared stable sphericity [71] where the sums of squares replace the sums squared. Starting from the thrust axis \(e_1\), a direction normal to it, \(e_2\), is identified by again making the energy flow maximum:

\[
F_{\text{Major}} = \max_e \frac{\sum_i |p_i \cdot e_1|}{\sum_i |p_i|}, \quad (e \perp e_1)
\]  

(9-85)

the resulting \(e_2\) is called the major axis [320]. Finally:

\[
F_{\text{Minor}} = \frac{\sum_i |p_i \cdot e_3|}{\sum_i |p_i|}
\]  

(9-86)

where \(e_3\) is orthogonal to both \(e_1\) and \(e_2\). The oblateness is defined by:

\[
O_B = F_{\text{Major}} - F_{\text{Minor}}.
\]  

(9-87)

Similarly the acoplanarity [143] is given by:

\[
A = 4 \min \left(\frac{\sum_i |p_i \cdot e|}{\sum_i |p_i|}\right)^2
\]  

(9-88)
where the minimum is taken with respect to a variable plane. Note that the spherocity and thrust axis are not coincident in general and the same is true for the acoplanarity and minor planes (on jettiness variables see also: \[57, 381, 385, 142, 360\]).

The event shape can also be studied in terms of variables not requiring an extremum procedure \[195, 349, 152\]. For example one can introduce the event tensor:

\[
\theta_{ab} = \frac{\sum_i (p_i^a p_i^b / |p_i|)}{\sum_i |p_i|} \tag{9-89}
\]

where \(p_i^a\) is the a component of the \(i\)th particle momentum. Note that both sums are linear in momentum. In terms of the eigenvalues of the \(\theta\) tensor \(\lambda_1, \lambda_2, \lambda_3\), with \(\lambda_1 + \lambda_2 + \lambda_3 = 1\), one defines:

\[
C = 3(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3) \tag{9-90}
\]

\[
D = 27\lambda_1 \lambda_2 \lambda_3 \tag{9-91}
\]

where normalizations are such that \(0 \leq C, D \leq 1\). For a linear event \(C = D = 0\), for a planar event \(D = 0\).

At order \(\alpha(Q^2)\) the differential distribution in eq. (9-39) for a \(q\bar{q}G\) event is all that is needed to derive distributions and averages of the jettiness variables in the limit, asymptotically approached, of negligible hadronization effects. For a three quantum final state the thrust and spherocity axis are coincident with the fastest parton and one has

\[
T = \max(x_1, x_2, x_3); \quad \frac{2}{3} \leq T \leq 1 \tag{9-92}
\]

\[
S = \frac{64}{\pi^2 T^2} (1-x_1)(1-x_2)(1-x_3); \quad 0 \leq S \leq \frac{16}{3\pi^2} \tag{9-93}
\]

\[
C = 6(1-x_1)(1-x_2)(1-x_3) \frac{x_1 x_2 x_3}{x_1 x_2 x_3}. \tag{9-94}
\]

Note that \(x_{\perp}^2\), where \(x_{\perp} = 2k_{\perp}/Q\) is the fractional transverse momentum of either parton in the opposite hemisphere of the fastest parton (with respect to its direction), is simply proportional to \(S\) for a three quantum state:

\[
x_{\perp}^2 = \frac{\pi^2}{16} S. \tag{9-95}
\]

Consider for example the calculation of the thrust distribution at order \(\alpha(Q^2)\). We subdivide the phase space in three regions according to \(x_q, x_q\) and \(x_G\) being the largest fraction. The contribution of the region \(T = x_q\) is given by \((N = 3)\):

\[
\frac{1}{\sigma_{\text{TOT}}} \frac{d\sigma}{dT} \bigg|_{T=x_q} = \frac{2\alpha}{3\pi} \int dx_q dx_G \frac{\delta(2-x_q-x_G-T) \theta(T-x_q) \theta(T-x_G)}{(1-x_q)(1-x_G)} \frac{T^2+x_q^2}{(1-x_q)(1-x_G)(1-T)} \tag{9-96}
\]

\[
= \frac{2\alpha}{3\pi} \int_{2(1-T)}^{T} dx \frac{T^2+x^2}{(1-T)(1-x)} = \frac{2\alpha}{3\pi} \left[ 1 + \frac{T^2}{1-T} \ln \frac{2T-1}{1-T} + \frac{3T^2-14T+8}{2(1-T)} \right]. \quad (T < 1)
\]
The contribution of $T = x_q$ is obviously the same and finally that of $T = x_G$ is given by:

$$\frac{1}{\sigma_{TOT}} \frac{d\sigma}{dT} \bigg|_{T=x_G} \approx \frac{2\alpha}{3\pi} \left[ \frac{2(1+T)^2}{T} \ln \frac{2T-1}{1-T} + 4 - 6T \right].$$

(9-97)

Note that the integral of this last expression between $\frac{2}{3}$ and 1 is the finite probability at order $\alpha(Q^2)$ of finding the gluon as the most energetic parton:

$$\frac{1}{\sigma_{TOT}} \int_{2/3}^{1} \frac{d\sigma}{dT} \bigg|_{T=x_G} \ dT = \frac{\alpha(Q^2)}{\pi} 0.611 \ldots$$

(9-98)

The probability that a quark or an antiquark is the most energetic parton is then given by $(1 - P_{T=x_G})$. As in other similar cases, in order to obtain this last result directly one should introduce a regularization and add the virtual contributions at $T = 1$.

By adding eq. (9-97) to twice eq. (9-96) one obtains the thrust distribution at order $\alpha(Q^2)$ [143]:

$$\frac{1}{\sigma_{TOT}} \frac{d\sigma}{dT} \bigg|_{T=x_G} = \frac{2\alpha(Q^2)}{3\pi} \left[ \frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right], \quad (T < 1).$$

(9-99)

The two parton configuration in the final state contributes in addition a term proportional to $\delta(1-T)$. The hadronization of the parton into a jet smears this contribution in a range of $(1-T)$ of a width which decreases as a power of the energy.

From eq. (9-99) the average value of $(1-T)$ can easily be obtained:

$$\langle 1-T \rangle = 1.05 \frac{\alpha(Q^2)}{\pi} + \cdots$$

(9-100)

However this value is much affected by the details of the $T$ distribution at small $(1-T)$, up to high energies. As already mentioned, what matters for QCD is not the absolute value of such a quantity but rather its rate of change with $Q$ from which, in principle, the leading term can be extracted.

Near $T = 1$ eq. (9-99) diverges as $\log(1-T)/(1-T)$. Actually this singular behaviour is quenched in higher orders [68, 234]. In fact this suppressing effect consists in a Sudakov factor $\exp[-\text{const} \cdot (\alpha/\pi) \cdot \log^2(1-T)]$ (expressing the exponentially small probability of no radiation from the parton) which multiplies the leading term at $T = 1$. As the region near $T = 1$ is obscured anyway by non perturbative effects, the value of this improvement is simply to provide an expression for the thrust distribution with a more reasonable extrapolation into the non perturbative region near $T = 1$.

Distributions at order $\alpha$ for spherocity, $C$ and so on can be derived in a quite similar way. On the other hand the distribution for quantities which at parton level start at order $\alpha^2$, like acoplanarity or $D$ (eqs. (9-88, 9-91)), are computed from tree diagrams with four partons in the final state [9]. Essentially $(1-T)$, $S$, $C$ are measures of the spread in $(k_\perp)_m$, while $A$ and $D$ are clearly determined by the distribution in $(k_\perp)_{out}$.

Further quantitative testing of the three jet distribution in eq. (9-39) can be done by measuring angular distributions, as for example for the test proposed by Ellis and Karliner [167], or for the pointing (not Poynting) vector distribution [143]. In the E.K. test one selects events at small fixed $T$. A boost is made along the thrust axis to the rest frame of the fat jet. In this frame the event looks as two
back to back jets. Each of them should look as a jet of a two jet event at lower energy, to the extent that a gluon jet looks like a quark jet. The angular distribution of the new thrust direction with the boost direction can easily be computed and compared to experiment (fig. 22). The pointing vector distribution refers to energy versus polar angle in the plane of the event:

\[ P(Q, T, \theta) = p \frac{d\sigma}{dT d\cos^2 \theta} \] or \[ p \frac{d\sigma}{dT d\theta} \]  \hspace{1cm} (9-101)

\( \theta \) is defined as zero in the thrust direction; then \( \frac{1}{2} \pi \leq \theta \leq \pi \) contains the next hardest candidate jet and finally \( \pi \leq \theta \leq \frac{3}{2} \pi \) is the quadrant of the softest candidate jet. At the parton level the pointing vector distribution is simply obtained by kinematics and change of variables from eq. (9-39). One can add a model for the hadronization smearing (see fig. 21). Alternatively an angular binning à la Sterman-Weinberg can be considered.

Besides the distributions in the plane of the event considered so far, one can also study angular distributions with respect to the beam direction. The angle between the normal to the event plane and the beam direction is an example. The distribution is expected to be of the form \( 3 - \cos^2 \theta \) and it would be the same also for scalar gluons [165]. More in general antenna patterns around the beam direction have been considered [57].

The question of how accurate the leading approximation is for the three jet matrix element has been investigated by very complicated calculations aiming at the corrections of order \( \alpha^2 \) to distributions like \( d\sigma/dC \) or \( d\sigma/dT \) which start at the parton level at order \( \alpha \). The results indicate that the corrections are indeed large. As an illustration for the \( C \) distribution it was found that [175]:

\[ \frac{1}{\sigma} \int_{1/2}^{1} \frac{d\sigma}{dC} dC = 1.4 \frac{\alpha(Q^2)}{\pi} \left[ 1 + (1.3 \pm 0.3) \frac{\alpha(Q^2)}{\pi} + \cdots \right] \]  \hspace{1cm} (\alpha = \overline{\text{MS}})  \hspace{1cm} (9-102)

and:

\[ \langle C \rangle = \frac{\alpha(Q^2)}{\pi} C_F \left( 2 \pi^2 - \frac{33}{2} \right) \left[ 1 + 8.5 \frac{\alpha(Q^2)}{\pi} + \cdots \right] \]  \hspace{1cm} (\alpha = \overline{\text{MS}}). \hspace{1cm} (9-103)
Similarly for the thrust distribution [174a, 406] (see also [299a]) (the following relation was evaluated for me by Vermaseren):

\[
\frac{1}{\sigma} \int_{0.5}^{0.85} \frac{d\sigma}{dT} dT = \left( \frac{1}{\sigma} \int_{0.5}^{0.85} \frac{d\sigma}{dT} dT \right)_{\text{order } \alpha} \left[ 1 + 17.1 \frac{\alpha(Q^2)}{\pi} + \cdots \right] \quad (\alpha = \text{MS}). \tag{9-104}
\]

The two groups mentioned worked with completely different regularization and numerical methods and their results are in agreement. Another group, instead of computing the thrust (or other similar) distribution as if the final state was made up of partons, used a different procedure and obtained appreciably different results [179]. They first define jets à la Sterman–Weinberg in terms of energy fractions \(1 - \epsilon\) and opening angles \(\delta\). Then at fixed \(\epsilon, \delta\) they compute the thrust distribution by treating each jet as a single particle. For values of \(\epsilon\) and \(\delta\) corresponding to \(M = 5\) GeV, \(M\) being the maximum invariant mass of the jets, the correction is \(\frac{1}{3} + \frac{2}{3}\) of that in eq. (9-104) and with the same sign. The two classes of computations should approach each other for \(\epsilon, \delta \to 0\). This dependence on the treatment of what is inside the Sterman–Weinberg jets is unfortunately an indication of (hadronization) model dependence for these corrections.

Taken at face value these results are a blow against hopes of a fast convergence of the expansion and of a fair accuracy of the leading approximations. However the situation can be made to look somewhat better by considering the structure and the origin of the large corrective terms. First, the bulk of the correction amounts to a rescaling of the lowest order result, so that the shape of the distribution is not much altered. Second, three effects, which it is conceivable to control to all orders, according to [372], explain a large component of the correction. They are (a) a change of scale for the running coupling which replaces the total squared energy \(s\) with \(s_1s_2s_3/s\); (b) the large logarithms near \(T = 1\) (or equivalently \(C = 0\)) and (c) a large \(\pi^2\) term left over after cancellation of the leading infrared double poles between virtual corrections to the three parton final state and real four parton diagrams. If tentatively these effects are extrapolated to all orders and resummed by educated guesses, then the remaining corrections are reasonable. In conclusion the question is far from being settled and it clearly deserves further attention.

To introduce a different direction of development we recall that in leptoproduction we have seen that transverse momentum distributions can be studied in the two scale regime \(\mu^2/Q^2 \ll k^2_1/Q^2 \ll 1\). The same technique can be applied in \(e^+e^-\) annihilation to the energy-energy correlation function defined as [58]:

\[
\frac{dW}{d\cos \theta} = \sum_{a,b} x_a x_b \frac{1}{\sigma} \frac{d\sigma}{dx_a dx_b} \frac{d\sigma}{d\cos \theta} dx_a dx_b \tag{9-105}
\]

where \(x_a\) and \(x_b\) are energy fractions carried by any two hadrons and \(\theta\) is the angle between their directions such that \(\theta = 0\) in the back to back configuration. This quantity is independent of fragmentation functions, which drop away as a consequence of the energy conservation sum rule in eq. (9-24), and is infrared safe in the massless theory. Since for small \(\theta\) one has \(\theta^2 \sim k^2/Q^2\) the approach developed by Dokshitzer, Dyakonov and Troyan [150] and completed by Parisi and Petronzio [352] can be extended to this case as well for predicting the region of small \(\theta\) [43]. The Parisi, Petronzio formulation has been improved by Ellis and Stirling [176] and a quite systematic procedure of resummation of sequences of non leading terms has been further developed by Collins and Soper [126, 127]. The comparison with experiment is still not sufficiently developed (see [422]).
10. Heavy quarkonium decays

We summarize in this section the main results on heavy quarkonium decays (flavour neutral, heavy \(q\bar{q}\) bound states: \(J/\psi,\ Y\) families) which are of direct interest to perturbative QCD while we leave aside the bound state description and the level spacing predictions in potential or bag models (for reviews see [32, 340]).

For sufficiently high mass of the constituent quarks a description of the quarkonium decay rates in terms of wave functions at (or near) the origin and of perturbative decay amplitudes becomes appropriate. Recently new arguments in support of this intuitive view have been brought up [159]. In the simplest version of this approach one neglects spin dependence and all relativistic effects and considers the decay amplitudes to lowest order in \(\alpha\).

According to the parton model rules for totally inclusive processes, it is sufficient to compute the amplitude for a \(q\bar{q}\) pair, in the appropriate partial wave, into partonic final states. The minimal final state is determined by recalling the (a) while both \(C\)-even and \(C\)-odd states \((C\) = charge conjugation) can go to three gluons, only \(C\)-even states can couple to two gluons (real or virtual); (b) \(J = 1\) states cannot decay into two on shell massless gluons. It follows that the hadronic widths of the quarkonium states are determined to lowest order in \(\alpha\) by the following parton modes (fig. 23):

\[
\begin{align*}
0^{++}, 0^{++}, 2^{++} &\rightarrow 2G \\
1^{--}, 1^{++} &\rightarrow 3G \\
1^{++} &\rightarrow 3G, Gq\bar{q}
\end{align*}
\]

where the lowest lying quarkonium states are labeled by \(J^{PC}\). In general we refer to the lowest radial excitations with \(n = 1\) (\(n\) is the principal quantum number), but most formulae in the following could be applied to \(n > 1\) as well.

To the extent that the wave functions are independent of the total spin and angular momentum for given \(n\) and \(L\), one has [36, 53a, 53b]:

\[
\begin{align*}
\Gamma(1^{--} \rightarrow \ell^+ \ell^-) &= 4\epsilon_4^2 \alpha^2 \frac{|R_0(0)|^2}{M_0^2}; \quad (M\ell\ell = 0) \\
\Gamma(1^{--} \rightarrow 3G) &\simeq \frac{40}{81\pi} (\pi^2 - 9) \alpha^3 \frac{|R_0(0)|^2}{M_0^2} = 10 \frac{(\pi^2 - 9)}{81\pi^2} \frac{\alpha^3}{\alpha_{em}} \Gamma(1^{--} \rightarrow \ell^+ \ell^-) \\
\Gamma(0^{++} \rightarrow 2G) &\approx \frac{8}{3} \alpha^2 \frac{|R_0(0)|^2}{M_0^2}
\end{align*}
\]

Fig. 23. Heavy quarkonium decay amplitudes.
\[
\Gamma(0^{++} \to 2G) = 96\alpha^2 \frac{|R^i_2(0)|^2}{M^4_\perp}
\]
\[
\Gamma(1^{++} \to Gq\bar{q}) = \frac{128}{3} \frac{\alpha^3}{3\pi} \frac{|R_1^i(0)|^2}{M^4_\perp} \ln \frac{4m^2_\perp}{4m^2_q - M^2_\perp}
\]
\[
\Gamma(2^{++} \to 2G) = \frac{128}{5} \frac{\alpha^2}{9\pi} \frac{|R_1^i(0)|^2}{M^4_\perp}
\]
\[
\Gamma(1^{-} \to 3G) = \frac{320}{9\pi} \frac{\alpha^3}{\pi} \frac{|R_1^i(0)|^2}{M^4_\perp} \ln \frac{4m^2_\perp}{4m^2_q - M^2_\perp}
\]

in previous formulae \( R_e(r) \) is the radial wave function (\( \int_0^\infty |R_e|^2 r^2 dr = 1 \)) and \( R^i_e(r) \) its derivative, while \( M_\perp \) is the quarkonium mass and \( e_q, m_q \) the constituent quark charge and mass. \( \Gamma(1^{++}) \) takes its dominant contribution, shown in eq. (10-8), from the \( Gq\bar{q} \) final state (note in fact the proportionality to \( f \), the number of flavours) which shows the typical zero binding logarithmic singularity also present in \( \Gamma(1^{--}) \). The presence of these singularities makes the relevance of the perturbative approximation more questionable for these particular channels.

The following results for real photons in the final state are also valid in the same approximations [34, 110, 342, 83]:

\[
\Gamma(1^{-} \to \gamma GG) = \frac{36}{5} \frac{\alpha e^2_q}{\alpha_{em}}
\]

\[
\frac{\Gamma(0^{-} \to \gamma\gamma)}{\Gamma(0^{-} \to 2G)} = \frac{\Gamma(0^{++} \to \gamma\gamma)}{\Gamma(0^{++} \to 2G)} = \frac{\Gamma(2^{++} \to \gamma\gamma)}{\Gamma(2^{++} \to 2G)} = \frac{9}{2} \left( \frac{\alpha_{em}}{\alpha} \right)^2 e^4_q.
\]

This list of lowest order results for quarkonium decay rates is concluded by recalling that of course for the \( 1^{-} \) state a fraction of hadronic decays occurs through the one photon channel:

\[
\Gamma(1^{-} \to \gamma^* \to \text{hadrons}) = R(M_\perp^2) \Gamma(1^{-} \to \ell^+ \ell^-)
\]

where \( R \) was defined in eq. (9-4).

The corrections of next order in \( \alpha \) to a number of the quarkonium modes have been computed. The first calculation of this kind to be carried through [51] was for the ground state (para) quarkonium and led to the result (\( N = 3 \)):

\[
\Gamma(0^{-} \to 2G) = \frac{2}{9e^4_q} \left( \frac{\alpha(\mu^2)}{\alpha_{em}} \right)^2 \Gamma(0^{-} \to 2\gamma) \left\{ 1 + \frac{\alpha(\mu^2)}{\pi} \left[ \frac{61}{2} - \frac{13}{8} \pi^2 - \frac{2}{3} \ln 2 + \frac{25}{3} \ln \frac{\mu}{2m_q} \right] + \cdots \right\}
\]

\( \alpha = \overline{\text{MS}} \).

In this equation \( \mu \) is the scale at which the running coupling is taken. For each \( \mu \) the non leading correction is then specified. For example the coefficient of \( \alpha/\pi \) in eq. (10-14) is 14 or 8.22 for \( \mu = 2m_q \) or \( \mu = m_q \) respectively: a smaller scale is preferred. As \( \Gamma(0^{++} \to 2G) \) is proportional to \( \alpha^2(\mu^2) \) the
Guido Altarelli, Partons in Quantum Chromodynamics

coefficient of $\alpha/\pi$ in the correction term is particularly sensitive to the choice of prescription for $\alpha$. For example, in a version of momentum subtraction advertised in refs. [104, 106] the previous figures are changed as $14.0 \to 7.56$ and $8.22 \to 1.78$. But this fiddling around is of little relevance, as we shall see.

Recently the next to leading corrections to other quarkonium modes have become available. This allows the comparison of different channels and the elimination of the dependence on the prescription for $\alpha$. The results are given here in numerical form as in the first paper by Barbieri, Caffo, Gatto and Remiddi [50] while the analytic expressions can be found in the second paper of the same reference. Defining $B(J^{PC}) = \Gamma(J^{PC} \to \gamma\gamma)/\Gamma(J^{PC} \to GG)$ one has:

$$\frac{B(0^{-}^{+})}{B(0^{++})} \approx 1 + 0.9 \frac{\alpha}{\pi} + \cdots \left(1 + 2.1 \frac{\alpha}{\pi} + \cdots\right)$$  \hspace{1cm} (10-15)

$$\frac{B(2^{-}^{++})}{B(0^{++})} \approx 1 + 6.5 \frac{\alpha}{\pi} + \cdots \left(1 + 4.0 \frac{\alpha}{\pi} + \cdots\right)$$  \hspace{1cm} (10-16)

$$\frac{\Gamma(0^{++} \to 2\gamma)}{\Gamma(2^{++} \to \gamma\gamma)} = \frac{15}{4} \left(1 + 5.5 \frac{\alpha}{\pi} + \cdots\right)$$  \hspace{1cm} (10-17)

$$\frac{\Gamma(1^{-}^{-} \to e^{+}e^{-})}{\Gamma(0^{++} \to \gamma\gamma)} = \frac{1}{3e_q^2} \left(1 - 1.96 \frac{\alpha}{\pi} + \cdots\right).$$  \hspace{1cm} (10-18)

These results refer to the $J/\psi$ and $Y$ families; when different the $Y$ results have been reported in parenthesis. Note that eqs. (10-16, 10-17) also imply:

$$\frac{\Gamma(0^{++} \to 2G)}{\Gamma(2^{++} \to 2G)} = \frac{15}{4} \left(1 + 12 \frac{\alpha}{\pi} + \cdots\right) \left[\frac{15}{4} \left(1 + 9.5 \frac{\alpha}{\pi} + \cdots\right)\right].$$  \hspace{1cm} (10-19)

It is clear that one cannot make the corrections small in all channels by a suitable choice of prescription for $\alpha$. It is also rather clear that a different choice of scale for each of the above very similar channels is also not appealing. Finally it is manifest that the leading approximation, at least for the charmonium family, is quite crude.

Quite recently the important calculation of the subleading corrections to the three gluon mode of orto-quarkonium was completed [317a]. The result can be cast in the form (see eq. (10-5)) for the $J/\psi$ and the $Y$ respectively:

$$\frac{81\pi e_q^2\alpha_{em}^2}{10 \pi^2} \Gamma(1^{-}^{-} \to 3G) = \alpha^3(M_\phi) \left[1 + (10.2 \pm 0.5) \frac{\alpha(M_\phi)}{\pi} + \cdots\right]$$

$$\left(\alpha^3(M_Y)\left[1 + (9.1 \pm 0.5) \frac{\alpha(M_Y)}{\pi} + \cdots\right]\right) [\alpha = \overline{MS}].$$

Comparison of the $Y$ prediction with experiment leads to the rather precise value $\Lambda_{\overline{MS}} = 100 \pm 25$ MeV, because of the cubic dependence on $\alpha$. The scale at which the first correction vanishes turns out to be $M = (0.48 \pm 0.02)M_Y$. The same value of $\Lambda_{\overline{MS}}$ reproduces the $J/\psi$ data only within a factor of two, which can be attributed to $o(v^2/c^2)$ relativistic corrections. When and if the top-onium will be discovered, the
determination of $\Lambda_{\overline{MS}}$ from a simultaneous fitting of the $Y$ and top orto-quarkonia will be very convincing, if both turn out to be compatible with the QCD prediction.

The final state analysis of $1^{--}$ (orto-quarkonium) decays is extremely interesting because it should allow the study of events with three gluon jets. However more energy is in general necessary to reveal jets than for studying bulk properties like rates. Thus charmonium is out of the question for this purpose and b-onium barely marginal.

The three gluon distribution obtained from the diagram in fig. 23b is given by De Rujula, Ellis, Floratos and Gaillard [143], Krasemann [296], Koller and Walsh [289], Koller, Krasemann and Walsh [288]:

$$\frac{1}{T_{\nu}} \frac{dI_{\nu}}{dx_1 dx_2} = \frac{6}{\pi^2 - 9} \left[ \frac{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2 + x_3^2(1-x_3)^2}{x_1^2 x_2^2 x_3^2} \right] \tag{10-20}$$

where the normalization was chosen as to reproduce $1$ by integration over $1/6$ of the total phase space. The thrust distribution is derived in the same way as for eq. (9-99) and is given by:

$$\frac{1}{\sigma} \frac{d\sigma}{dT} = \frac{3}{\pi^2 - 9} \left[ \frac{4(1-T)(5T^2 - 12T + 8)}{T^2(2-T)^3} \ln \frac{2(1-T)}{T} + \frac{2(3T - 2)(2-T)}{T^3(2-T)^2} \right]. \tag{10-21}$$

The average value of $T$ is of order 1 in this case:

$$\langle T \rangle = \frac{3}{\pi^2 - 9} \left[ 6 \ln \frac{2}{3} - \frac{3}{2} + \frac{4}{3} \pi^2 + 20 \int_0^1 dz \ln \frac{z}{2+z} \right] = 0.889. \tag{10-22}$$

In $e^+e^-$ annihilation (with unpolarized beams) the distribution in the cm angle $\varphi$ between the beam direction and the normal to the three gluon plane is $(3 - \cos^2 \varphi)$ [83]. Similarly the distribution in the angle $\theta$ between the most energetic gluon and the beam axis can be studied. It is of the form:

$$\frac{dN}{dT d\cos \theta} = f(T) \left[ 1 + g(T) \cos^2 \theta \right] \tag{10-23}$$

![Fig. 24. The jet axis angular distribution in $Y$ decays and the theoretical curves for vector and scalar gluons. $\theta$ is the angle between the thrust and the beam axis.](image-url)
and \( f(T) \) and \( g(T) \) are given in ref. [287]. In particular \( g(T = 1) = 1 \) in QCD while it is \(-1\) for scalar gluons (see fig. 24).

Eq. (10-20) is also relevant for the distribution of the \( \gamma gG \) final state in or-ko-quarkonium decay. In particular the direct photon spectrum is readily obtained [343]:

\[
\frac{dN}{dx} = \frac{2}{\pi^2} \left[ \frac{x(1-x)}{(2-x)^2} - \frac{2(1-x)^2}{(2-x)^2} \ln(1-x) + \frac{2-x}{x} + \frac{2(1-x)}{x^2} \ln(1-x) \right].
\] (10-24)

The experimental spectrum in \( J/\psi \) decays is softer than that predicted from eq. (10-24). At such low energies we cannot rely on an asymptotic prediction too much and the difference between a massless gluon and a massive jet can well explain [352a] the observed rather soft spectrum.

Before concluding this section we recall that quarkless bound states of gluons (gluonium or glueballs) are also expected in QCD (for a recent discussion see, for example, [70a, 131a]). Their experimental identification is however not easy. It is clear that the discovery of glueballs would be an important breakthrough in establishing QCD.

11. Drell–Yan processes

In this section we consider the process of deep inelastic lepton pair production in hadron–hadron collisions:

\[
H_1 + H_2 \rightarrow \ell^+ \ell^- + X.
\] (11-1)

We shall mostly refer to the case of charged leptons produced by a virtual photon (for recent reviews, including the data, see for example [316a, 321a, 307]). Over a decade ago a parton description for this process was proposed [156] based on the annihilation of a quark from one hadron and an antiquark from the other with point-like cross section. At the naive parton model level one has:

\[
\frac{d\sigma}{dQ^2} = \frac{4\alpha em} {3Q^2 S N} \int_0^1 dx_1 \int_0^1 dx_2 \sum_{h=1}^f e_h^2 [q_{0h}(x_1) \bar{q}_{0h}(x_2) + 1 \leftrightarrow 2] \delta(x_1x_2 - \tau). 
\] (11-2)

In this formula \( \sqrt{S} \) is the invariant mass of the incoming hadron system, \( Q \) is the virtual mass of the (timelike) photon and hence of the produced lepton pair and:

\[
\tau = Q^2/S.
\] (11-3)

In the deep inelastic region, i.e. \( Q^2 \) and \( S \) large with fixed \( \tau \), where eq. (11-2) should be valid apart from terms down by powers, the range of \( \tau \) is given by \( 0 \leq \tau \leq 1 \). The structure of the formula is clear: it is the convolution of the pointlike annihilation cross section \( 4\alpha em e_h^2 /3Q^2 \) with the \( \bar{q} \) densities and \( \delta(x_1x_2S - Q^2) \) projects out the correct mass squared for the initial \( q\bar{q} \) system. As usual in the naive parton model the transverse momentum of the quarks is neglected and \( q \) and \( \bar{q} \) are taken as collinear along the \( H_1H_2 \) line. The important colour factor \( 1/N \) (in the following we shall set \( N = 3 \)) is there because a quark of given colour can only annihilate with an antiquark of the corresponding colour.

More differential cross sections are often considered in terms of the Feynman \( x_F \) or the rapidity. In the
hadron–hadron center of mass where the photon four momentum is in general given by:

\[ q = (E; q_x, q_y) \]  

one has:

\[ x_F = 2q_x/\sqrt{S} \]  

\[ y = \frac{1}{2} \ln \frac{E + q_z}{E - q_z}. \]  

In the same approximation as for the total cross section (for a useful summary of formulae on Drell–Yan processes see [169]):

\[ \frac{d\sigma}{dQ^2 dx_F} = \frac{4\pi\alpha_s^2}{9Q^2 S} \frac{1}{x_1 + x_2} \sum_{h=1}^f e_h^2 \left[ q_h^{[1]}(x_1) \bar{q}_h^{[2]}(x_2) + (1 \leftrightarrow 2) \right] \]  

\[ x_1 = \frac{1}{2}[x_F + \sqrt{x_F^2 + 4\tau}] ; \quad x_2 = \tau/x_1 = \frac{1}{2}[-x_F + \sqrt{x_F^2 + 4\tau}]. \]  

In terms of the rapidity variable the differential cross section is given by:

\[ \frac{d\sigma}{dQ^2 dy} = \frac{4\pi\alpha_s^2}{9Q^2 S} \sum_{h=1}^f e_h^2 \left[ q_h^{[1]}(\sqrt{-\tau} e^+) \bar{q}_h^{[2]}(\sqrt{-\tau} e^-) + (1 \leftrightarrow 2) \right]. \]  

These processes are very important as they provide a crucial, quite non trivial test of the validity of the parton approach and of its extension in QCD through the factorization theorem. One expects that the same parton densities as measured in lepton production for a given hadron target should be relevant to make predictions on other hard processes where the same hadron is involved. Drell–Yan processes are especially fit for a test of the parton picture as the cross sections are bilinear in the parton densities and no fragmentation functions appear because a totally inclusive sum over the hadronic component of the final state is made. This allows, on one hand, an absolute computation of the cross sections in the channels P–Nucleon or P–Nucleon from the densities measured in lepton production, and on the other hand, it provides us with a unique possibility of measuring (up to now) otherwise inaccessible parton densities as those of pions and kaons. The fact that the cross sections are quadratic in the parton densities implies testing the parton model in a particularly non trivial dynamical situation. There are effects which may destroy parton results associated with non linear quantities while preserving linear predictions (for example instantons [166]; soft (wee) partons [127]; also at the level of terms down by powers there are other possible infrared difficulties [153]).

The strategy in testing QCD must be based on a sequence of natural steps. First one must establish the approximate validity of the naive parton model. Once it has been verified that the parton mechanism is indeed dominant it makes sense to study the structure of the deviations from the naive parton dynamics and compare these violations with the QCD predictions. Drell–Yan processes have the advantage of offering very clear cut signatures for the underlying parton mechanism. We recall the main tests of the Drell–Yan dynamics.

(a) \textit{Intensity rules.} We know from lepton production that valence quarks in the nucleon are dominant over sea quarks, especially at large \( x \). We have no reasons to doubt that this ought to be true for
other hadrons as well (in fact this has been confirmed in the case of pions by studying lepton pair production in \( \pi^\pm - \)Nucleon collisions). Then the Drell–Yan formula predicts much larger cross sections for processes where the lepton pair can be produced by valence–valence annihilation (\( \pi^\pm - N, K^- - N, \bar{P} - N \)) than processes where only valence-sea annihilation is possible (\( K^+ - N, P - N \)). This is expected to be increasingly true as the larger is \( \tau \) because the bulk of the production is at small \( y \), where \( x_1 \sim x_2 \sim \sqrt{\tau} \). Similarly for large enough \( \tau \) where the sea can be neglected we can also expect, for example, that on a isoscalar target:

\[
\frac{\sigma[\pi^+ N(I = 0)]}{\sigma[\pi^- N(I = 0)]} \rightarrow \frac{1}{4}.
\]

(c) **Approximate scaling.** In the limits of validity of the naive parton model all adimensional quantities, like for example \( Q^4 \frac{d\sigma}{dQ^2} \), \( Q^4 \frac{d\sigma}{dQ^2} dx_F \) or \( Q^4 \frac{d\sigma}{dQ^2 dy} \), should scale, that is should be functions of the scaling variables \( \tau, x_F \) or \( y \), independent of \( Q^2 \). This is a stringent test in view of the very steep dependence of the individual cross sections.

(c) **Angular distribution of leptons.** Because the production is through one photon exchange the angular distribution of either \( \ell^+ \) or \( \ell^- \) in their center of mass must be of the form, after azimuthal integration (more in general see \[273, 124, 119\]):

\[
\frac{d\sigma}{dQ^2 d\cos \theta} \sim W_T(Q^2, \tau) (1 + \cos^2 \theta) + W_L(Q^2, \tau) \sin^2 \theta.
\]

In the limit of negligible transverse momentum of \( \gamma^* \) in the lab. frame the reference axis for the definition of \( \theta \) coincides with the \( H_1 H_2 \) line. As a consequence of the spin-1/2 nature of quarks, the Drell–Yan mechanism predicts \( W_L/W_T \to 0 \) and that the angular distribution becomes predominantly \( (1 + \cos^2 \theta) \).

(d) **Atomic number dependence.** The cross section being proportional to the number of quarks or antiquarks in the target nucleus, each contribution adding up incoherently in the parton picture, one expects a linear \( A \) dependence in the Drell–Yan domain of validity.

---

Fig. 25. Correction of order \( \alpha \) to lepton pair production; a) \( q + \bar{q} \to \ell^+ \ell^- + G \) (the small \( \vee \) is the lepton pair); b) \( q \) (or \( \bar{q} \)) \( + G \to \ell^+ \ell^- + q \) (or \( \bar{q} \)).
The available experimental evidence, at sufficiently large energies and for masses of the pair beyond the $J/\psi$, supports the validity of all the previous distinctive predictions [307]. This neat success of the parton model strengthens the point that the general theoretical framework for its derivation is not limited to a study of the leading light cone singularities but rather, at the naive scaling level, is to be found in a diagrammatic analysis of softened field theories as in the original derivation by Drell and Yan. The validity of the same type of analysis is carried on to QCD with the only difference that the artificial softening of the theory is removed, thus giving up exact scaling, and in a sense it is replaced by the unique features of asymptotic freedom.

Also in this case of lepton pair production, in QCD one replaces the point-like cross section of the naive parton model by the full cross section with radiative corrections of all orders, as familiar by now. At order $\alpha$ the relevant parton processes are (fig. 25):

\begin{align}
q + \bar{q} &\rightarrow \gamma^* \\
q + \bar{q} &\rightarrow \gamma^* + G \\
G + q(\text{or } \bar{q}) &\rightarrow G + q(\text{or } \bar{q}).
\end{align}

By explicit computation one finds for the cross section the following form:

\begin{equation}
\frac{d\sigma}{dQ^2} \sim 4\pi \alpha_\text{em}^2 \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \left\{ \frac{f}{2\pi} \delta(1-z) + \theta(1-z) \frac{\alpha}{2\pi} (2P_{q\bar{q}}(z)t + f_{q\bar{q}}^D(z)) \right\} \left(1 \leftrightarrow 2\right) \left(\sum_{h=1}^f e_h^2 (q_h^{[1]}(x_1) \bar{q}_h^{[1]}(x_2) + (1 \leftrightarrow 2)) \right) \left(\sum_{h=1}^f e_h^2 (q_h^{[2]}(x_1) \bar{q}_h^{[2]}(x_2)) G^{[2]}_0(x_2) + (1 \leftrightarrow 2) \right) \left(1 \leftrightarrow 2\right)
\end{equation}

where the variable $z$ is given by:

\begin{equation}
z = \frac{\not{Q}^2}{x_1 x_2 S} = \frac{\not{T}}{x_1 x_2}.
\end{equation}

The logarithmic terms are obviously given by the same splitting functions as in leptoproduction in agreement with the factorization theorem, as first remarked in this context by Politzer [363]. The factor of two in front of $P_{q\bar{q}}$ appears because the zeroth order term is quadratic in the quark densities. Thus the leading logarithms can be factored out and included into a redefinition of parton densities. By recalling eq. (4-33) we see that in the LLA the effective quark densities so constructed are the same as in leptoproduction. In this same approximation, i.e. when all terms vanishing with $\alpha(Q^2)$ are neglected, the naive Drell–Yan formula is reproduced in terms of $Q^2$ dependent effective parton densities which satisfy the evolution equations of section 4:

\begin{equation}
\frac{d\sigma}{dQ^2} = \frac{4\pi \alpha_\text{em}^2}{9Q^2 S} \int_0^1 dx_1 \int_0^1 dx_2 \sum_{h=1}^f e_h^2 \left[q_h^{[1]}(x_1, Q^2) \bar{q}_h^{[2]}(x_2, Q^2) + (1 \leftrightarrow 2)\right] \delta(x_1 x_2 - T) + o[\alpha(Q^2)]
\end{equation}

Thus the first and theoretically most evident of the QCD predictions is the presence of logarithmic
scaling violations. However these scaling violations are not directly visible in the data, which cannot, at this stage, be precise enough to show such delicate effects. This is true in lepton pair production because of the very steep dependence of the cross section on $Q^2$ which makes the problem of a precise normalization quite difficult.

Fortunately there are more efficient ways of detecting the same physical effect in different forms than from scaling violations. As in lepton production and in $e^+e^-$ annihilation, it is through the study of the transverse momentum distribution of the lepton pair that one can detect the hard parton origin of scaling violations. We are talking here of the total transverse momentum of the pair, that is that of $\gamma^*$, with respect to the $H_1H_2$ reference line. Most of the lepton pairs are produced with small transverse momentum together with two hadronic jets at opposite ends in rapidity arising from the fragments of the incoming hadrons. However a small fraction of events of order $\alpha(Q^2)$ should exhibit a third hadronic jet and a lepton pair at large $q_\perp \sim Q$. This tail is obtained from the parton processes in eqs. (11-13, 11-14) when the final quanta are produced at large angles, by convolution with effective parton densities.

The hard component of the transverse momentum distribution can be studied in terms of $q_\perp$ moments. For example the average $q_\perp^2$ is predicted to rise linearly with $S$ at fixed $\tau$, apart from logarithms:

$$\langle q_\perp^2 \rangle = \alpha(Q^2) S f(\tau, Q^2) + \cdots$$  (11-18)

and similarly for other moments [259, 366, 205, 24, 272, 249]. In the previous formula the dots indicate terms which do not increase with $S$ and are affected by the non perturbative intrinsic transverse momentum. The $Q^2$ dependence in the slope function is due to scaling violations in parton densities. The explicit form of the slope function in eq. (11-18) to lowest order in $\alpha(Q^2)$ can be obtained from:

$$\langle q_\perp^2 \rangle Q^2 \frac{d\sigma}{dQ^2} = \frac{\alpha^2 m^2 \alpha(Q^2)}{27} \int_0^1 dx_1 \int_0^1 dx_2 \theta(1-z)(1-z)^3 \left\{ \sum_{h=1}^f e_h^2 [q_h^{(1)}(x_1, Q^2)\bar{q}_h^{(2)}(x_2, Q^2) + (1 \leftrightarrow 2)] \right\} \times \left[ \frac{16}{3} + \frac{16z}{(1-z)^2} \right] + \left[ \sum_h e_h^2 (q_h^{(1)}(x_1, Q^2)\bar{q}_h^{(2)}(x_1, Q^2)) G(x_2, Q^2) + (1 \leftrightarrow 2) \right] \times \left[ \frac{3}{2} \frac{1}{1-z} + \frac{1}{4} (1-z) - 2z \right].$$  (11-19)

A remarkable feature of Drell–Yan processes is that the relevant $q_\perp$ distribution is rather easily measured. It is in fact determined by the total transverse momentum of the lepton pair which is directly equal to that of the annihilating parton pair. This is to be contrasted to other processes, for example lepton production, where this test demands a difficult reconstruction of the parton $k_\perp$ from the sum of all momenta of the hadrons in a jet, as discussed in section 8. Thus it is not surprising that, while the direct observation of scaling violations in Drell–Yan processes was so far not possible, the increase of the average $q_\perp$ or $q_\perp^2$ with $\sqrt S$ or $S$ at fixed $\tau$ is instead apparent both in P–N collisions (where the very high energy ISR data are extremely useful in this respect) and in $\pi^-–N$ data (fig. 26). This is a clear cut deviation from the naive parton model in agreement with QCD expectations.

An important task for the near future is the improvement of the above test from the qualitative to the quantitative stage by precise measurement and comparison to the theory of the slope functions $f(\tau, Q^2)$ in eq. (11-18) for different $q_\perp$ moments and different processes. At present there are still gaps to
be filled both in theory and experiment before one can be really conclusive on this point. More statistics and more energy binnings are demanded to experiment. On the theoretical side there are still uncertainties in the slope prediction which may amount to a factor up to 2–3 or so. In P–N or K–N collisions an important source of ambiguity is due to our relative ignorance of the sea and gluon densities which determine the result in these cases. Otherwise in P–N or π–N the ambiguity is essentially due to the inadequacy of the LLA. For example we wrote \( \alpha(Q^2) \) in eqs. (11-18, 11-19) but we could as well have written \( \alpha(\langle q_{\perp}^2 \rangle) \). As \( Q^2 \) is typically of order 50–60 GeV^2 and \( \langle q_{\perp}^2 \rangle \approx 2 \) GeV^2 in the available data the difference makes about a factor of 2, for reasonable values of \( \Lambda \). This is because the LLA is strictly valid only in presence of a single energy scale, namely, in this case, when \( q_{\perp} \approx Q \). An additional related problem has to do with the anomalously large size of the non leading correction to the Drell–Yan cross section, a problem which will be treated in detail later in this section.

One direction of improvement in the theory of the transverse momentum distributions is the very involved study of the next to the leading terms of order \( \alpha^2(Q^2) \). The calculations are still in progress but some results already appeared [172]. These results indicate that a smaller scale than \( Q^2 \) for the running coupling is more appropriate and that the \( K \) factor (see later) on the l.h.s. of eq. (11-19) appears to be roughly compensated by a similar but different effect on the r.h.s.

A different important progress is the resummation of the \( \log(q_{\perp}^2/Q^2) \) terms, according to [150] later modified by Parisi and Petronzio [352] as discussed in section 8, which is of relevance in the kinematic region:

\[
\mu^2 \ll q_{\perp}^2 \ll Q^2. \tag{11-20}
\]

The relevant formulae can be readily adapted to the case of Drell–Yan processes by starting from the definition of \( \Sigma \) in eq. (8-22). At the parton level \( \Sigma \) is determined by a Sudakov form factor squared according to eq. (8-34). The counterpart of \( \Sigma \) for the actual hadronic process is simply obtained by folding with effective parton densities at the scale \( q_{\perp}^2 \) (because it is the maximum allowed transverse momentum that fixes the scaling violations as in eq. (4-14)). Finally the \( q_{\perp} \) distribution is obtained by differentiation of \( \Sigma \). At zero rapidity the result is for example (often referred to as the DDT formula):

\[
\frac{d\sigma}{dQ^2 dq_{\perp}^2 \, dy} \bigg|_{y=0} = \frac{4\pi\alpha^2_{em}}{9Q^2S} \frac{d}{dq_{\perp}^2} \left\{ R^2(Q^2, q_{\perp}^2) \left[ \sum_h e_h^2(q_h^{[1]}(\sqrt{\tau}, q_{\perp}^2) q_h^{[2]}(\sqrt{\tau}, q_{\perp}^2) + (1 \leftrightarrow 2) \right] \right\}. \tag{11-21}
\]
As discussed in section 8, where the set of references can be found, a further improvement of this formula is obtained by transformation to the impact parameter space, where the conservation of transverse momentum in the otherwise independent soft multiple emissions can most simply be taken into account. In analogy with eq. (8-37), by using:

$$\int_0^{2\pi} d\phi \ e^{\pm i \Phi} = 2\pi J_0(z)$$  \hspace{1cm} (11-22)

one finds:

$$\frac{d\sigma}{dQ^2 \ dq^2_\perp \ dy} \bigg|_{y=0} = \frac{1}{2} \int b \ db \ J_0(bq_\perp) \ \frac{4\pi\alpha^2_{em}}{9Q^2 S} \ \sum_k e_k^2 \left[ q_k^{[1]} \left( \sqrt{\tau}, \frac{1}{b^2} \right) q_k^{[2]} \left( \sqrt{\tau}, \frac{1}{b^2} \right) + (1 \leftrightarrow 2) \right]$$

$$\times \exp \left( \frac{1}{\pi} \int \frac{d^2p_\perp}{p_\perp} \frac{\alpha(p_\perp^2)}{\pi} C_F \ \ln \frac{Q^2}{p_\perp} \ [\exp(ibp_\perp) - 1] \right).$$  \hspace{1cm} (11-23)

These developments are of great theoretical interest and still the subject of active research. Recently, for example, a systematic expansion method for the subleading corrections to the DDT formula has been discussed [126, 127]. However, since the kinematical inequalities in eq. (11-20) are to be interpreted in a logarithmic sense, the practical usefulness of the two scale approach is still limited. Presumably by merging the results from the explicit calculations of the terms of order $\alpha^2(Q^2)$ in the $q_\perp$ distributions together with the mentioned expansion of non leading corrections to the DDT formula (the former fixing the unknown parameters of the latter) it will be possible to further sharpen the QCD predictions.

Besides the scaling violations and the increase with energy of the average $q_\perp$ a third effect is also of great interest for establishing the relevance of QCD in Drell–Yan processes. This is the issue of normalization for the cross section. We have seen that the parton formulae in terms of effective parton densities are valid in QCD within the LLA. In order to evaluate the first order corrections it is preliminarily necessary to precisely specify what is meant by quark densities beyond the LLA. As discussed in section 5, this is conveniently done by specifying that the quark densities are to be measured from the structure function $F_2$ in leptoproduction at the same absolute value of the virtual photon mass. The relation between the effective parton densities so defined and the bare densities which appear in eq. (11-15) for the Drell–Yan cross section is given by:

$$q(x, t) = \int_0^1 \frac{dy}{y} \left[ q_0(y) \left\{ \delta \left( 1 - \frac{x}{y} \right) + \frac{\alpha}{2\pi} \left( t p_q \left( \frac{x}{y} \right) + f_q \left( \frac{x}{y} \right) \right) \right\} \right] + G_0(y) \frac{\alpha}{2\pi} \left( t p_G \left( \frac{x}{y} \right) + f_G \left( \frac{x}{y} \right) \right)$$  \hspace{1cm} (11-24)

as obtained from eq. (5-5) and an equation of the form of eq. (4-30) written for $F_2$. By replacement in eq. (11-15) of the bare with the effective densities, in analogy with eqs. (5-10, 5-11), one finds:
\[ \frac{d\sigma}{dQ^2} \approx \frac{4\pi\alpha_s^2}{9Q^2S} \int_0^1 dx_1 \int_0^1 dx_2 \left\{ \left[ \sum_n e_n^2 q_n^{11}(x_1, Q^2)q_n^{22}(x_2, Q^2) + (1 \leftrightarrow 2) \right] \right. \\
\times \left[ \delta(1-z) + \theta(1-z) \frac{\alpha(Q^2)}{2\pi} \left( f_{qG}^{Dy}(z) - 2f_G^2(z) \right) \right] \\
+ \left[ \sum_n e_n^2 (q_n^{11}(x_1, Q^2) + \bar{q}_n^{11}(x_1, Q^2)) G_n^{22}(x_2, Q^2) + (1 \leftrightarrow 2) \right] \theta(1-z) \frac{\alpha(Q^2)}{2\pi} \left( f_{G}^{Dy}(z) - f_G^2(z) \right) \right\}. \]

(11-25)

While the functions \( f_{qG}^{Dy} \) and \( f_G^2 \) have no physical meaning by themselves and in fact depend on the details of the regularization method, the differences in eq. (11-25) are completely well defined. This is a direct consequence of the factorization theorem that makes the leading logarithmic series to disappear when one process is referred to the other. It is also evident from the fact that these differences appear in a relation between measurable quantities. In fact to the accuracy required the gluon density is only needed in the LLA and to that level it also is well defined and measurable, for example from the longitudinal structure function in leptoproduction (eq. (5-16)). An explicit computation leads to ([15, 297], see also [252, 129, 265, 383]):

\[ \frac{\alpha}{2\pi} \left[ f_{qG}^{Dy}(z) - 2f_G^2(z) \right] = \frac{\alpha}{2\pi} C_F \left\{ \frac{1 + 4\pi^2}{3} \delta(1-z) + 2(1+z^2) \left( \ln(1-z) \right) + \frac{3}{1-z} - 6 - 4z \right\}, \]

(11-26)

\[ \frac{\alpha}{2\pi} \left[ f_G^{Dy}(z) - f_G^2(z) \right] = \frac{\alpha}{2\pi} \frac{T}{f} [(z^2 + (1-z)^2) \ln(1-z) + \frac{3}{2} z^2 - 5z^2 + \frac{3}{2}]. \]

(11-27)

Had we chosen to define quark densities by a different combination of structure functions rather than by \( F_2 \) only the polynomial terms in the previous equations would have been modified. This is because of eqs. (5-12 to 5-14) relating the various structure functions.

It turns out that the gluon term, eq. (11-27), is of normal size. It can be neglected at present energies and not too large values of \( \tau \) in all processes where the quark leading term is of valence-valence type, while it makes a non negligible contribution to valence-sea processes. On the other hand the quark correction is quite large at present energies with currently accepted values of \( \alpha(Q^2) \). The physical origin of these large corrections is quite clear: it can be traced back to the continuation of \( q^2 \) from the spacelike region in leptoproduction to the timelike region in lepton pair production and to the difference in phase space between the two processes (the heavy photon is in the initial state in one case and in the final state in the other case).

In fact the large corrections in eq. (11-26) are due to the \( \delta \) function and to the logarithmic term. Most of the contribution to the \( \delta \) function term arises from the continuation of the vertex diagram from \( q^2 < 0 \) to \( q^2 > 0 \). Precisely a term \((\alpha/2\pi)C_F \pi^2 \delta(1-z)\) is due to this effect as was already derived in eq. (9-66). The origin of the logarithmic term is immediately traced back to the expression of the maximum allowed transverse momentum in deep inelastic scattering and in Drell–Yan processes (recall the discussion leading to eq. (5-47)).
Detailed studies of the first order correction show that it is approximately a constant in the limited \( \tau \) range covered by the data and in most of the rapidity range \([15, 16, 298]\). Taken at face value the effect of the first order correction should amount at current energies and \( \tau \) values to a rescaling of the cross section upward by an approximately constant factor of about two (called the \( K \) factor) with respect to parton model prediction.

Note that all tests of the Drell–Yan mechanism discussed in the first part of this section are not affected by the above results. This is immediately clear for the predictions of a linear dependence on the atomic number, of approximate scaling and of an approximately linear rise of \( q_{\perp} \) with energy at fixed \( \tau \) (although the value of the slope is affected). As for the intensity rules they are essentially left unaltered because the whole quark correction is homogeneous in \( q \bar{q} \) in the same way as the leading term. Finally the dominance of the transverse component in the angular distribution is not altered because the vertex diagram and the leading logarithms are not present in the longitudinal component.

The presence of a nearly universal and constant factor of the right magnitude and in the right direction is impressively confirmed by the data in all the measured channels (see for example \([307]\)). What is most impressive, at first sight, is that the experiments appear to closely reproduce all the quantitative features of the first order result, although its magnitude should correspond to a breakdown of the perturbative expansion in \( \alpha(Q^2) \). The answer to this puzzle is to some extent provided by the observation that the largest contributions to this effect can be resummed and exponentiated \([351, 137, 138]\). The terms arising from the continuation of the form factor from spacelike to timelike values of \( q^2 \) can be treated to all orders according to eq. (7-5):

\[
\left| \frac{f(q^2)_{\mathrm{D}Y}}{f(q^2)_{\mathrm{DIS}}} \right|^2 \sim \exp \left[ C_F \frac{\alpha}{2\pi} \pi^2 \right] = 1 + C_F \frac{\alpha}{2\pi} \pi^2 + \cdots
\]  

(11-30)

Similarly the exponentiation of the logarithms arising from the phase space effect follows from arguments as those leading to eq. (5-47). For moments in the \( \tau \) variable one obtains a factor:

\[
\exp \int_0^1 dx \left( x^{n-1} - 1 \right) C_F \frac{1+x^2}{1-x} \int_{Q^2(1-x)/4x}^{Q^2(1-x)^2/4x} \frac{dk^2_{\perp} \alpha(k^2_{\perp})}{k^2_{\perp}}.
\]  

(11-31)

While the possibility of factorizing out these sequences of terms to all orders is quite well founded at the moment, it certainly remains a conjecture that the residual expansion, which starts with small terms in lowest order, is also well behaved in higher orders. An obvious task for the theory is to put these arguments on a more solid basis in order to understand the \( K \) factor more completely. On the experimental side it is important to study this effect in more detail, for example by detecting its dependence on the scaling variables and on \( Q^2 \).

Continuum lepton pair production does not exhaust the lines of research of main interest in the field. We mention for example the study of heavy quark production (see for example \([225, 298a, 55a]\)).
particular the comparison of continuum versus resonance production clearly illustrates the difference in the underlying production dynamics (see for example [44]). Also very important in practice is the extensive work in preparation to the $W^+$ and $Z_0$ production experiments at the $p\bar{p}$ collider (see for example [40]). Then a particular mention is deserved by the analysis of real photon production at large $q_\perp$ in hadron–hadron collisions, which is of interest in perturbative QCD in that class of phenomena that only starts at order $\alpha$ and thus has no naive parton model analogue (see for example [128, 250]). Finally the wide class of large $p_\perp$ phenomena in hadronic physics is in principle within reach of perturbative QCD (see for example [170]).

12. Electro-weak form factors of hadrons

One of the most important applications of perturbative QCD outside the domain of deep inelastic phenomena is the prediction of the asymptotic behaviour of hadron form factors. The theoretical status of these predictions has been gradually consolidated over the last years. The theory at present is completely satisfactory for the meson form factor. For baryon form factors only the magnetic form factor $G_M$, which dominates at large $q^2$, appears to be perhaps computable in perturbation theory. In the following we shall discuss the pion form factor in detail.

For a colour singlet composite particle the asymptotic behaviour of the form factor is completely unaffected by the Sudakov double logarithms. This is because the double logarithms arise from the region of soft gluons. Long wavelength gluons see the particle in its totality and as a consequence of the overall colour neutrality of the particle the total double logarithmic singularity is cancelled order by order.

Simple dimensional counting rules [85, 322], based on the idea that the hard tail of the hadronic wavefunction is dictated by the minimum number of gluon exchanges between constituent legs, fix the power behaviour of spin averaged form factors, apart from logarithmic corrections, to the simple law:

$$F_{\pi}(Q^2) \sim (Q^2)^{(n-1)}$$

(12-1)

where $n$ is the number of constituent quarks and antiquarks. This corresponds to the naive parton model limit in this case and we shall now address the problem of computing the logarithmic departures from this counting rule in QCD in the simplest case of the pion form factor.

The delicate point is the proof in the limit of large $Q^2$, when all terms down by powers are neglected, of the factorization of the meson form factor according to:

$$F_{\pi}(Q^2) = \int_0^1 dx \int_0^1 dy \phi^+(y, Q^2) T(x, y, Q^2) \phi(x, Q^2)$$

(12-2)

as is illustrated in fig. 27. $\phi(x, Q^2)$ can be described as an effective amplitude density, as seen by the photon with $Q^2 = |q^2|$, of finding a constituent quark with fraction $x$ of the pion momentum (the antiquark carrying the remaining fraction $(1-x)$) integrated in transverse momentum up to $\sim Q^2$. Components of the pion wavefunction with more than two constituents are suppressed by powers, in agreement with the counting rules. As for effective parton densities in lepton production, the $Q^2$ dependence in $\phi^+$ and $\phi$ is generated by multiplication, by a $Q^2$ dependent factor including all leading
logarithmic terms and all mass singularities, of the bare wave function in the infinite momentum frame. These two factors, one for each side, are extracted from the 5-point function \( q + \bar{q} + \gamma \to q + \bar{q} \), projected over the proper spin-parity state, which contains the hard parton physics. In a physical gauge the leading logarithmic series of each singular factor is generated by a dressed ladder of gluon rungs, as shown in fig. 27. Gluons along the sides of the ladder are ruled out because the pion carries isospin and thus it is non singlet under the flavour group. \( T(x, y, Q^2) \) is the reduced transition amplitude from a quark with fraction \( x \) to a quark with fraction \( y \) after removal of the singular factors. As it does not contain the leading logarithmic terms, its perturbative expansion can be written down as a series in the running coupling \( \alpha(Q^2) \). The amplitude \( T \) is the analogue of the reduced coefficient \( C_\gamma \gamma[\alpha(Q^2)] \) in the case of moments of structure functions in leptoproduction. The basic formula in eq. (12-2) and the corresponding factorization property have been proven by diagrammatic methods [88] and also by operator expansion and RGE techniques [86, 162, 158].

The first term in the expansion of \( T(x, y, Q^2) \) can be obtained from the Born diagrams shown in the central part of fig. 27:

\[
T(x, y, Q^2) = \frac{16\pi C_F \alpha(Q^2)}{Q^2(1-x)(1-y)}.
\]

(12-3)

The effective amplitude \( \phi(x, Q^2) \) satisfies the evolution equation

\[
Q^2 \frac{d}{dQ^2} \phi(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_0^1 du \phi(u, Q^2) V(u, x)
\]

(12-4)

(we take \( \phi \) real in the following) with a kernel \( V(x, y) \), describing the amplitude for a \( q\bar{q} \) pair with momentum fractions \( x \) and \( 1-x \) to evolve by exchange of a gluon into a \( q\bar{q} \) pair with fractions \( y \) and \( 1-y \), given by [109]:

\[
V(x, y) = C_F \left\{ \frac{1-y}{1-x} \left( 1 + \frac{1}{y-x} \right) \theta(y-x) + \frac{y}{x} \left( 1 + \frac{1}{x-y} \right) \theta(x-y) \right\}.
\]

(12-5)

where the "+" means here, in analogy with eq. (4-65):

\[
\int_0^1 dy \{ \} \varphi(y) = \int_0^1 dy \{ \} [\varphi(y) - \varphi(x)].
\]

(12-6)

Note the relation:

\[
\frac{1-x}{1-y} V(x, y) = \frac{y}{x} V(y, x).
\]

(12-7)

The differential equation for \( \phi \) can be solved by first observing that the kernel \( V(x, y) \) satisfies the eigenvalue equation:
Guido Altarelli, Partons in quantum chromodynamics

\[ \int_0^1 dy \, V(x, y) \, G^{3/2}_n(2y - 1) = A_{n+1}^{qg} G^{3/2}_n(2x - 1) \]  
(12-8)

where \( A_{n+1}^{qg} \) are the moments of the \( P_{qq} \) splitting function, as defined in table 2; \( G^{3/2}_n(z) \) are the Gegenbauer polynomials of order 3/2 which satisfy the relations:

\[ \int_{-1}^1 dz \, (1 - z^2) \, G^{3/2}_n(z) \, G^{3/2}_m(z) = \frac{2(n + 1)(n + 2)}{2n + 3} \delta_{nm} \]  
(12-9)

\[ G^{3/2}_0(x) = 1 \]  
(12-10)

\[ G^{3/2}_1(z) = 3z \]  
(12-11)

\[ (n + 1) \, G^{3/2}_{n+1}(z) = (2n + 3)z G^{3/2}_n(z) - (n + 2) G^{3/2}_n(z). \]  
(12-12)

The connection of \( V(x, y) \) with \( P_{qq} \) is made less mysterious by observing that \( V \) is proportional to the square of the amplitude for emitting or absorbing a gluon.

Then the solution of the evolution equation for \( \phi(x, Q^2) \) can be written down as:

\[ \phi(x, Q^2) = x(1 - x) \sum_{n=0} a_n \, G^{3/2}_n(2x - 1) \left[ \frac{\alpha(\mu^2)}{\alpha(Q^2)} \right]^{d_{n+1}^{qq}} \]  
(12-13)

where \( d_{n+1}^{qq} \) are the non singlet exponents of table 2 and:

\[ a_n = \frac{4(2n + 3)}{(n + 1)(n + 2)} \int_0^1 dy \, G^{3/2}_n(2y - 1) \phi(y, \mu^2). \]  
(12-14)

For a pion, isospin invariance forces the symmetry of \( \phi \) under the exchange \( x \leftrightarrow (1 - x) \). As a consequence only even values of \( n \) are present in the sum of eq. (12-13).

By combining eqs. (12-2, 12-3, 12-13, 12-14) and using the relation valid for all \( n \):

\[ \int_0^1 dx \, x \, G^{3/2}_n(2x - 1) = \frac{1}{2} \]  
(12-15)

one obtains:

\[ F_\pi(Q^2) = \frac{4\pi G_F \alpha(Q^2)}{Q^2} \sum_{n=0,2,4,\ldots} a_n \left[ \frac{\alpha(\mu^2)}{\alpha(Q^2)} \right]^{d_{n+1}^{qq}} \left\{ 1 + O(\alpha(Q^2)) \right\}. \]  
(12-16)

All \( d_{n+1}^{qq} \) being negative except for \( n = 0 \) in which case \( d_1^{qq} = 0 \), it is precisely this term which is dominant at sufficiently large \( Q^2 \). The constant \( a_0 \) is six times the wavefunction at the origin as seen
from eqs. (12-10, 12-14) and can be obtained from the weak decay $\pi \rightarrow \mu \nu$:

$$a_0 = \frac{3}{\sqrt{N}} f_\pi \quad (f_\pi \approx 0.94 \text{ MeV}) \ . \quad (12-17)$$

Thus one finally obtains an absolute prediction for the asymptotic value of the pion form factor:

$$F_\pi(Q^2) \underset{\sigma \rightarrow \infty}{\longrightarrow} \frac{\alpha(Q^2) C_F}{Q^2 N} \frac{36 \pi f_\pi^2}{2}$$

(12-18)

(for a kaon $f_\pi \leftrightarrow f_K$ all the rest being the same). This asymptotic behaviour was first obtained in [183] and [161]. It is not expected to hold but at very large $Q^2$ values. The present data do not allow a test of this prediction. This is also confirmed by a recent study of the next to leading corrections to eq. (12-16) (the factor in curly bracket) in the limit of a pion wavefunction coincident with its asymptotic form [188]. The size of the corrections is rather large in the MS prescription and decreases with taking a smaller scale for the running coupling than $Q^2$.

Interesting results for vector mesons, e.g. the asymptotic expression for the form factor of a longitudinally polarized $\rho$ which dominates at large $Q^2$ over that of a transversely polarized $\rho$, can likewise be derived [89].

A list of related applications to other exclusive processes include exclusive decays of heavy quarkonia [159] and large angle exclusive hadron scattering (still a problematic area) [89, 350, 301, 158].

13. QCD effects in weak non leptonic amplitudes

We mention in this section the important implications of perturbative QCD in the theory of weak non leptonic amplitudes. We shall limit our summary here to recalling the main points and to listing a set of references.

We start by considering a non leptonic weak process induced by charged currents. In the lowest order in the weak coupling the transition matrix element is given by the time ordered product of two weak charged currents folded with the $W$ propagator:

$$H_{FI} \approx g_W^2 \int d^4x D_W(x^2, M_W^2)(F[T[J^\mu (x), J_\mu (0)]]$$

(13-1)

where $M_W$, $g_W$ and $D_W$ are the $W$ boson mass, coupling and propagator respectively. For flavour changing amplitudes the leading contributions in the limit $M_W \rightarrow \infty$ arises from the four fermion operators of dimension six in the short distance operator expansion for the $T$-product [420]. In the massless theory the relevant terms are of the form:

$$H_{FI} \approx \frac{G_F}{\sqrt{2}} \{C_+(t_w, \alpha)(F|O_+(0)|I) + C_-(t_w, \alpha)(F|O_-(0)|I)\}$$

(13-2)

where
\[ t_w = \ln \frac{M^2}{\mu^2} \]  

(13-3)

and \( O_\pm \) are given in terms of current times current operators with appropriate quantum numbers. If we refer for definiteness to charm changing transitions, \( O_\pm \) are given by:

\[ O_\pm = \frac{1}{4} [(s' c)_L(\bar{u} d')_L \pm (\bar{s} d')_L(\bar{u} c)_L] = \frac{N \pm 1}{2N} (s' c)_L(\bar{u} d')_L \pm \sum_A (s' t^A c)_L(\bar{u} t^A d')_L \]  

(13-4)

where the shorthand notation for left-handed (right-handed) currents,

\[ (\bar{q}_1 q_2)_{L,R} = \bar{q}_1 \gamma_\mu (1 \mp \gamma_5) q_2 \]

\[ [q_1 t^A q_2]_{L,R} = \bar{q}_1 \gamma_\mu (1 \mp \gamma_5) t^A q_2 \]  

(13-5)

was used here; \( s' \) and \( d' \) are the Cabibbo-like quark mixtures coupled to the \( c \) and \( u \) quark respectively [282]. The second equality in eq. (13-4) was obtained by Fierz rearrangement. In the free field limit (naive parton limit) the coefficients \( C_\pm \) of the operators \( O_\pm \) in the short distance expansion for the effective Hamiltonian reduce to:

\[ C_+ = C_- = 1 \quad \text{(free fields)} \]  

(13-6)

and correspondingly the effective Hamiltonian is of the simplest current times current form:

\[ H_{\text{eff}}^{A=1} = \frac{G_F}{\sqrt{2}} (s' c)_L(\bar{u} d')_L \quad \text{(free fields).} \]  

(13-7)

\( C_\pm \) obey RGE as true in general for coefficient functions of short distance or light cone operator expansions. \( O_\pm \) have been introduced because they are multiplicatively renormalizable, that is the anomalous dimension matrix is diagonal in this basis. The reason is that \( O_+ \) and \( O_- \) are symmetric and antisymmetric respectively under interchange of the two quarks and, separately, of the two antiquarks. As a consequence they transform according to definite and different irreducible representations of the flavour group \( SU(3) \), which is a symmetry of the massless theory. This symmetry prevents any mixing between the two operators from occurring.

In the LLA from an explicit computation of the diagrams of fig. 28 with \( O_+ \) and \( O_- \) at the four fermion vertex one finds [19, 303]:

\[ C_{\pm}^{\text{LLA}}(t_w) = \left[ \frac{\alpha}{a(t_w)} \right]^{d_\pm} \left[ \frac{\ln(M_\pm^2/A^2)}{\ln(\mu^2/A^2)} \right]^{d_\pm} \]  

(13-8)

Fig. 28. Diagrams for the one loop anomalous dimension matrix of four fermion operators. The two fermion lines are separated to remind the shrinking of the W line.
with
\[ d_\pm = \gamma_\pm^{(1)}/b \] (13-9)
\[ \gamma_\pm^{(0)} = \mp \frac{3}{4\pi} \frac{N \mp 1}{N}. \] (13-10)

Note that in this approximation for \( N = 3 \):
\[
C_{\pm}^{\text{LLA}} = \left[ \frac{\ln(M_\pm^2/A^2)}{\ln(\mu^2/A^2)} \right]^{12/(33-2f)}; \quad C_{\pm}^{\text{LLA}} = (C_{\pm}^{\text{LLA}})^{-1/2} \] (13-11)

so that \( O_- \) is enhanced and \( O_+ \) is suppressed (by a smaller amount) with respect to the free field case.

Recently the next to the leading corrections to the effective Hamiltonian of non leptonic amplitudes have also been computed [13]. At this level of accuracy the effective Hamiltonian can be cast into the form (see eqs. (3-57 to 3-60)):
\[
H_{\text{eff}} = G_F \sqrt{2} \sum \bar{O}_i^{\text{FI}}(\alpha) C_i^{\text{LLA}}(t_\omega) \left[ 1 + \frac{\alpha - \alpha(t_\omega)}{\pi} \rho_i + \ldots \right] \] (13-12)

where
\[
\bar{O}_i^{\text{FI}} = (F_i | O_i | 1)(1 + C_i^{\text{(1)}\alpha}) \] (13-13)

and
\[
\rho_\pm = \frac{\pi}{b} \left[ \gamma_\pm^{(2)} - b C_\pm^{(1)} - b' \gamma_\pm^{(1)} \right]. \] (13-14)

An explicit calculation leads to (in the \( \overline{\text{MS}} \) prescription):
\[
\gamma_\pm^{(2)} - b C_\pm^{(1)} = \frac{1}{(4\pi)^2} \frac{N \mp 1}{N} \left[ \mp \frac{223}{12} + \frac{21}{4N} + \frac{57}{4N} - \frac{10}{3f} \right] \] (\( \alpha = \overline{\text{MS}} \)). (13-15)

Numerically in QCD for \( f = 4 \) (the \( f \) dependence is mild):
\[
C_\pm^{\text{LLA}}(t_\omega) \left[ 1 + \frac{\alpha - \alpha(t_\omega)}{\pi} \rho_\pm \right] = \left[ \frac{\alpha}{\alpha(t_\omega)} \right]^{-0.240} \left[ 1 + \frac{\alpha - \alpha(t_\omega)}{\pi} \left( -0.469 \right) \right] \] (13-16)

Obviously the improved form of \( \alpha(t_\omega) \), including the \( b' \) term, as given in eq. (3-31) has to be replaced in \( C_\pm^{\text{LLA}} \) in eqs. (3-12, 3-16). The magnitude of the corrections is normal and their signs are such as to reinforce the results obtained in the LLA, including the particular pattern of enhancement and suppression.

The make connection with the physical content of the above results we first observe that for \( \mu = M_\omega \)
the effective Hamiltonian is essentially coincident with its naive parton model limit. In fact this statement is clearly true in the LLA, as seen from eq. (13-8), and becomes a matter of definitions beyond it, because we can always define the renormalized $O_\pm$ operators in such a way as to coincide with the free field ones at $\mu = M_w$. However, what one is really interested into is a comparison between the QCD improved Hamiltonian and its naive parton model counterpart at the energy scale of relevance in a typical physical situation for a given channel. For example for charm decays the natural energy scale is of order $m_c$, for strange decays of order $m_s$. A necessary condition for the validity of the RGE results, as obtained in the massless theory, on the relation between the description at the scale $\mu$ and at the scale $M_w$, is clearly that the scale $\mu$ is sufficiently large.

In particular the massless approximation can be justified for charm decays. This is completely clear in a model with four flavours, because in most of the range of energy between $m_c$ and $M_w$ the neglect of all quark masses would certainly be allowed. In presence of heavier flavours we can still maintain the previous statement with some words of justification. The first point is that mixing with operators such as $(\bar{b}c)_L$, $(\bar{u}b)_L$ are irrelevant, at least for Cabibbo allowed transitions, because of the existing bounds on the couplings of heavy quarks. The remaining effects of heavy quarks amount to some distortions at virtual momenta near the $b$ and $t$ thresholds, in the transition region from one value to the other of $f$, the number of excited flavours, which appears in the relevant formulae of the massless theory. Thus a reasonable approximation is to break the range from $m_c$ to $M_w$ into subintervals where the massless theory results can be applied with $f = 4, 5, 6$ respectively. Alternatively one can take some effective value of $f$ between 4 and 6 and simply forget about the existence of heavy quarks.

The situation is different for strange particle decays. In this case $m_s$ is too low to be directly reached in perturbation theory. Moreover below $m_s$, effects of the "large" parameter $\log (m_s/\mu)$ become important [384]. Then a first consequence is that operators with different transformation properties under SU(3), within a given representation of SU(4), acquire different anomalous dimensions. Also "penguin" contributions come into play. These arise from the diagrams in fig. 29 which lead to operators of the form:

$$\sum_A (\bar{q}_1 t^A q_2)_L \sum_f (\bar{q}_f t^A q_f)_L + R$$

where the sums over $A$ and $f$ refer to colour and flavour respectively. In the massless theory these terms are absent because of the GIM cancellation. Their coefficient, apart from mixing angles, is of order $(\alpha/12\pi) \log (m_s/\mu)$, that is rather small, for $\mu \geq m_s$.

The complicated interplay of effects which is present for strangeness changing amplitudes leads to the following schematic picture (for recent discussions see [222, 246]). The most reliable part of the computation is that on the effect of virtual momenta between $m_c$ and $M_w$. The results are practically coincident with those arising from a calculation of $C_\pm$ in the massless theory. For strangeness changing

![Fig. 29. Penguin diagrams.](image-url)
transitions $O_\pm$ are given by:

$$O_\pm \sim \frac{1}{2} \left[ (\bar{s}u)_L (\bar{u}d)_L \pm (\bar{s}d)_L (\bar{u}u)_L \right].$$

(13-17)

It is easily seen that $O_-$ is pure at $\Delta I = 1/2$ ($I =$ isospin), because $u$ and $d$ are combined antisymmetrically leading to $I = 0$ which is then added to the isospin $1/2$ of $\bar{u}$. On the other hand $O_\pm$ contains both $\Delta I = 1/2$ and $\Delta I = 3/2$. From the results on $C_\pm$ in eqs. (13-8 to 13-11) and (13-16) it follows that the short distance effect works in favour of the $\Delta I = 1/2$ rule. While for kaons and hyperons the observed ratio between $\Delta I = 1/2$ and $\Delta I = 3/2$ amplitudes is of order 20, the short distance effect can at most explain the square root of this factor, even including an estimate of the contribution of the range from $m_s$ to $m_\pi$ to the coefficients of the various SU(3) subcomponents which are generated by the splitting of $O_\pm$. The remaining of the observed effect is attributed to low energy effects in the matrix elements (see for example: [354a, 327, 294, 378]). Among these a particular role may be played by the penguin operators. In fact while their coefficient is small, plausible arguments, based on current algebra and chiral symmetry, can however be given to suggest that their matrix elements might be anomalously large, because of the large ratio $m_u/m_d$ where $m_d$ is the current algebra mass of the $u$ quark (of the order of 10 MeV or so) [384]. The discussion on the importance of this effect is still under way ([257, 401]; see also [423]). The experimental observation that the $\Delta I = 3/2$ enhancement is substantially less pronounced in the $\Omega^-$ case, lends support to the idea that much of the $\Delta I = 1/2$ rule for kaons and hyperons is due to low energy effects.

In conclusion the form of the effective Hamiltonian for flavour changing amplitudes is believed to be essentially known. Implications for $CP$ violating amplitudes are treated in [223]; for $\Delta S = 0$ parity violating amplitudes in [14, 90]. The difficulty of testing this Hamiltonian arises from our ignorance of the non perturbative matrix elements, so that up to now no firm experimental evidence for the QCD effect could be established. For the decay of a sufficiently heavy quark eventually it should become possible to evaluate the matrix elements in a partonic approximation as well. This was hoped to be true with fair approximation for charmed particles [304, 12, 278, 164, 92, 397, 91], but evidence of large deviations from the parton picture of matrix elements has been found (a marked deviation from unity of the ratio of the $D^0$ and the $D^+$ lifetimes). Theoretical discussion of charm decays in view of these experiments can be found in [49, 206, 66, 245, 13]. In any case important elements will be acquired as soon as a clear picture of the pattern of decay properties of charmed particles (and of $b$ particles as well) will emerge from the experiments under way.

14. Outlook

It is almost a decade by now since QCD has been formulated as a theory of strong interactions. Within the general framework of relativistic quantum field theories, QCD is the most direct extrapolation into the domain of strong interactions of these principles of renormalizability and gauge invariance that have been established for electro-weak interactions. Over the last years a large amount of sophisticated theoretical and experimental work has been devoted to developing and testing predictions of QCD, mainly in the domain of deep inelastic phenomena, where the property of asymptotic freedom allows perturbative calculations to become predictive. The theory has met so far considerable qualitative and quantitative successes and no significant piece of counter evidence has ever been found. While it is true, in practice, that no single experiment provides by itself a clear cut, uncontroversial and completely quantitative test of QCD, it is also true that the totality of the experimental evidence gathered from many different processes
amounts by now to a quite substantial support in favour of QCD. This continued effort should lead in the near future to a quantitative determination of $\alpha$ at different energy scales from several processes. Other important experimental tasks to be pursued are, for example, the experimental distinction of gluon and quark jets and the disentangling from scaling violations of the gluon to gluon splitting function arising from the three gluon non Abelian gauge coupling. Finally crucial breakthroughs might hopefully occur on the way, as the identification of glueballs or the discovery and the study of top quarkonia. Thus the study of the phenomenology of deep inelastic phenomena and of QCD is certainly to remain one of the most important and active subjects of particle physics in the next decade.

It is a pleasure to acknowledge the collaboration and advice from my friends R.K. Ellis, G. Martinelli, G. Parisi, R. Petronzio. I am also really grateful to Mrs. Lidia Paoluzi and Miss Gianna Vacri for their excellent and dedicated work on the editing of this manuscript.

References

Guido Altarelli, Partons in quantum chromodynamics

Guido Altarelli, Partons in quantum chromodynamics

Partons in quantum chromodynamics