Axial-Vector Vertex in Spinor Electrodynamics

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Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- and higher-order perturbation theory. A corollary is that the radiative corrections to $\pi\Delta$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, the axial-vector current is not conserved. In an Appendix we demonstrate the uniqueness of the triangle diagrams, and discuss a possible connection between our results and the $\pi^0 \rightarrow 2\gamma$ and $\eta \rightarrow 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the $\pi^0$ and the $\eta$ two-photon decays.

INTRODUCTION

The axial-vector vertex in spinor electrodynamics is of interest because of its connections (i) with radiative corrections to $\pi\Delta$ scattering and (ii) with the $\gamma\gamma$ invariance of massless electrodynamics. We will show in this paper, within the framework of perturbation theory, that the axial-vector vertex has anomalous properties which disagree with those found by the formal manipulation of field equations. In particular, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- and higher-order perturbation theory. A corollary is that the radiative corrections to $\pi\Delta$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, the axial-vector current is not conserved.

In Sec. I we derive the usual formulas for the axial-vector divergence and Ward identity, and then show how they are modified by the presence of triangle diagrams. In Sec. II we discuss various consequences of the additional term found in Sec. I. In the Appendix we show that it is not possible to redefine the triangle diagram in a physically acceptable way so as to eliminate the anomalous behavior discussed in Secs. I and II. We also discuss in the Appendix a possible connection between our results and the $\pi^0 \rightarrow 2\gamma$ and $\eta \rightarrow 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the $\pi^0$ and the $\eta$ two-photon decays.

I. AXIAL CURRENT DIVERGENCE AND WARD IDENTITY

We work in the usual spinor electrodynamics, described by the Lagrangian density

$$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma \cdot \nabla - m_0)\psi(x) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad (1)$$

$$F_{\mu\nu}(x) = \frac{\partial A_\mu(x)}{\partial x^\nu} - \frac{\partial A_\nu(x)}{\partial x^\mu}, \quad \gamma \cdot \nabla \equiv \gamma^\mu \frac{\partial}{\partial x^\mu}.$$  

We define the axial-vector current $j_\mu^A(x)$ and the pseudoscalar density $j^S(x)$ by

$$j_\mu^A(x) = \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x),$$  

$$j^S(x) = \bar{\psi}(x) \gamma_5 \psi(x);$$  

the corresponding vertex parts $\Gamma_\mu^A(p,p')$ and $\Gamma^S(p,p')$ are defined by

$$S_f'(p) \Gamma_\mu^A(p,p') S_f'(p')$$  

$$= - \int d^4x d^4y \, e^{ip' \cdot x - ip \cdot y} V(T(\psi(x) j_\mu^A(0) \bar{\psi}(y))_0, \quad (3)$$

$$S_f'(p) \Gamma^S(p,p') S_f'(p')$$  

$$= - \int d^4x d^4y \, e^{ip' \cdot x - ip \cdot y} V(T(\psi(x) j^S(0) \bar{\psi}(y))_0.$$  

Using the equations of motion which follow from Eq. (1), the divergence of the axial-vector current may

1 We use the notation and metric conventions of J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields (McGraw-Hill Book Co., New York, 1965), pp. 377-390. Note that $\epsilon_{0123} = - \epsilon^{0123} = 1.$
easily be calculated to be

$$\frac{\partial}{\partial x} j^\mu(x) = 2im\gamma_5 j^\mu(x).$$

(4)

From Eqs. (3) and (4), we obtain the usual axial-vector Ward identity

$$(p-p')^a \Gamma^b(p,p') = 2m_0 \Gamma^b(p,p') + S_{p'}^{-1}(p) - \gamma_5 \gamma_5 S_{p'}^{-1}(p')^{-1}.$$  

(5)

Our task in this section is to see whether Eqs. (4) and (5), which we have formally derived from the field equations, actually hold in perturbation theory.

To this end, let us rederive Eq. (5) in perturbation theory. It is convenient to write

$$\Gamma^a = \gamma^a \gamma_5 + \Lambda^a,$$

$$\Gamma^b = \gamma^b + \Lambda^b,$$

$$S_{p'}^{-1}(p) = p - m_0 - \Sigma(p),$$

where the vertex corrections $\Lambda^a$ and $\Lambda_b$ and the proper self-energy part $\Sigma(p)$ are calculated using $(p - m_0)^{-1}$ as the free propagator. (Use of the bare mass $m_0 = m - \delta m$ in the free propagator automatically includes the mass-renormalization counter terms.) In terms of $\Lambda^a$, $\Lambda^b$, and $\Sigma$, Eq. (5) becomes

$$(p-p')^a \Lambda^b(p,p') = 2m_0 \delta^a_b(p,p') - \Sigma(p) \gamma_5 \gamma_5 \Sigma(p').$$

(7)

In order to derive Eq. (7), let us divide the diagrams contributing to $\Lambda^a\Lambda^b(p,p')$ into two types: (a) diagrams in which the axial-vector vertex $\gamma_5 \gamma_5$ is attached to the fermion line beginning with external four-momentum $p'$ and ending with external four-momentum $p$; (b) diagrams in which the axial-vector vertex $\gamma_5 \gamma_5$ is attached to an internal closed loop [See Figs. 1(a) and 1(b), respectively]. A typical contribution of type (a) has the form

$$\sum_{k=1}^{2n-1} \prod_{j=1}^{2n-1} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(a)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(b)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(c)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(d)} \left[ \frac{1}{p+p_j-m_0} \right],$$

(8)

where we have focused our attention on the line to which the $\gamma_5 \gamma_5$ vertex is attached and have denoted the remainder of the diagram by $(\cdots)$. Multiplying Eq. (8) by $(p-p')^a$ and making use of the identity

$$\frac{1}{p+p_a-m_0} \frac{1}{p'+p_a-m_0} \gamma_5 = \frac{1}{p+p_a-m_0} \frac{1}{p'+p_a-m_0} (2m_0 \gamma_5)$$

(9)

gives, after a little algebraic rearrangement,

$$\sum_{k=1}^{2n-1} \prod_{j=1}^{2n-1} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(a)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(b)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(c)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(d)} \left[ \frac{1}{p+p_j-m_0} \right]$$

$$\times \prod_{j=1}^{2n-1} \left[ \frac{1}{p'+p_j-m_0} \right] \gamma^{(c)} \left[ \frac{1}{p'+p_j-m_0} \right] \gamma^{(d)} \left[ \frac{1}{p'+p_j-m_0} \right] \gamma^{(e)} \left[ \frac{1}{p'+p_j-m_0} \right] \gamma^{(f)} \left[ \frac{1}{p'+p_j-m_0} \right] \gamma^{(g)} \left[ \frac{1}{p'+p_j-m_0} \right] \gamma^{(h)} \left[ \frac{1}{p'+p_j-m_0} \right] (\cdots)$$

$$- \gamma_5 \prod_{j=1}^{2n-1} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(a)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(b)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(c)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(d)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(e)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(f)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(g)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(h)} \left[ \frac{1}{p+p_j-m_0} \right] \gamma^{(i)} \left[ \frac{1}{p+p_j-m_0} \right] (\cdots).$$

(10)

The first, second, and third terms in Eq. (10) are, respectively, the type-(a) piece of $\Lambda^a$, and the pieces of $-\Sigma(p) \gamma_5$ and $-\gamma_5 \Sigma(p')$ corresponding to the type-(a) piece of $\Lambda^a$ in Eq. (8). Summing over all type-(a) contributions to $\Lambda^a$, we get

$$(p-p')^a \Lambda^a(p,p') = 2m_0 \delta^a_b(p,p') - \Sigma(p) \gamma_5 - \gamma_5 \Sigma(p').$$

(11)

We turn next to contributions to $\Lambda^a$ of type (b). A
typical term is
\[
\int d^4r \text{Tr} \left[ \sum_{k=1}^{2n} \prod_{j=1}^{k-1} \left( \frac{r}{r+p_j-m_0} \frac{1}{r+p_{k-m_0}} \gamma_\alpha \gamma_5 \right) \frac{1}{r+p_{k-m_0}} \right] 
\times (-\cdots). \tag{12}
\]

Multiplying by \((p-p')^\alpha\) and using Eq. (9) gives
\[
\int d^4r \text{Tr} \left[ \sum_{k=1}^{2n} \prod_{j=1}^{k-1} \left( \frac{r}{r+p_j-m_0} \frac{1}{r+p_{k-m_0}} \gamma_\alpha \gamma_5 \right) \frac{1}{r+p_{k-m_0}} \right] 
\times (-\cdots) + \int d^4r \text{Tr} \left[ \sum_{k=1}^{2n} \prod_{j=1}^{k-1} \left( \frac{r}{r+p_j-m_0} \frac{1}{r+p_{k-m_0}} \gamma_\alpha \gamma_5 \right) \frac{1}{r+p_{k-m_0}} \right] 
\times (-\cdots) = 2 \text{tr} \left[ \gamma_\alpha \gamma_5 \right] (\cdots). \tag{13}
\]

The first term in Eq. (13) is the type-(b) contribution to \(A\) corresponding to Eq. (12), while making the change of variable \(r \rightarrow r+p'-p\) in the integration in the second term causes the second and third terms to cancel. This gives, when we sum over all type-(b) contributions,
\[
(p-p')^\alpha A_\alpha^{(b)} (p,p') = 2m_0 A_\alpha^{(b)} (p,p'). \tag{14}
\]

The Ward identity of Eq. (7) is finally obtained by adding Eqs. (11) and (14).

Clearly, the only step of the above derivation which is not simply an algebraic rearrangement is the **change of integration variable** in the second term of Eq. (13). This will be a valid operation provided that the integral is at worst superficially logarithmically divergent, a condition that is satisfied by loops with four or more photons, that is, loops with \(n \geq 2\). However, when the loop is a triangle graph with only two photons emerging (see Fig. 2) we have \(n = 1\), and the integral in Eq. (13) appears to be quadratically divergent. Actually, since
\[
\text{tr} (\gamma_\alpha \gamma_5 \gamma_\beta \gamma_5) = 0,
\]
the integral in the \(n=1\) case is superficially linearly divergent. Since it is well known that translation of a linearly divergent integral is not necessarily a valid operation,\(^5\) we must check carefully to see whether Eq. (14) holds for the triangle graph.

To do this we make use of an explicit expression for the triangle graph calculated by Rosenberg.\(^2\) The sum of the diagram illustrated in Fig. 2 and the corresponding diagram with the two photons interchanged is
\[
\frac{-ie_0^2}{(2\pi)^4} R_{\text{ppp}} = 2 \int d^4r \left( \frac{(-i)}{r+1} \right) \left( \frac{i}{r+1} \right) \left( \frac{-i}{r-m_0} \right) \left( \frac{i}{r-K_3-m_0} \right). \tag{16}
\]

Evaluation of Eq. (16) by the usual regulator techniques leads to the following expression for \(R_{\text{ppp}} [A_i] \) [\(A_i\) denotes \(A_i(k_1,k_2)\)]:
\[
R_{\text{ppp}} (k_1,k_2) = A_1 k_1 e_{k_1 k_2} + A_2 k_2 e_{k_1 k_2} + A_3 k_3 e_{k_1 k_2} + A_4 k_4 e_{k_1 k_2} + A_5 k_5 e_{k_1 k_2},
\]
\[
A_1 = k_1^2 A_4 + k_4^2 A_4,
A_2 = k_2^2 A_4 + k_4^2 A_4,
A_3 = k_3^2 A_4 + k_4^2 A_4,
A_4 = k_4^2 A_4 + k_4^2 A_4,
A_5 (k_1,k_2) = -A_5 (k_3,k_4) = -16 \pi^2 I_{12} (k_1,k_2),
I_{12} (k_1,k_2) = \int d^4r \int 1^{-\infty} dy \; x^y (1-y) k_1^2 + \infty (1-x) k_2^2 + 2xy k_1 k_2 - m_0^2)^{-1}. \tag{18}
\]


\(^{2}\) L. Rosenberg, Phys. Rev. 129, 2766 (1963). In Eq. (16) and Fig. 2, we have labeled the legs of the triangle in accordance with Rosenberg’s notation, which differs from the labeling convention used in Eqs. (12) and (13). Because the integral defining the triangle graph is linearly divergent, the value of the triangle graph is ambiguous and depends on the labeling convention and the method of evaluation of the integral. For example, if Eq. (16) is evaluated by symmetric integration about the origin in \(r\) space, the value of \(R_{\text{ppp}}\) so obtained satisfies the usual axial-vector Ward identity (but is not gauge-invariant with respect to the vector indices). If, on the other hand, Eq. (16) is evaluated by symmetric integration around some other point in \(r\) space, say \(r = k_1\) [or, alternatively, if we integrate symmetrically around \(r = 0\) but label the triangle using the convention of Eqs. (12) and (13)], then the result has an anomalous axial-vector Ward identity. The value in Eq. (17) which we have assigned to \(R_{\text{ppp}}\) is the unique value which is gauge-invariant with respect to the vector indices. Further discussion of the ambiguity in the definition of Eq. (16), and a justification of the specific choice in Eq. (17), are given in the Appendix.
We will also need an expression for the triangle graph with $\gamma_1\gamma_2$ replaced by $2m_0\gamma_0$. Defining

$$\frac{-ie^2}{(2\pi)^4}2m_0R_{\gamma}\gamma = 2\int \frac{d^4q}{(2\pi)^4}(-1)\mathrm{Tr}\left\{ \frac{i}{q + k_1 + m_0} \right\} \times \frac{1}{q - k_2 - m_0} \right\}, \quad (19)$$

we find that

$$R_{\gamma}\gamma = \frac{1}{16\pi^2}m_0\delta_{j0}(k_1, k_2). \quad (20)$$

We are now ready to calculate the divergence of the axial-vector triangle diagram. If the Ward identity holds, we should find

$$- (k_1 + k_2)^\nu R_{\gamma}\gamma = 2m_0R_{\gamma}\gamma, \quad (21)$$

but from Eqs. (16)–(20) we find, instead,

$$- (k_1 + k_2)^\nu R_{\gamma}\gamma = 2m_0R_{\gamma}\gamma + 8\pi^2k_1k_2\epsilon_{\xi\xi'\eta'\rho'}. \quad (22)$$

We see that the axial-vector Ward identity fails in the case of the triangle graph. The failure is a result of the fact that the integration variable in a linearly divergent Feynman integral cannot be freely translated.

The breakdown of the axial-vector Ward identity which we have just found is related to another anomalous property of the triangle graph. To see this, let us consider the behavior of the general axial-vector loop diagram with $2n$ photon vertices (See Fig. 3), as the $2n-1$ independent photon momenta $k_1, \ldots, k_{2n-1}$ approach infinity simultaneously in the manner

$$k_j = n_1j, \quad j = 1, \ldots, 2n-1; \quad q_j \text{ fixed, } \xi \to \infty, \quad (23)$$

while the momentum $p - p'$ carried by the axial-vector current is held fixed. According to Weinberg's theorem, the asymptotic behavior of the loop graph in this limit is

$$\xi^n(\ln \xi)^\beta, \quad (24)$$

where $\beta$ is undetermined by Weinberg's analysis and where $\alpha$ is the maximum of the superficial divergences $\alpha(g)$ of the subgraphs linking the $2n$ photon lines (i.e., linking the momenta which are becoming infinite). For the diagram of Fig. 3 there are two such subgraphs, illustrated in Fig. 4, with superficial divergences $\alpha(1) = -2n+1$ and $\alpha(2) = -2n+3$. Thus, the asymptotic coefficient $\alpha$ is $\alpha(2) = -2n+3$, and comes from the subgraph in which all propagators in the loop are involved. Now Weinberg's theorem always tells us what the maximal asymptotic power of a graph is, but it does not guarantee that the coefficient of the maximal term is nonvanishing. In fact, in the case of the axial-vector loop diagram we will show that the coefficient of the $\xi^{-2n+3}(\ln \xi)^\beta$ term does vanish, so that the leading asymptotic behavior is $\xi^{-2n+3}(\ln \xi)^\beta$, one power lower than is predicted by naive power counting. Let us denote by $L(p - p', m_0; p_1, \ldots, p_{2n-1})$ the graph illustrated in Fig. 3,

$$L(p - p', m_0; p_1, \ldots, p_{2n-1}) = \int d^4q \mathrm{Tr}\left\{ \sum_{h=1}^{2m} \left[ \gamma^{(1)} \frac{1}{r + p_j - m_0} \right] \right\} \times \gamma^{(2)} \frac{1}{r + p - m_0} \gamma^{(3)} \frac{1}{r + p + p' - m_0} \gamma^{(4)} \frac{1}{r + p + p' - m_0} \right\}. \quad (25)$$

S. Weinberg, Phys. Rev. 118, 838 (1960). For a simplified exposition of Weinberg's results, see J. D. Bjorken and S. D. Drell, Ref. 1, pp. 317–330 and pp. 364–368. Weinberg's theorem applies for arbitrary space-like four-vectors $q_i$. There can also be powers of $\ln \xi$, $\ln \ln \xi$, etc., in Eq. (24), which we do not indicate explicitly.

The superficial divergence of the subgraph is obtained, as usual, by adding $1$ for each internal fermion line, $-2$ for each internal boson line, and $+4$ for each internal integration. For the precise definition of subgraph in the general case, see Ref. 4.
Clearly we can write
\[ L(p-p',m_o; p_1, \cdots, p_{2n-1}) \]
\[ = L(p-p', m_o; p_1, \cdots, p_{2n-1}) - L(0,m_o; p_1, \cdots, p_{2n-1}) \]
\[ + L(0, m_o; p_1, p_{2n-1}) - L(0, m_o; p_1, \cdots, p_{2n-1}) \]
\[ + L(0,0; p_1, \cdots, p_{2n-1}). \]

Because differencing the loop graph with respect to either the axial-vector current four-momentum \( p-p' \) or the fermion mass \( m_o \) decreases the degree of divergence by one, terms (A) and (B) on the right-hand side of Eq. (26) have \( n(2) = -2n+2 \), and therefore behave asymptotically as \( \xi^{-2n+2}(\ln \xi)^{\alpha} \). Term (C) on the right-hand side of Eq. (26) can be rewritten as
\[ L(0,0; p_1, \cdots, p_{2n-1}) = 0, \]
proving that the asymptotic behavior of the loop graph is one power better than given by Weinberg's theorem.

The only nonalgebraic step in this proof is the integration by parts with respect to \( r \), an operation which is valid provided that the integration variable in
\[ \int d^4 r \text{ Tr} \left( \frac{\gamma_5}{r+\gamma_5} \right) \]
can be freely translated. This is the same condition as we found above for validity of the axial-vector Ward identity. Thus again, our proof is valid for \( n \geq 2 \), but we expect possible trouble in the case of the triangle graph (\( n=1 \)). From the explicit expression for the triangle graph in Eqs. (17) and (18), we see that if we write \( k_1 = \xi', k_3 = -\xi', \) then as \( \xi \rightarrow \infty \) we find
\[ R_{\gamma \alpha} (k_1, k_3) = -\frac{\alpha}{4\pi} \gamma_5 \epsilon_{\gamma \alpha} = +O(\ln \xi). \]

In other words, the asymptotic power is \( \alpha = -2n+3 \), as given by Weinberg's rules, rather than one power lower, as is the case for the loop graphs with \( n \geq 2 \). It is easy to check that when Eq. (29) is multiplied by \( -(k_1+k_3) \), the term with the anomalous asymptotic behavior agrees, for large \( \xi \), with the term in Eq. (22) which violates the Ward identity. Thus, the breakdown of the axial-vector Ward identity in the triangle graph and the anomalous asymptotic behavior of the triangle graph are basically the same phenomenon.

It is clear that the breakdown of the Ward identity for the basic triangle graph will also cause failure of the Ward identity for any graph of the type illustrated in Fig. 5, in which the two photon lines coming out of the triangle graph join onto a “blob” from which \( 2f \) fermion and \( b \) boson lines emerge. From Eq. (22) for the divergence of the basic triangle graph, it is possible to show that the breakdown of the axial-vector Ward identity in the general case is simply described by replacing Eq. (4) for the axial-vector-current divergence (which we have shown to be incorrect) by
\[ \frac{\partial}{\partial x} j_{\mu} (x) = 2i m_0 \delta_{\mu}^5 (x) + \frac{\alpha_0}{4\pi} \Gamma_{\mu}^5 (x) \delta_{\mu}^5 (x) : \epsilon_{\mu \nu \rho \sigma} \psi (y) \psi (y) : \]
II. CONSEQUENCES OF THE EXTRA TERM

In this section we investigate the consequences of the extra term which we have found in the axial-vector-current divergence [Eq. (30)] and in the axial-vector-current Ward identity [Eq. (32)]. We consider, in particular, the questions of (A) renormalization of the axial-vector vertex, (B) radiative corrections to $\gamma d$ scattering, and (C) the connection between $\gamma d$ invariance and a conserved axial-vector current in massless quantum electrodynamics.

A. Renormalization of the Axial-Vector Vertex

Recently, Preparata and Weisberger\(^7\) have proved the following theorem: If a local current, constructed as a bilinear product of fermion fields, is conserved apart from mass terms, then the vertex parts of both the current and its divergence are made finite by multiplication by the wave-function renormalization constants of the fields from which the current is constructed. If Eq. (4) correctly described the divergence of the axial-vector current in spinor electrodynamics, then the theorem of Preparata and Weisberger would apply in this case. However, we have seen that the divergence is actually given by Eq. (30), and involves an additional term which is not a mass term. The effect of this extra term, we shall see, is to cause the Preparata-Weisberger argument to break down.

First let us review how the Preparata-Weisberger result could be derived if Eq. (4), and the corresponding Ward identity of Eq. (5), were true. Since both $j_0^a$ and $\partial^a$ are local bilinear products of fermion fields, the vertex parts $\Gamma_0^a$ and $\Pi^a$ are multiplicatively renormalizable. Thus we can write

\[
\begin{align*}
\Gamma_0^a(p,p') &= Z_A^{-1} \Gamma_0^a(p,p'), \\
\Pi^a(p,p') &= Z_D^{-1} \Pi^a(p,p'), \\
S_0(p) &= Z_S S_0(p),
\end{align*}
\]

where the tilde quantities are finite (cutoff-independent) and where $Z_A$, $Z_D$, and $Z_S$ are cutoff-dependent renormalization constants. Substituting Eq. (32) into Eq. (5) we get

\[
(p-p')^a \Gamma_0^a(p,p') = (2m_A Z_A/Z_D) \Pi^a(p,p') + \frac{1}{2} \gamma d S_0(p) \left[ (p-p')^a \gamma d S_0(p') \right]^{-1}.
\]

and varying the cutoff gives

\[
0 = \delta(2m_A Z_A/Z_D) \Pi^a(p,p') + \delta(Z_A/Z_d) \\
\times \left[ \gamma d(p) \gamma d(p') \right]^{-1}.
\]

Putting $p, p'$, or both on mass shell then implies that

\[
\delta(2m_A Z_A/Z_D) = \delta(Z_A/Z_d) = 0,
\]

which means that both $2m_A Z_A/Z_D$ and $Z_A/Z_d$ are cutoff-independent, and hence finite. Thus, if Eqs. (4) and (5) were correct, multiplication by the wave-function renormalization constant $Z_A$ would make $\Gamma_0^a$ and $\Pi^a$ finite.

Let us now consider the actual situation, in which the divergence of the axial-vector current is given by Eq. (30) and the axial-vector Ward identity by Eq. (32). The extra term in Eq. (32) first appears in order $\alpha_0^2$ of perturbation theory. [See Fig. 7.] This lowest-order contribution is already logarithmically divergent; introducing a cutoff by replacing the photon propagator $1/(q'^2+i\epsilon)$ with $[1/(q'^2+i\epsilon)]^\nu A^2(\Lambda'-(\Lambda' q'^2+i\epsilon)]$, we find

\[
-I(\alpha_0/4\pi)P(p,p') = -\frac{2}{3} \alpha_0(\alpha_0/\pi)^2 \ln(\Lambda/m^2) (p-p)^a \\
\times \gamma d \gamma d + \alpha_0^2 \ln a + O(\alpha_0^3).
\]

We will also need part of the expression for $\Pi^a(p,p')$ to order $\alpha_0^2$.

\[
\Pi^a(p,p') = \gamma d \left[ \gamma d + (p-p')^a \gamma d \gamma d \right] + O(\alpha_0^2),
\]

Comparing Eqs. (37) and (38), we see that it is impossible to cancel away the divergence in Eq. (37) by adding to it a constant multiple of Eq. (38): A constant counter term of order $\alpha_0^2$ multiplying the leading $\gamma d$ term in Eq. (38) cannot cancel the divergence in Eq. (37), because the latter is proportional to $(p-p)^a \gamma d \gamma d$, while a constant counter term of order $\alpha_0$ multiplying the $(p-p)^a \gamma d \gamma d$ term in Eq. (38) cannot cancel the divergence in Eq. (37) because of the nontrivial functional dependence of $I(p,p')$ on $p$ and $p'$. In other words, the axial-vector divergence with the extra term included,

\[
2m_A \Pi^a(p,p') = -i(\alpha_0/4\pi)P(p,p'],
\]

is not multiplicatively renormalizable.

\*,\* We show in the Appendix that this extra term cannot be eliminated by redefining the triangle graph.

\(1\) G. Preparata and W. I. Weisberger, Phys. Rev. 175, 1965 (1968), Appendix C.
Since multiplicative renormalizability of the divergence was essential to the Preparata-Weisberger argument outlined above, this argument no longer applies. We expect, then, that even after multiplication by $Z_2$, there will still be logarithmically divergent terms in the axial-vector vertex. Such terms first appear in order $\alpha^2$ of perturbation theory, as a result of the diagram shown in Fig. 8; the divergence of Fig. 8 is just a consequence of the anomalous asymptotic behavior of the triangle graph pointed out in Sec. I. Introducing a cutoff in the photon propagator as above, we find that

$$Z_2 \Gamma_{\mu}(p, p') = \gamma_\mu \gamma_4 \left[ 1 - \frac{3}{4} (\alpha / \pi)^3 \ln (\Lambda^2 / m^2) \right] + \alpha_0 \Gamma_{\mu}^{\text{finite}} - \alpha_2 \Gamma_{\mu}^{\text{finite}} + O(\alpha^3).$$

Equation (40) shows explicitly that the axial-vector vertex, while still multiplicatively renormalizable, is not simply made finite by multiplication by the wave-function renormalization constant $Z_2$. Rather, we have [see Eq. (33)]

$$Z_A = Z_2 \left[ 1 + \frac{3}{4} (\alpha_0 / \pi)^3 \ln (\Lambda^2 / m^2) + O(\alpha^3) \right].$$

**B. Radiative Corrections to $\nu \ell$ Scattering**

As an application of Eq. (40), let us consider the radiative corrections to $\nu \ell$ scattering, where $\ell$ is a $\mu$ or an $e$. According to the usual local current-current theory, the leptonic weak interactions are described by the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = (G / \sqrt{2}) j_\mu j_\mu^\dagger,$$

where $G = 10^{-3} / M_{\text{proton}}^2$ is the Fermi constant and where

$$\begin{align*}
\Gamma_{\mu}^{\text{finite}} &= \frac{\bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \nu_\mu \gamma^\lambda (1 - \gamma_5) e}{2},
\end{align*}$$

is the leptonic current. In addition to the usual terms describing muon decay, Eq. (42) contains the terms

$$\begin{align*}
(G / \sqrt{2}) [ \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) &\nu_\mu \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) e ] \\
+ \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) &\nu_\mu \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) e ],
\end{align*}$$

which describe elastic neutrino-lepton scattering. In order to study radiative corrections to the basic $\nu \ell$ scattering process, it is convenient to use a Fierz transformation to rewrite Eq. (44) in the form (the so-called "charge retention ordering")

$$\begin{align*}
&\left[ \frac{G / \sqrt{2}}{2} \right] \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) \nu_\mu \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) e ] \\
+ \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) &\nu_\mu \bar{\nu}_\mu \gamma_5 (1 - \gamma_5) e \right].
\end{align*}$$

The radiative corrections to Eq. (45) may then be obtained simply by calculating the radiative corrections to the charged lepton currents $\bar{\nu}_\mu \gamma_5 (1 - \gamma_5) \nu_\mu$ and $\bar{\nu}_\mu \gamma_5 (1 - \gamma_5) e$, with any reference to the neutrino currents.

Now, application of standard electrodynamics perturbation theory shows that the effect of the radiative corrections to the charged lepton currents is to replace the matrix elements $\bar{\nu}_\mu \gamma_5 (1 - \gamma_5) \nu_\mu$ by $\bar{\nu}_\mu \gamma_5 (1 - \gamma_5) e$ (we use $\nu, e$ to denote spinors here) by

$$\bar{\mu} Z_2^{(\alpha)} \Gamma_{\lambda}^{(\alpha)} \Gamma_{\lambda}^{(\alpha)} = \gamma_\lambda \gamma_4 \left[ 1 - \frac{3}{4} (\alpha_0 / \pi)^3 \ln (\Lambda^2 / m^2) + O(\alpha^3) \right].$$

In Eq. (46), $\Gamma_{\lambda}^{(\alpha)}$ and $\Gamma_{\lambda}^{(\alpha)}$ denote the proper vector and axial-vector vertices, while the wave-function renormalization factors $Z_2^{(\alpha)}$ come from self-energy insertions on the external lepton lines which run into and out of the proper vertices. From the usual electrodynamics Ward identity for the vector part, we know that $Z_2^{(\alpha)} \Gamma_{\lambda}^{(\alpha)}$ and $Z_2^{(\alpha)} \Gamma_{\lambda}^{(\alpha)}$ are finite. On the other hand, Eq. (40) tells us that

$$Z_2^{(\alpha)} \Gamma_{\lambda}^{(\alpha)} \Gamma_{\lambda}^{(\alpha)} = \gamma_\lambda \gamma_4 \left[ 1 - \frac{3}{4} (\alpha_0 / \pi)^3 \ln (\Lambda^2 / m^2) + O(\alpha^3) \right],$$

which means that, on account of the presence of axial-vector triangle diagrams, the radiative corrections to $\nu \mu$ and $\nu \ell$ scattering diverge in the fourth order of perturbation theory. This result contrasts sharply with the fact that the radiative corrections to muon decay or to the scattering reaction $\nu_n + e^- \rightarrow \nu_e + \mu$ are finite to all orders in perturbation theory.\(^8\) The crucial difference between the two cases, of course, is that because of separate muon and electron-number conservation, the current $\bar{\nu}_\lambda (1 - \gamma_5) e$ cannot couple into closed electron or muon loops, and thus the troublesome triangle diagram is not present.

Two points of view can be taken towards the divergent radiative corrections in $\nu \ell$ scattering. One viewpoint is that we know, in any case, that the local current-current theory of leptonic weak interactions cannot be correct, since this theory leads at high energies to nonunitary matrix elements, and since it gives divergent results for higher-order weak-interaction effects.\(^8\) Thus, it is entirely possible that the modifications in Eq. (44) necessary to give a satisfactory weak-interaction theory will also cure the disease of infinite radiative corrections in $\nu \ell$ scattering. The other viewpoint is that we should try to make the radiative corrections to $\nu \ell$ scattering finite, within the framework of a local weak-interaction theory. It turns out that this

---

\(^8\) For recent discussions of the sickness of the local current-current theory and their possible remedies, see N. Christ, Phys. Rev. 176, 2086 (1968); and M. Gell-Mann, M. L. Goldberger, N. M. Kroll, and F. E. Low, Phys. Rev. (to be published).
is possible, if we introduce \( \nu_{\mu} \) and \( \nu_{\nu} \) scattering terms into the effective Lagrangian, so that Eq. (44) is replaced by

\[
(G/\sqrt{2})[\bar{\nu}\gamma_\mu(1-\gamma_5)\mu - \gamma_\nu(1-\gamma_5)\nu] \\
\times [\bar{\nu}\gamma_\lambda(1-\gamma_5)\nu - \gamma_\nu(1-\gamma_5)\nu].
\] (48)

This works because the troublesome extra term in Eq. (30) is independent of the bare mass \( m_0 \), so that it cancels between the muon and electron terms in Eq. (48), giving

\[
\frac{\partial}{\partial x_\lambda} [\bar{\nu}\gamma_\lambda \gamma_\mu - \gamma_\nu(1-\gamma_5)\nu] = 2im_0(\nu)\bar{\nu} \gamma_\mu - 2im_0(\nu)\gamma_\nu \gamma_5. \] (49)

Application of the Preparata-Weisberger argument to Eq. (49) then shows that the radiative corrections to Eq. (48) are finite in all orders of perturbation theory. Experimentally, it will be possible to distinguish between Eq. (48) and Eq. (44) by looking for elastic scattering of muon neutrinos from electrons.

C. Connection Between \( \gamma_5 \) Invariance and a Conserved Axial-Vector Current in Massless Electrodynamics

Finally, let us discuss the effects of the axial-vector triangle diagram in the case of massless spinor electrodynamics [Eq. (1) with \( m_0 = 0 \)]. We will find that the triangle diagram leads to a breakdown of the usual connection between symmetries of the Lagrangian and conserved currents. As in our previous discussions, we begin by describing the standard theory, which holds in the absence of singular phenomena. Let \( \{ \Phi(x) \} = \{ \Phi_1(x), \Phi_2(x), \ldots \} \) and \( \{ \partial_\mu \Phi \} \) be a set of canonical fields and their space-time derivatives, and let us consider the field theory described by the Lagrangian density

\[
\mathcal{L}(x) = \mathcal{L}[\{ \Phi \}, \{ \partial_\mu \Phi \}].
\] (50)

To establish the connection between invariance properties of \( \mathcal{L} \) and conserved currents, we make the infinitesimal, local gauge transformation on the fields,

\[
\Phi_j(x) \to \Phi_j(x) + \Lambda(x) G_j[\{ \Phi(x) \}],
\] (51)

and define the associated current \( J^a \) by

\[
J^a = - \delta \mathcal{L} / \delta (\partial_\mu \Lambda). \] (52)

Then, by using the Euler-Lagrange equations of motion of the fields, we easily find\(^8\) that the divergence of the current is given by

\[
\partial_\mu J^a = - \delta \mathcal{L} / \delta \Lambda. \] (53)

In particular, if the gauge transformation of Eq. (51), with constant gauge function \( \Lambda \), leaves the Lagrangian invariant, then \( \delta \mathcal{L} / \delta \Lambda = 0 \) and the current \( J^a \) is conserved. Thus, to any continuous invariance transformation of the Lagrangian there is associated a conserved current. It is also easily verified that the charge \( Q(t) = \int d^3x J^a(x,t) \) associated with the current \( J^a \) has the properties

\[
dQ(t)/dt = 0,
\]

\[
[Q, \Phi_j(x)] = ig\partial_\mu \Phi_j(x). \] (54a)

Equation (54b) states that \( Q \) is the generator of the gauge transformation in Eq. (51), for constant \( \Lambda \).

Let us now specialize to the case of massless electrodynamics, with Eq. (51) the gauge transformation

\[
\psi(x) \to [1 + i\gamma_\lambda \Lambda(x)]\psi(x). \] (55)

When \( \Lambda \) is a constant and \( m_0 = 0 \), this transformation leaves the Lagrangian of Eq. (1) invariant, so that according to Eq. (53), the associated current \( J^a \) should be conserved. But calculating \( J^a \), we find

\[
J^a = - \delta \mathcal{L} / \delta (\partial_\mu \Lambda) = \psi^\dagger \gamma^\mu \gamma_5 \psi,
\] (56)

which according to Eq. (30) has the divergence

\[
\partial_\mu J^a = (\alpha_0 / 4\pi) F^\mu(\nu) F^\nu(x) \epsilon_{\mu\nu\rho\sigma}. \] (57)

Thus, Eq. (53), which was obtained by formal calculation using the equations of motion, breaks down in this case. We see that because of the presence of the axial-vector triangle diagram, even though the Lagrangian (and all orders of perturbation theory) of massless electrodynamics are \( \gamma_5 \) invariant, the axial-vector current associated with the \( \gamma_5 \) transformation is not conserved.

However, it is amusing that even though there is no conserved current connected with the \( \gamma_5 \) transformation, there is still a generator \( Q^a \) with the properties of Eq. (54). To see this, let us consider the quantity \( j^a \) defined by

\[
\oint j^a(x) dx = \int j^a(x) - \frac{\alpha_0}{\pi} \frac{\partial A^\mu(x)}{\partial x_\mu} \epsilon_{\mu\nu\rho\sigma},
\] (58)

referred to Eq. (30), we see that

\[
\partial_\mu j^\mu(x) = 0.
\] (59)

Although \( j^a \) is conserved, it is explicitly gauge-dependent and therefore is not an observable current operator. But the associated charge

\[
Q^a = \int d^3x j^a(x)
\]

\[
= \int d^3x \left[ \psi^\dagger(x) \gamma^\mu \psi(x) \frac{\alpha_0}{\pi} \frac{\partial A^\mu(x)}{\partial x_\mu} \right].
\] (60)
is gauge-invariant and therefore observable. According to Eq. (59), $Q^j$ is time-independent, and its commutator with $\psi(x)$ (calculated formally by use of the canonical commutation relations) is

$$[Q^j, \psi(x)] = -i\gamma_5 \psi(x) = i[\gamma_5 \psi(x)].$$

(61)

Comparison with Eq. (59) then shows that $\bar{Q}^j$ is the conserved generator of the $\gamma_5$ transformations.

After this manuscript was completed, we learned that Bell and Jackiw had independently studied the anomalous properties of the axial-vector triangle graph, in the context of the $\sigma$ model. In the Appendix we discuss certain questions raised both by the paper of Bell and Jackiw and in conversations with Professor S. Coleman.

Note added in proof. (1) All field quantities appearing in the paper denote *unrenormalized* fields, with the one exception that in Eqs. (A29), (A30), and (A34), $\phi_\sigma^s$ and $\phi_\eta^t$ denote, respectively, the renormalized pion and $\eta$ fields.

(2) It is our claim that Eq. (30) is an *exact* result, valid to all orders in electromagnetism, and similarly that the $\sigma$-model analog, Eq. (A22), is exact to all orders in both the electromagnetic and strong couplings. These conclusions follow in our diagrammatic analysis from the fact that electromagnetic or strong radiative corrections to the basic triangle always involve axial-vector loops with more than three vertices, which satisfy the normal axial-vector Ward identities. A more detailed discussion of this question will be given by the author and W. A. Bardeen (to be published).

(3) Field-theoretic derivations of Eq. (30) have been given by C. R. Hagen (to be published), R. Jackiw and K. Johnson (to be published), B. Zumino (to be published), and R. A. Brandt (to be published). J. Schwinger, Phys. Rev. 82, 664 (1951).

(4) In Eq. (A1) we state that the general form of the triangle diagram is $R_{\sigma\rho}$, Rosenberg’s gauge-invariant expression, plus an arbitrary multiple of $\gamma_5(k_1-k_2)$. We infer this form for the extra term by studying how the triangle graph is changed by shifts in the integration variable. It is easy to see that this is the *only allowed form* for the ambiguity, by noting that the extra term must satisfy the following conditions. (i) The extra term must have the dimensions of a mass; (ii) the extra term must be a three-index ($\sigma\rho\mu$) Lorentz pseudotensor; (iii) the extra term must be symmetric under interchange of the photon variables ($k_1, \sigma$) and ($k_2, \rho$); (iv) the extra term must have *no singularities* in any of the variables $k_1^2, k_2^2, k_1, k_2$ and $m_0$, since the discontinuities of the triangle diagram across its singularities involve no linear divergences and hence are unambiguously contained in Rosenberg’s expression $R_{\sigma\rho}$.

(5) The statement in Ref. 20, that the simultaneous presence of isoscalar and isovector vector mesons affects the $\pi^0 \to 2\gamma$ prediction, is not correct. There will, of course, be an extra term of the form

$$\partial^B(I=1)/\partial x_{\pi}B^I(I=0)/\partial x_{\pi}$$

in the PCAC equation. However, the matrix element of this term relevant to the $\pi^0 \to 2\gamma$ low-energy theorem, when expressed in terms of Fourier transforms of the vector-meson fields, is proportional to

$$\int d^k(k_1, k_2) \gamma(k_1, k_2) |B_{k_1+k_2}|^2 (I=1)B_{-k}^I(I=0)|0\rangle \times (k_1+k_2)^{2k^2} \epsilon_{\pi^0}.$$

Because of photon gauge invariance, the matrix element

$$\langle \gamma(k_1, k_2) |B_{k_1+k_2}|^2 (I=1)B_{-k}^I(I=0)|0\rangle$$

is proportional to $k_1k_2$, and so the two-vector meson term is of order $k_1k_2(k_1+k_2)$. Since the low-energy theorem involves only terms of order $k_1k_2$, the two-vector meson contribution is of higher order and does *not* affect our result. This also means that the extra terms in the PCAC equation proposed recently by R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Letters 27B, 657 (1968), do not in fact lead to a non-null PCAC prediction for $\pi^0 \to 2\gamma$.

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**APPENDIX**

We discuss here the following questions raised both by the recent paper of Bell and Jackiw and in conversations with Professor S. Coleman: (1) Is the expression $R_{\sigma\rho}$ [see Eq. (17)] which we have used for the triangle graph unique, or is it possible to redefine $R_{\sigma\rho}$ by a subtraction in such a way as to eliminate the anomalies discussed in the text? (2) What is the connection between our results and the $\sigma$-model discussion of Bell and Jackiw, and between our results and the physical $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$ decays?

**A. Uniqueness of the Triangle Graph**

The expression for $R_{\sigma\rho}$ in Eq. (17) is obtained from Eq. (16) by the regulator technique of subtracting from

---

Footnotes:

8 Because of an implicit photon field dependence of $j_\sigma(x)$ implied by Eq. (30), $\bar{Q}^j$ does commute with all the photon field variables. The details of showing this are complicated, and will be given elsewhere.

9 J. S. Bell and R. Jackiw (unpublished).
Eq. (16) a loop with \( m_0 \) replaced by \( M \), performing the \( r \) integration, and then letting \( M \to \infty \). Clearly, any mass-independent terms in Eq. (16) will be lost in this process. That a mass-independent term is present can be seen from the fact that when we make the change of integration variable \( r \to r + a k_1 + b k_2 \) in Eq. (16), the result is not left invariant, but rather is changed by multiples of \( \varepsilon_{\text{reg}} k_1^\alpha \) and \( \varepsilon_{\text{reg}} k_2^\alpha \). If we are careful to preserve symmetry with respect to the photon variables, the change will be proportional to \( \varepsilon_{\text{reg}} (k_1 - k_2)^\tau \). The noninvariance of the triangle graph under changes of integration variable is of course just a result of the linear divergence in Eq. (16), and means that in a nonregulator calculation the results obtained for the triangle graph will depend on how the external momenta \( k_1 \) and \( k_2 \) are taken to run through the internal lines. We may express this ambiguity formally by writing that the general expression for the triangle graph is

\[
R_{\text{reg}}[\gamma] = R_{\text{reg}}[\tau] + \varepsilon_{\text{reg}} (k_1 - k_2)^\tau ,
\]

(A1)

with \( R_{\text{reg}} \) the regulator value in Eq. (17).

We easily find the following properties of \( R_{\text{reg}}[\tau] \):

(i) vector index divergence:

\[
k_1^\tau R_{\text{reg}}[\tau] = -\varepsilon_{\text{reg}} k_1^\tau ;
\]

\[
k_2^\tau R_{\text{reg}}[\tau] = -\varepsilon_{\text{reg}} k_2^\tau ;
\]

(A2)

(ii) axial-vector index divergence:

\[
-(k_1 + k_2)^\tau R_{\text{reg}}[\tau] = 2 m_0 R_{\text{reg}} + (8 \varepsilon^3 - 2 \varepsilon) k_1 k_2^\tau \varepsilon_{\text{reg}} ;
\]

(A3)

(iii) asymptotic behavior: Writing \( k_1 = \varepsilon q, k_2 = -\varepsilon q + \rho' - \rho, \) as \( \varepsilon \to 0 \)

\[
R_{\text{reg}}[\tau] \to -\varepsilon (8 \varepsilon^3 - 2 \varepsilon) q^\tau \varepsilon_{\text{reg}} ;
\]

(A4)

(iv) axial-vector meson to two-photon matrix element: If \( l \cdot (k_1 + k_2) = e_1 k_1 + e_2 k_2 = k_1^2 = k_2^2 = 0, \) then

\[
l^\mu e_1^\nu e_2^\nu R_{\text{reg}}[\tau] = \varepsilon l^\mu e_1^\nu e_2^\nu (k_1 - k_2)^\nu \varepsilon_{\text{reg}} ;
\]

(A5)

(v) large \( m_0 \) behavior:

\[
\lim_{m_0 \to 0} R_{\text{reg}}[\tau] = -\varepsilon_{\text{reg}} (k_1 - k_2)^\tau .
\]

(A6)

Referring first to Eqs. (A2)–(A4), we see that when \( \varepsilon = 0 \), which is the case discussed in the text, the triangle graph is gauge-invariant with respect to the photon indices but has an anomalous axial-vector Ward identity and anomalous asymptotic behavior. By contrast, when \( \varepsilon = 4 \pi^2 \) there is no longer gauge invariance with respect to the photon indices, but the axial-vector Ward identity and the asymptotic behavior as \( \varepsilon \to \infty \) are normal. Since the formal proof of gauge invariance for the triangle graph suffers from the same difficulties as does the formal proof of the axial-vector Ward identity, there is no a priori reason to demand gauge invariance with respect to the photon indices as opposed to a normal axial-vector Ward identity, or, for that matter, to demand either. In other words, as long as we consider only the divergence properties of \( R_{\text{reg}}[\tau] \), there is no requirement fixing \( \varepsilon \).

There are, however, two additional restrictions on \( R_{\text{reg}} \) which force us to choose \( \varepsilon = 0 \). First of all, we recall that two real photons can never be in a state with total angular momentum 1, which means that the matrix element for an axial-vector meson to decay into two photons must vanish. In order for our triangle graph to satisfy this requirement, we must have \( l^\mu e_1^\nu e_2^\nu R_{\text{reg}}[\tau] = 0 \) when \( l \) is an axial-vector meson polarization vector satisfying \( l \cdot (k_1 + k_2) = 0 \) and when the photon variables satisfy \( e_1 k_1 = e_2 k_2 = k_1^2 = k_2^2 = 0 \). Referring to Eq. (A5), we see that this requirement forces us to choose \( \varepsilon = 0 \). To check that, even with the constraints on \( l, e, \) etc., the expression \( l^\mu e_1^\nu e_2^\nu R_{\text{reg}}[\tau] \) is in general nonvanishing, choose \( k_1 = (-1, 1, 0, 0), e_1 = (0, 0, 1, 0), \)

\[
k_2 = (-2, 0, 1, 0), e_2 = (0, 1, 0, 0), k_1 + k_2 = (-3, 1, 2, 0),
\]

\[
il = (0, 0, 1), k_1 - k_2 = (1, 1, -2, 0). \]

Secondly, it is physically unreasonable that a loop diagram such as our triangle graph should influence low-energy phenomena in the limit as the mass of the loop fermion becomes infinite. In other words, we expect

\[
\lim_{m_0 \to \infty} R_{\text{reg}}[\tau] = 0, \quad k_1, k_2 \text{ fixed}
\]

(A7)

which according to Eq. (A6) again requires \( \varepsilon = 0 \). Thus, there are strong physical restrictions which uniquely select the regulator value for the triangle graph; in particular, it is not permissible to make the choice \( \varepsilon = 4 \pi^2 \) which eliminates the anomalies discussed in the text.

### B. Connection with Bell and Jackiw and with \( \pi^0 \to 2 \gamma \) and \( \eta \to 2 \gamma \) Decay

In a recent paper, Bell and Jackiw discuss \( \pi^0 \to 2 \gamma \) in the \( \sigma \) model; they find and attempt to resolve a paradox arising from the presence of triangle diagrams. We briefly summarize their work, and then discuss our own interpretation of the paradox, which differs from theirs.\(^{18}\)

Bell and Jackiw use a truncated version of the \( \sigma \) model, in which the charged pion and the neutron fields are omitted. Letting \( \psi, \phi, \) and \( \sigma \) be, respectively, the fields of the proton, the neutral pion, and the scalar meson, the Lagrangian density is\(^8\)

\[
\mathcal{L} = \bar{\psi} [i \gamma \cdot \mathbf{D} - m_0 - g_0 (\sigma + i \phi \gamma_5)] \psi + \frac{1}{2} [m_0 + \phi \gamma_5 - \frac{1}{2} (\phi^2) + (\sigma \gamma_5)^2]
\]

\[
- \frac{1}{2} \mu \phi^2 - \frac{1}{2} (\mu_0 + 2 \lambda_0 f_\sigma^2) \sigma^2 - \lambda_0 (\phi^2 + \sigma^2)^2
\]

\[
- 2 f_\sigma^2 \sigma (\phi^2 + \sigma^2) - \frac{1}{2} f_{\sigma^2} \phi^2 - e_0 \phi \gamma_5 \Lambda \phi ,
\]

(A8)

with the coupling constant \( f_0 \) given by

\[
f_0 = g_0 / (2 m_0) .
\]


18 Our results do not contradict those of Bell and Jackiw, but rather complement them. The main point of Bell and Jackiw is that the \( \phi \) model interpreted in the conventional way, does not satisfy the requirements of PCAC. Bell and Jackiw modify the \( \sigma \) model in such a way as to restore PCAC. We, on the other hand, stay within the conventional \( \sigma \) model, and try to systematize and exploit the PCAC breakdown.
The axial-vector current is

$$j_{5\mu}(x) = \overline{\psi}(x)i\gamma_5\gamma_\mu\psi(x) + 2\left[\sigma(x)\frac{\partial}{\partial x^\mu}\phi(x) - \phi(x)\frac{\partial}{\partial x^\mu}\sigma(x)\right] - f_0^{-1}\frac{\partial}{\partial x^\mu}\phi(x),$$  \hspace{1cm} (A10)

and the divergence of the axial-vector current, as calculated by formal use of the equations of motion, is

$$\frac{\partial}{\partial x_\mu}j^\mu_{5\mu}(x) = \frac{\mu_0^2}{f_0}\phi(x).$$  \hspace{1cm} (A11)

This is, of course, the usual operator PCAC equation.

The paradox noted by Bell and Jackiw is obtained by applying Eq. (A11) to the calculation of \(\pi^0 \to 2\gamma\) decay. Let us concentrate first on the left-hand side of Eq. (A11). The matrix element \(\mathcal{M}_\mu\) of the axial-vector current between the vacuum and a state with two photons has the following general structure, imposed by the requirements of Lorentz invariance, gauge invariance, and Bose statistics [cf. Eq. (17)]:

\[
\mathcal{M}_\mu = e_1^a e_2^b S_{\epsilon\epsilon_{\mu\rho}}(k_1,k_2),
\]

\[
S_{\epsilon\epsilon_{\mu\rho}}(k_1,k_2) = C_1 k_1^\epsilon x_{\mu} + C_2 k_1^\epsilon x_{\rho} + C_3 k_1^\rho x_{\mu} + C_4 k_2^\rho x_{\mu} + C_5 k_1^\mu x_{\rho} + C_6 k_2^\mu x_{\rho} + C_7 \epsilon^\mu \epsilon^\rho x_{\epsilon} x_{\epsilon_{\rho\mu}},
\]

\[
C_1 = k_1^\epsilon k_2^\rho + k_1^\mu k_2^\mu,
\]

\[
C_2 = k_1^\nu k_2^\nu + k_1^\rho k_2^\rho,
\]

\[
C_3 = k_1^\nu k_2^\nu + k_1^\rho k_2^\rho,
\]

\[
C_4 = -C_5(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu),
\]

\[
C_5 = -C_6(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu),
\]

As in Eq. (17), \(k_1\) and \(k_2\) denote the photon four-momenta. The matrix element of the divergence of the axial-vector current is proportional to \((k_1+k_2)^\mu\mathcal{M}_\mu\), and a straightforward algebraic rearrangement using Eq. (A12) shows that

\[
(k_1+k_2)^\epsilon e_1^a e_2^b S_{\epsilon\epsilon_{\mu\rho}}(k_1,k_2) |_{k_1^\mu=0,k_2^\mu=0} = \frac{1}{2}(C_1-C_5)(k_1^\epsilon k_2^\rho + k_1^\rho k_2^\epsilon) + C_6 \epsilon^\mu \epsilon^\rho x_{\epsilon} x_{\epsilon_{\rho\mu}}.
\]

Thus, if we write the matrix element for \(\pi^0 \to 2\gamma\) in the form

\[
\mathcal{M}(\pi^0 \to 2\gamma) = k_1^\epsilon k_2^\rho x_{\epsilon} x_{\epsilon_{\rho\mu}},
\]

then Eqs. (A11) and (A13) tell us that in the \(\sigma\) model (or any other PCAC model), \(F\) vanishes when the pion mass \(m_\pi\) is extrapolated to zero. This statement, of course, must hold in each order of perturbation theory. So let us check by calculating \(\mathcal{M}(\pi^0 \to 2\gamma)\) directly in the \(\sigma\) model in lowest-order perturbation theory, where the only diagram which contributes is the pseudoscalar coupling triangle diagram (i.e., Fig. 2 with \(\gamma_5\gamma_\mu\) replaced by the pion-nucleon coupling \(ig_\sigma\gamma_5\)). We find, comparing with Eqs. (19) and (20), that

\[
\mathcal{M}(\pi^0 \to 2\gamma)_{\text{lowest order}} = \frac{-ie_0^2}{(2\pi)^4}ig_\sigma e_\mu e_\nu R_{\mu\nu},
\]

\[
= k_1^\epsilon k_2^\rho x_{\epsilon} x_{\epsilon_{\rho\mu}},
\]

so that

\[
F_{\text{lowest order}} = \frac{2a_0}{\pi}g_{\text{m_0}0}(k_1,k_2)|_{k_1^\mu=k_2^\nu=x_0}.
\]

Setting \((k_1+k_2)^\mu=0\) then gives

\[
F_{\text{lowest order}} |_{(k_1+k_2)^\mu=0} = \frac{a_0 g_0}{m_0},
\]

which does \textit{not} \textit{vanish}, contradicting the conclusion obtained indirectly from PCAC. The nonzero value of Eq. (A17) is the paradox of Bell and Jackiw.

Bell and Jackiw attempt to circumvent this contradiction by introducing a regulator nucleon field \(\psi_i\), which is quantized with commutators rather than anticommutators. The coupling of the regulator field to the mesons is described by the interaction Lagrangian density

\[
\bar{\psi}i_5(\sigma+i\gamma_5\gamma_\mu)\psi_1;
\]

to maintain the PCAC equation the regulator coupling and mass must satisfy the relation

\[
g_1/m_1 = g_0/m_0.
\]

Thus, as the regulator mass approaches infinity, the regulator coupling to the mesons becomes infinite as well. As a consequence, even in the limit of infinite regulator mass the regulator field triangle diagram makes a contribution to the amplitude for \(\pi^0 \to 2\gamma\) decay,

\[
F_{\text{regulator triangle diagram}} = \frac{a_0 g_1}{m_1} = \frac{a_0 g_0}{m_0}.
\]

The total amplitude is the sum of Eqs. (A16) and (A20), and \textit{does} \textit{vanish} at \((k_1+k_2)^\mu=0\), in accord with the PCAC prediction.

Unfortunately, however, the regulator procedure of Bell and Jackiw leads to grave difficulties when we turn to purely strong interaction phenomena. Let us, in particular, consider the regulator loop contribution to the scattering of 2\(\pi\) \(\sigma\) particles. In the limit of large regulator mass, this loop is proportional to

\[
g_2^{-n}\int d^4x Tr \left[ \frac{1}{x-x_0} \right]^{2n} \approx m_1^2 g_1^{-2n},
\]

and thus, on account of Eq. (A19), becomes infinite as \(m_1 \to \infty\). This means that the regulator procedure of Bell and Jackiw introduces unrenormalizable infinities into the strong interactions in the \(\sigma\) model, and therefore is not satisfactory.
We now suggest a different resolution of the paradox, utilizing the ideas developed in the text.18 As we saw, when triangle graphs are present we cannot naively use the equations of motion to calculate the divergence of the axial-vector current. Rather, we must infer the correct divergence equation from perturbation theory, which tells us that the extra term of Eq. (30) is present. In the σ model, the effect of this extra term is to replace Eq. (A11) by
\[
\frac{\partial}{\partial x_\mu}\phi(x) = \frac{\mu_0^2}{f_0} \left( \rho(x) + F^{\mu \rho} \epsilon_{\mu \rho \nu} \right). \quad (A22)
\]
In other words, the PCAC equation must be modified in the presence of electromagnetic interactions. As a result, the argument leading to the conclusion that \( F \) vanishes at \((k_1 + k_2)^2 = 0\) must be modified. As before, we conclude that the matrix element of the left-hand side of Eq. (A22) between vacuum and two photons vanishes at \((k_1 + k_2)^2 = 0\). But instead of implying that \( \text{PCAC}(\pi^0 \to 2\gamma) \) vanishes, this now tells us that
\[
\text{Re}(\pi^0 \to 2\gamma) = Z_\gamma^{1/2} \times \text{matrix element of } (\mu^2 \phi) = -\mu^2 (\phi') \phi Z_\gamma^{-1/2} \times \text{matrix element of } \left( [\phi' / f_\gamma] F^{\mu \rho} \epsilon_{\mu \rho \nu} \right) \]}
\[
= -\frac{\mu^2}{\mu_0^2} Z_\gamma^{-1/2} \left( -\frac{\alpha}{\pi m_0} \right) k_1 k_2 \epsilon^\nu \epsilon_{\rho \nu} \epsilon_{\rho \nu} \epsilon_{\rho \nu} \epsilon_{\rho \nu} \epsilon_{\rho \nu} \epsilon_{\rho \nu} \epsilon_{\rho \nu} ; \quad (A23)
\]
in other words,
\[
F|_{(k_1+k_2)^2=0} = -\frac{\mu^2}{\mu_0^2} Z_\gamma^{-1/2} \left( -\frac{\alpha}{\pi m_0} \right) . \quad (A24)
\]
[In Eqs. (A23) and (A24), \( Z_\gamma \) is the \( \pi^0 \) wave-function renormalization constant.] To lowest order in perturbation theory, Eq. (A24) agrees with Eq. (A17), so our modified PCAC equation leads to no paradox. In addition, Eq. (A22) yields a bonus: From the derivation of Eq. (A24) it is clear that Eq. (A24) is not just a lowest-order perturbation theory result, but in fact is an exact statement in the σ model. We can reexpress Eq. (A24) in terms of physical quantities using the equation
\[
\frac{\text{g}_\sigma \mu^2}{m_0 \mu_0^2} \frac{Z_\gamma^{-1/2}}{m_N \text{g}_A} = \frac{\text{g}_\sigma(0)}{m_N \text{g}_A} , \quad (A25)
\]
where \( m_N, \text{g}_A(0), \text{g}_A \) are, respectively, the renormalized nucleon mass, the renormalized pion-nucleon coupling constant (evaluated at pion mass zero), and the nucleon axial-vector coupling constant in the σ model. Thus Eq. (A24) becomes
\[
F|_{(k_1+k_2)^2=0} = -\frac{\alpha}{\pi m_N g_A} \frac{\text{g}_\sigma(0)}{m_N g_A} . \quad (A26)
\]
Let us now make the standard PCAC assumption that \( F \) is slowly varying as the pion mass \((k_1 + k_2)^2 \) is varied from \( \mu^2 \) to 0, so that we can use Eq. (A26) for the physical \( \pi^0 \) decay matrix element. We also replace \( g_\sigma(0) \) by the on-shell coupling constant \( g_\sigma \). Using the physical values for \( \mu, m_N, g_\sigma, g_A \),17 we find for the pion lifetime
\[
\tau^{-1} = (\mu^2 / 64 \pi) F^2 = 9.7 \text{ eV} , \quad (A27)
\]
in good agreement with the experimental value\(^{18}\)
\[
\tau^{-1} = (1.12 \pm 0.22) \times 10^{-14} \text{ sec}^{-1} \]}
\[
= (7.37 \pm 1.5) \text{ eV} . \quad (A28)
\]
So we see that the σ model, as interpreted with Eq. (A22), gives a reasonable account of \( \pi^0 \to 2\gamma \) decay.\(^{19}\) This also makes it clear that the use of regulators to cancel away the triangle graph contribution to \( F \) up to terms of order \( \mu^2 / m_N \) will tend to give much too small a value for the \( \pi^0 \to 2\gamma \) matrix element.

The above ideas are readily extended to other field theoretical models, and hopefully, to the physical axial-vector current as well. Let \( \gamma^{13} \) be the third component of the axial-vector octet. (It corresponds to \( \frac{1}{2} \gamma^{13} \) in the model discussed above.) Let us suppose that the world is really described by a field theory, and that there are only spin-0 or spin-\( \frac{1}{2} \) elementary fields.\(^{20}\) We then make the following two assumptions:

(i) The usual PCAC equation,
\[
\frac{\partial}{\partial x^\lambda} F^{\mu \rho} = C_{\mu} \phi \phi_{\rho} , \quad C_{\mu} = m_N g_A / g_{\sigma}(0) , \quad (A29)
\]

17 We take \( g_\sigma = 13.4, g_A = 1.8 \). If we used \( g_\sigma = 2.4 \), then we would get \( \tau^{-1} = 8.9 \text{ eV} \). We can also evaluate Eq. (A26) by using the relation \( g_\sigma(0) / (m_N g_A) = 2q_\sigma / f_\gamma \), with \( f_\gamma \) the charged-pion decay amplitude and \( m_N \) the charged-pion mass. (See S. L. Adler and R. F. Dashen, Ref. 11, pp. 41-43.) This gives \( F|_{(k_1+k_2)^2=0} = -\alpha / (2 \pi \sigma q_\sigma / f_\gamma) \). Using the experimental value \( f_\gamma = 0.96 \mu_\gamma \), we find from Eq. (A27) that \( \tau^{-1} = 7.4 \text{ eV} \).


19 Comparing Eqs. (A26) and (A17), we see that apart from a factor of \( g_\sigma \), our PCAC expression for the \( \pi^0 \) lifetime is the same as the expression obtained from the pseudoscalar coupling triangle graph if one uses the physical nucleon mass and pion-nucleon coupling rather than the bare mass and coupling appearing in Eq. (A17). That the triangle graph, evaluated using physical quantities, gives a good value for \( \pi^0 \to 2\gamma \) decay has been noted by J. Steinberger, Phys. Rev. 76, 1180 (1949); and J. Steinberger (private communication).

20 This assumption is not strictly necessary for the calculation of the \( \pi^0 \to 2\gamma \) rate. If there is also a single elementary neutral vector-meson field \( B^0 \), then there will be an additional term in Eq. (A30) proportional to \( F^{0} B^{0} / q_{e_{\pi} e_{\pi}} \). However, because the gauge-invariant coupling of a massive vector boson to a physical photon vanishes [G. T. Feldman and P. T. Matthews, Phys. Rev. 132, 823 (1963)], this term makes no contribution to the physical \( \pi^0 \to 2\gamma \) decay. In general, there will be no change in the \( \pi^0 \to 2\gamma \) prediction if only isoscalar vector mesons or only isovector vector mesons are present. If both isoscalar and isovector vector mesons are present, there will be additional terms like \( \partial B^{0} (I = 1) / \partial x_{\pi} \partial B^{0} (I = 0) / \partial x_{e_{\pi} e_{\pi}} \), which do affect the \( \pi^0 \to 2\gamma \) prediction.

should, on account of triangle graphs, be replaced by

$$\frac{\partial}{\partial \xi^3} \Phi_{3, \lambda} = C_{\mu} \alpha_{\phi} + S - \frac{\alpha_0}{4\pi} F_{\pi} F_{\pi} \epsilon_{\xi \tau \rho}, \quad (A30)$$

with $S$ a constant.$^{21}$

(ii) If $\Phi_{3, \lambda}$ is expressed in terms of the elementary fields by

$$\Phi_{3, \lambda} = \sum_i g_i \psi_i \gamma^\lambda \psi_i + \text{meson terms}, \quad (A31)$$

then $S$ is given by

$$S = \sum_i g_i Q_i^3, \quad (A32)$$

where the charge of the $i$th fermion is $Q_i$. Equation (A32) means that we count only triangle graphs of the elementary fermions, but do not include triangles involving nonelementary bound states. It may be possible to decide in model calculations whether this rule, which we conjecture, is really correct.

Using Eq. (A30) to calculate the $s^\nu \rightarrow 2\gamma$ matrix element then gives

$$P = - (\alpha/\pi) 2S (g_0/m_{NGA}). \quad (A33)$$

The experimentally measured $s^\nu$ lifetime corresponds$^{22}$ to $|S| = 0.44$; for comparison, $S$ in the $\sigma$ model is $\frac{1}{2} - \frac{1}{2} = \frac{1}{2}$, while $S$ in the quark model is $\frac{1}{2} - \frac{1}{2} = \frac{1}{2}$. More generally, in any triplet model in which the electromagnetic current is a $U$-spin singlet, the triplet charges will be $(Q, Q, Q) = (Q, Q - 1, Q - 1)$ and we have $S = \frac{1}{2} Q^2 - \frac{1}{2} (Q - 1)^2 = Q - \frac{1}{4}$. That is, in triplet models we have $S = (Q, Q)$, where $(Q, Q)$ is the average charge of the triplet particles taking part in both the $\Delta S = 0$ weak $V-A$ current and the $|\Delta S| = 1$ weak $V-A$ current. This means that the condition $(Q, Q) = 2$ necessary for the radiative corrections to the $\Delta S = 0$ and $|\Delta S| = 1$ weak currents to be finite, also predicts a $s^\nu \rightarrow 2\gamma$ rate in good accord with experiment.$^{24}$

$^{21}$ In Eq. (A30), $\phi_{3, \lambda}$ does not necessarily mean a canonical pion field, but only a suitable interpolating field for the pion. For example, in the quark model, $\phi_{3, \lambda}$ would be proportional to $\gamma \tau \gamma \pi$. The separation of $\delta \Phi_{3, \lambda}$ into two terms in Eq. (A30) is made unique by the requirement that $\phi_{3, \lambda}$ and the photon field be dynamically independent, in the sense that $[\phi_{3, \lambda}, A] = [\phi_{3, \lambda}, \xi] = 0$ at equal times.

$^{22}$ If we use instead of Eq. (A33) the formula $P = -(\alpha/\pi)(2S) \propto (\Sigma M_i^2 f_i^2)$, as in Ref. 17, then the experimentally measured $s^\nu$ lifetime gives $|S| = 0.30$.


$^{24}$ This result was noted previously, in the context of the vector dominance model, by N. Cabibbo, L. Maiani, and G. Preparata, Phys. Letters 25B, 51 (1967).

The two-photon decay $\eta \rightarrow 2\gamma$ can be treated in a similar manner. The analog of Eq. (A30) for $\Phi_{3, \lambda}$ is

$$\frac{\partial}{\partial \xi^3} \Phi_{3, \lambda} = C_{\mu} \alpha_{\phi} + S - \frac{\alpha_0}{4\pi} F_{\pi} F_{\pi} \epsilon_{\xi \tau \rho}, \quad (A34)$$

where $S$ is the same constant as in Eq. (A30) and where the factor $3^{-1/2}$ appears because the electromagnetic current is a $U$-spin singlet.$^{25}$ If there were no $\eta \rightarrow X^0$ mixing, then $\phi_{3, \lambda}$ would be the $\eta$ field; in the presence of mixing, $\phi_{3, \lambda}$ would be a mixture of the $\eta$ and $X^0$ fields. In the SU3 limit, one has, of course, $C_\eta = C_\pi$. To get a prediction for the $\eta \rightarrow 2\gamma$ rate from Eq. (A34), we sandwich Eq. (A34) between the $\eta$ state and a two-photon state and make the following three approximations: (i) We neglect $\eta \rightarrow X^0$ mixing; (ii) we take $C_\eta = C_\pi$; (iii) we neglect the left-hand side of Eq. (A34), which makes a contribution of order $\mu_\pi^2$ [equivalently, we assume that the exact prediction $F_x(\mu_\pi^2 = 0) = - (\alpha/\pi) \propto (2S / 3\sqrt{3})/(1/C_\eta)$ can be smoothly extrapolated from $\mu_\pi^2 = 0$ to the physical $\eta$ mass]. These approximations give the standard $SU3$ prediction$^{26}$

$$\Gamma(\eta \rightarrow 2\gamma) = \frac{3}{2} (\mu_\pi/\mu) \Gamma(s^\nu \rightarrow 2\gamma) = (165 \pm 34) \text{ eV}, \quad (A35)$$

about a factor of 8 smaller than the experimental value of

$$\Gamma(\eta \rightarrow 2\gamma) = (1210 \pm 260) \text{ eV}. \quad (A36)$$

In view of the approximations made, the discrepancy is not too disturbing; in particular, the terms of order $\mu_\pi^2$ are by no means negligible, and could easily make a contribution to the $\eta \rightarrow 2\gamma$ matrix element as important as the $S/\sqrt{3}$ term which we have retained.$^{27}$

$^{25}$ The correctness of the factor $1/\sqrt{3}$ is easily verified in the triplet model.

$^{26}$ The factor $\mu_\pi/\mu$ comes from phase space.

$^{27}$ We discuss briefly two other electromagnetic decays to which current algebra methods have been applied: $\omega \rightarrow \pi^0\pi^0$ and $\eta \rightarrow 3\pi$. In the case of $\omega \rightarrow \pi^0\pi^0$ it has been argued by D. G. Sutherland [Nucl. Phys. B2, 433 (1967)] that the usual PCAC equation [Eq. (A11)] implies vanishing of the decay amplitude at zero $s^\nu$ four-momentum. This conclusion, however, is erroneous, and results from the use by Sutherland of an insufficiently general form for the axial-vector-current-vector-meson-photonic vertex. The most general such vertex is given by Eq. (A12); an examination of the reasoning leading to Eq. (A13) shows that Eq. (A13) is valid only when $k_{\nu}^2 = k_{\tau}^2 = 0$. When one of the vectors is massive, as in the case of $\omega$ decay, we find instead that

$$\sum_{i=1}^{2} x^i e_i \sum_{i=1}^{2} x^i = \sum_{i=1}^{2} x^i e_i = 0,$$

contradicting Sutherland’s conclusion. This equation also means that our modified PCAC prediction for $s^\nu \rightarrow 2\gamma$ will be altered when one of the photons is virtual, as is the case in the Primakoff effect.

In the decay $\eta \rightarrow 3\pi$, the only point which we wish to make is that the triangle graphs which we have considered (involving either photons or strongly interacting vector mesons) cannot alter the usual PCAC predictions. The reason is the presence in all matrix elements coming from our extra term of the factor $k_\nu k_\tau \chi e_i e_i e_\tau$, which vanishes at zero four-momentum for the axial-vector vertex. (In the $s^\nu \rightarrow 2\gamma$ case we were always talking about the matrix element left after removal of this factor.)