GAUGE THEORIES*

Ernest S. ABERS and Benjamin W. LEE

Institute for Theoretical Physics, State University of New York,
Stony Brook, N.Y. 11790, USA

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*Permanent address: Department of Physics, University of California, Los Angeles, Calif. 90024, USA
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Introduction

The four-point Fermi theory of the weak interactions in the V–A form, together with the conserved vector current hypothesis, has long been known to be an incomplete theory. Even though it describes well $\mu$ and beta decay, it is not a renormalizable theory, and higher-order effects cannot be calculated. Physicists have long felt that mediating the interaction by vector boson exchanges would solve the problem, but until a few years ago have been unsuccessful at doing so.

Perhaps the most significant development in weak-interaction theory in the last few years, both from a purely theoretical viewpoint and for its possible impact on future experiments, has been the construction of renormalizable models of weak interactions based on the notion of spontaneously broken gauge symmetry. The basic strategy of this construction appeared in 1967 and 1968 in papers by Weinberg and by Salam. In these papers, the weak and electromagnetic interactions are unified in a Yang-Mills gauge theory with the intermediate vector bosons $W^\pm$ and the photons as gauge bosons. The idea itself was not new. What was new in the Weinberg-Salam strategy was to attribute the observed dissimilarities between weak and electromagnetic interactions to a spontaneous breakdown of gauge symmetry.

This mechanism has been studied by Higgs, Brout, Englert, Kibble, Guralnik, Hagen, and others since 1964. It takes place in a gauge theory in which the stable vacuum is not invariant under gauge transformations. In the absence of gauge bosons, non-invariance of the vacuum under a continuous symmetry implies the existence of massless scalar bosons, according to the Goldstone theorem. In a gauge theory, these would-be Goldstone bosons combine with the would-be massless gauge bosons (with two transverse polarizations) to produce a set of massive vector bosons (with three polarizations). In fact, the number of vector mesons which acquire mass exactly equals the number of Goldstone scalar mesons which disappear.

There are two attractive features of the model of Weinberg and Salam. The first is their elegant unifications of the electromagnetic and weak interactions. The second is the suggestion, stressed by these authors, that a theory of this kind might be renormalizable because the equations of motion are identical to those of an unbroken gauge theory. Not much was known about the renormalizability of these theories, and so the development of the Weinberg-Salam theory lay dormant for some years.

Two developments were responsible for the resurgence of interest in these models in 1971. The first was the quantization and renormalization of the Yang-Mills theory. After the pioneering works of Feynman, deWitt, Mandelstam, and Fadeev and Popov, vigorous studies on the renormalizability and the connection between massive and massless gauge theories were carried out by Boulware, Fadeev, Fradkin, Slavnov, J.C. Taylor, Tuytin, Van Dam and Veltman, among others. The second is the detailed study of the $\sigma$-model, which is the simplest field theory which exhibits spontaneous breakdown of symmetry. We learned from this study that the divergences of the theory were not affected by the spontaneous breakdown of symmetry so that the same renormalization counterterms remove the divergences from the theory whether or not the symmetry is spontaneously broken.

In 1971, G. 't Hooft presented a very important paper on manifestly renormalizable formulations of massive Yang-Mills theories wherein the masses of the gauge bosons arise from spontaneous breakdown of the gauge symmetry. His formulation takes explicit advantage of the gauge freedom afforded in such a theory.
Since then, there has been an explosion of interest in the subject, and the study of spontaneously broken gauge theories has become a major industry among theorists. Many models have been proposed and their implications explored. These models all predict new heavy vector mesons or heavy leptons, with interesting experimental implications. The fact that each model has a specific prediction for the properties of weak neutral currents has stimulated experimental interest in trying to detect them.

One of the most difficult problems has been to include hadrons naturally into the scheme. There have been many proposals, some of them very complicated. Surely this is an important subject for further research.

On the other hand, because the models are renormalizable, all higher order corrections are now calculable. There have been many calculations of radiative corrections to the muon anomalous magnetic moment and to weak decay rates, and, in some models, of electromagnetic mass differences. The possibility of doing such calculations has rendered the old "cutoff" methods obsolete.

In the fall of 1972, B.W. Lee gave a series of lectures on these subjects at the State University of New York at Stony Brook. What follows is based on these lectures as expanded and elaborated by both of us afterwards. They are divided into two parts. Part I describes the construction of models with spontaneously broken gauge symmetries, and some of their phenomenological implications. Part II describes the path-integral formulation of quantum field theory, and its application to the question of the renormalizability of these theories.

Part I begins by reviewing the theoretical tools needed to construct the models. Section 1 describes local gauge invariance and its application to non-Abelian gauge groups. Section 2 explains the spontaneous symmetry breaking mechanism and the origin of Goldstone bosons. In section 3 this idea is applied to locally gauge invariant theories, where instead of massless Goldstone bosons one obtains automatically massive vector gauge mesons, without introducing explicitly a symmetry-breaking mass term in the Lagrangian.

The next three sections are a brief review of the phenomenology of weak interactions and conventional theoretical ideas about them. They are far from a complete review of the subject; rather, their purpose was to make the series of lectures self-contained. The subjects covered include a few basic phenomena, the V–A theory, intermediate vector bosons, Cabibbo theory, and a few special topics which will be useful in later lectures.

Section 7 describes the original model of Weinberg and Salam in some detail. Section 8 discusses some experimental implications of this model, the inclusion of hadrons, and the question of neutral currents. Section 9 discusses a class of models with heavy leptons, and describes in some detail the model of Georgi and Glashow. Several other models are briefly described in section 10.

Part II is more mathematical. Its subject is the development of techniques for calculating higher order corrections to scattering amplitudes in spontaneously broken gauge theories, and, ultimately, to show why they are renormalizable. The subject is formulated in the language of path-integral quantization. Since this language is not very familiar to many physicists, we begin by reviewing it in detail.

Section 11 develops the integral-over-paths expression for the time-translation operator, following Feynman. In section 12, the method is extended to quantum field theory, and a general expression for the Green's functions is obtained. Using this principle, in section 13 we obtain the rules for calculating the Green's functions for the Yang-Mills theory in the Coulomb gauge.
The Coulomb gauge is the easiest to quantize in from first principles; what is really needed is the rule for calculating Green's functions and the S-matrix in any gauge. In section 14, the elegant, though somewhat intuitive, prescription for doing this, due principally to Fadeev and Popov, is described. Section 15 contains a formal proof that the Landau and Coulomb gauges give the same renormalized S-matrix.

In section 16, the generating functionals for the proper vertices are obtained and the idea of a superpotential is introduced. The \( \sigma \)-model is discussed in section 17, as an example of the usefulness of this approach in renormalizing theories with spontaneously broken symmetries. In section 18, we outline the renormalization scheme of Bogoliubov, Parasiuk, Hepp, and Zimmermann, whose topological analysis forms the basis for renormalizing gauge theories. The renormalization scheme of 't Hooft and Veltman is described in section 19, and the general application of all these methods to the renormalization of spontaneously broken gauge theories is discussed in section 20. Renormalization is done there in the Landau gauge, and the Feynman rules are derived. A more general class of gauges, called the \( R_f \) gauges, are derived in section 21, and in section 22 it is proved that the S-matrix is the same in all these gauges, and that the Goldstone bosons really do disappear in all gauges. As an illustration, the muon anomalous magnetic moment is computed in the last section, and shown explicitly to be gauge independent.

In the second half of Part II we fail to give a comprehensive review of all the work done by others (among them, notably, 't Hooft and Veltman; Ross and J.C. Taylor) towards proving the renormalizability and physical acceptability of spontaneously broken gauge theories. For the moment we are not equipped to do so. We apologize to our colleagues and the reader for presenting only our views and strategy. It would be presumptuous to assert that the renormalizability has been proved completely by us here or elsewhere. There are still some loose ends in our arguments for that. We do hope, however, to have marshalled sufficiently strong arguments for it, so that serious students of spontaneously broken gauge theories can accept their renormalizability as something more than just a working hypothesis.

These sections are not a final report on a closed subject. Rather, they are a reasonably self-contained course of study about a beautiful idea. Indeed the mathematical elegance and aesthetic appeal of this scheme for constructing models of weak interactions is what convinces many physicists that it may contain a germ of truth. The fact that some of the phenomenological implications of the various models may be tested in the near future is very exciting.

We have benefited greatly for our education in this field, from discussions and correspondence with many of our colleagues, among them: C. Albright, T. Appelquist, W.A. Bardeen, J.D. Bjorken, S. Coleman, C.G. Callan, H.H. Chen, R.R. Dashen, L.D. Fadeev, D.Z. Freedman, P. Freund, D. Fujikawa, H. Georgi, S. Glashow, D.J. Gross, R. Jackiw, W. Lee, Y. Nambu, A. Pais, E. Paschos, J. Primack, H. Quinn, A.I. Sanda, G. 't Hooft, S.B. Treiman, M. Veltman, S. Weinberg, M. Weinstein, L. Wolfenstein, C.N. Yang, J. Zinn-Justin, and B. Zumino. We would like to record our gratitude to them. One of us (ESA) would like to thank Professor C.N. Yang for the hospitality of the Institute of Theoretical Physics. We would like to thank Mrs. Dorothy DeHart and Mrs. Hannah Schlowsky for typing the difficult manuscript.
PART I

GAUGE MODELS OF WEAK AND ELECTROMAGNETIC INTERACTIONS

1. Gauge invariance in classical field theories

Now if we adopt the view that this arbitrary convention should be independently chosen at every space-time point, then we are naturally led to the concept of gauge fields.

C.N. Yang

In field theories one takes as the basic object the Lagrangian density \( \mathcal{L} \) which is a function of all the fields \( \phi_i(x) \) in the theory, and their gradients \( \partial_\mu \phi_i(x) \). The Lagrangian \( L \) itself is the space integral of \( \mathcal{L} \), and the integral over all space and time is called the action \( S \):

\[
S = \int_{-\infty}^{\infty} L(t) \, dt = \int d^4x \, \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)).
\] (1.1)

The equations of motion follow from Hamilton's principle,

\[
\delta \int_{t_1}^{t_2} L(t) \, dt = 0
\] (1.2)

for any \( t_1 \) and \( t_2 \), where the variations of the fields must vanish at \( t_1 \) and \( t_2 \). Hamilton's principle implies that the fields satisfy Euler's equations:

\[
\frac{\delta \mathcal{L}}{\delta \phi_i} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)}.
\] (1.3)

The idea of gauge transformations stems from the old observation that to every continuous symmetry of the Lagrangian there corresponds a conservation law. For example, suppose \( \mathcal{L} \) has no explicit time dependence: the form of \( \mathcal{L} \) is independent of the time \( x^0 \). Under an infinitesimal time translation, each of the fields \( \phi_i \) is changed by

\[
\delta \phi_i(x^0, x) = \phi_i(x^0 + \epsilon, x) - \phi_i(x) = \epsilon \partial_0 \phi_i/\partial x^0 \quad \text{and} \quad \delta (\partial_\mu \phi_i) = \epsilon \partial \mu \partial_\mu \phi_i/\partial x^0.
\] (1.4)

Similarly, \( \delta \mathcal{L} = \epsilon \partial \mathcal{L}/\partial x^0 \):

\[
\epsilon \frac{\partial \mathcal{L}}{\partial x^0} = \sum_i \left[ \frac{\delta \mathcal{L}}{\delta \phi_i} \partial_0 \phi_i + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \partial_\mu \partial_\mu \phi_i \right].
\] (1.5)

Using the equation of motion in the first term, one gets

\[
\epsilon \frac{\partial \mathcal{L}}{\partial x^0} = \epsilon \sum_i \left[ \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \right) \partial_\mu \phi_i + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \partial_\mu \frac{\partial \phi_i}{\partial x^0} \right].
\] (1.6)
or

\[ \frac{\partial \mathcal{L}}{\partial x^0} = \partial_\mu \sum_i \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial x^0} \]  \hspace{1cm} (1.7)  

which can be rewritten

\[ \frac{\partial \mathcal{L}}{\partial x^0} \left[ \mathcal{L} - \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial x^0} \right] = \nabla \cdot \sum_i \frac{\delta \mathcal{L}}{\delta (\nabla \phi_i)} \frac{\partial \phi_i}{\partial x^0}. \]  \hspace{1cm} (1.8)  

The bracket on the left-hand side is the Hamiltonian density \( \mathcal{H}(x) \). Since the fields are required to vanish sufficiently rapidly for large \( |x| \),

\[ \frac{\partial H}{\partial t} = 0 \]  \hspace{1cm} (1.9)  

where \( H = \int d^3x \mathcal{H}(x) \) is the Hamiltonian.

Continuing along these lines it is easy to see that in a Lorentz invariant theory, the energy, momentum and angular momentum can be defined and are conserved. In order for the equations of motion to be covariant, \( \mathcal{L} \) must be a Lorentz scalar density. This is one of the reasons that it is useful to work with \( \mathcal{L} \) instead of \( H \) in a relativistic field theory.

Here we will be interested in conservation laws that are not consequences of classical space-time symmetries. For every conserved quantum number one can construct a transformation on the fields which leaves \( \mathcal{L} \) invariant. The simplest example is electric charge. Suppose each field \( \phi_i \) has charge \( q_i \) (in units of e). Then define a group of transformations on the fields by

\[ \phi_i(x) \rightarrow \exp(-iq_i \theta) \phi_i(x). \]  \hspace{1cm} (1.10)  

The group is the group of unitary transformations in one dimension, \( U(1) \). It is not hard to see that \( \mathcal{L} \) must be invariant under these transformations. Every term in \( \mathcal{L} \) is a product of fields \( \phi_1...\phi_n \). Under the transformation above,

\[ \phi_1(x)...\phi_n(x) \rightarrow \exp \{-i(q_1 + q_2 + ...q_n)\theta\} \phi_1(x)...\phi_n(x). \]

Charge conservation requires that \( \mathcal{L} \) be neutral; therefore the sum \( q_1 + q_2 + ...q_n \) must vanish.

Some terms in \( \mathcal{L} \) contain gradients of the fields as well as the fields themselves. But since \( \theta \) is independent of \( x \), \( \partial_\mu \phi_i \rightarrow \exp(-iq_i \theta) \partial_\mu \phi_i \) as well, so these terms are also invariant. A transformation like (1.10) is called a gauge transformation, or more properly, a gauge transformation of the first kind. The invariance of \( \mathcal{L} \) under the gauge group is called gauge invariance of the first kind, or sometimes global gauge invariance (because \( \theta \) is independent of \( x \)).

The infinitesimal form of (1.10) is

\[ \delta \phi_i = -ieq_i \phi_i \]  \hspace{1cm} (1.11)  

where in (1.11) \( e \) is an infinitesimal parameter. Global gauge invariance can be succinctly stated:

\[ \delta \mathcal{L} = 0. \]  \hspace{1cm} (1.12)  

If \( \mathcal{L} \) depends only on \( \phi_i \) and on \( \partial_\mu \phi_i \), then eq. (1.12) gives
\[ 0 = \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi_i} \delta \phi_i + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \delta (\partial_\mu \phi_i) = -ie \frac{\partial}{\partial x_\mu} \left[ \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} q_i \phi_i \right]. \]

Thus for the operation (1.11) which leaves the Lagrangian invariant there is a conserved current \( J^\mu \):
\[
\partial J^\mu(x, q)/\partial x^\mu = 0
\]

with
\[
J^\mu = i \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} q_i \phi_i.
\]

The gauge group has an infinitesimal generator \( Q \). The \( q_i \) are just the eigenvalues of \( Q \), and \( \exp(-iq_0 \theta) \) is a one dimensional representation of \( U(1) \) generated by \( Q \). In quantized theory the operator \( Q \)
\[
Q = \int d^3J(x, t)
\]
is the charge operator, and
\[
\delta \phi_i = -ie [Q, \phi_i] = -ie q_i \phi_i.
\]

A theory may contain more than one conserved quantity, and be invariant under a more complicated group of transformations than \( U(1) \). The simplest non-Abelian example is isospin. In a theory with isospin symmetry, the fields will come in multiplets which form a basis for representations of the isospin group \( SU(2) \). Then we can define a gauge transformation by
\[
\phi \rightarrow \exp(-i L \cdot \theta) \phi
\]
where \( \phi \) is a column vector and \( L \) is the appropriate matrix representation of \( SU(2) \). For a doublet, for example, \( L = \frac{1}{2} \tau \) (\( \tau \) are the Pauli matrices). For a triplet
\[
L^{ij}_k = -ie \epsilon^{ijk}.
\]

Since the generators, \( T_i \) of the group satisfy
\[
[T_i, T_j] = i \epsilon^{ijk} T_k,
\]
the representation matrices satisfy the same rule : \( [L^i, L^j] = i \epsilon^{ijk} L^k \). The Lagrangian \( \mathcal{L} \) will be invariant under any of the transformations of the group.

Under an infinitesimal transformation,
\[
\delta \phi = -iL \cdot \epsilon \phi
\]
where we may think of \( \epsilon \) as three independent infinitesimal parameters.

Thus if \( \phi \) is a two component isospinor,
\[
\delta \phi = -\frac{i}{2} \tau \cdot \epsilon \phi,
\]
and if \( \phi_i \) are the components of an isovector,
\[ \delta \phi_i = \epsilon^{ijk} \epsilon^l \phi_k. \]

Isospin invariance requires \( \delta \mathcal{L} = 0 \) for all \( \epsilon^l \).

The idea is easily generalized to any internal symmetry Lie group \( G \). Let \( T_i \) be the group generators, and \( c_{ijk} \) the structure constants:

\[ [T_i, T_j] = i c_{ijk} T_k. \quad (1.15) \]

The fields \( \phi_i \) will transform according to some (generally reducible) representation of \( G \). The \( T_i \) are represented by the matrices \( L_i \). A finite gauge transformation is

\[ \phi \rightarrow \exp(-i L \cdot \theta) \phi \]

the corresponding infinitesimal one is

\[ \delta \phi = -i L \cdot \epsilon \phi \quad (1.17) \]

where the number of independent parameters \( \theta^i \) is the dimension of the group. The Lagrangian \( \mathcal{L} \) is invariant under the group: \( \delta \mathcal{L} = 0 \).

It is well known that electrodynamics possesses a formal symmetry larger than gauge transformations of the first kind. The gauge transformation can depend on the space-time point which is the argument of the field:

\[ \phi_i(x) \rightarrow \phi'_i(x) = \exp\{-i q_i \theta(x)\} \phi_i(x). \quad (1.18) \]

This is called a gauge transformation of the second kind, or local gauge transformation. The infinitesimal form of (1.18) is

\[ \delta \phi_i(x) = -i q_i \theta(x) \phi_i(x). \quad (1.19) \]

Here \( \theta(x) \) is an arbitrary infinitesimal function of \( x \). Terms in the Lagrangian which depend only on the fields are obviously invariant under (1.18). Terms with field gradients, such as the kinetic energy term, need more care. The reason is that, from (1.18)

\[ \partial_\mu \phi_i \rightarrow \exp\{-i q_i \theta(x)\} \partial_\mu \phi_i(x) - i q_i \left[ \partial_\mu \theta(x) \right] \exp\{-i q_i \theta(x)\} \phi_i(x). \quad (1.20) \]

The second term is the difference between the way \( \partial_\mu \phi_i \) and \( \phi_i \) transform; but the Lagrangian will be invariant only if it is a product of terms all of which transform like (1.10), with the sum of the \( q_i \) vanishing.

Electrodynamics is made invariant by introducing the photon field according to the following rule, usually called minimal coupling: A gradient of a charged field, \( \partial_\mu \phi_i \), is allowed to appear in \( \mathcal{L} \) only in conjunction with the photon field, \( A_\mu \), in the combination \( (\partial_\mu - ie q_i A_\mu)\phi_i \). \( A_\mu \) is the field of a spin-one meson — the photon — which is our first example of a gauge boson. We require it to transform under local gauge transformations in a special way, so that the combination \( (\partial_\mu - ie q_i A_\mu)\phi_i(x) \) transforms like \( \phi_i(x) \) in (1.10). That is,

\[ (\partial_\mu - ie q_i A_\mu)\phi'_i(x) = \exp\{-i q_i \theta(x)\} (\partial_\mu - ie q_i A_\mu)\phi_i(x). \quad (1.21) \]

Then \( \mathcal{L} \) will be invariant under local gauge transformations as well. Putting in what we know
for $\partial_{\mu} \phi_i(x)$, we get
\[
\exp\{-iq_i \theta(x)\} \partial_{\mu} \phi_i(x) - iq_i[\partial_{\mu} \theta(x)] \exp\{-iq_i \theta(x)\} \phi_i(x) - ieq_i A_{\mu}(x) \exp\{-iq_i \theta(x)\} \phi_i(x)
\]
\[= \exp\{-iq_i \theta(x)\} \partial_{\mu} \phi_i(x) - ieq_i A_{\mu}(x) \exp\{-iq_i \theta(x)\} \phi_i(x). \tag{1.22}
\]
The solution to (1.22) is
\[A_{\mu}'(x) = \frac{1}{e} \partial_{\mu} \theta(x) + A_{\mu}(x) \tag{1.23}\]
or
\[\delta A_{\mu}(x) = -\frac{1}{e} \partial_{\mu} \theta(x). \tag{1.24}\]
In addition to terms coupling the photon field to the charged particle fields, there could be quadratic kinetic energy and mass terms coupling $A_{\mu}$ only to itself. The solution is well-known.

Define the field-strength tensor $F_{\mu\nu}$:
\[F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{1.25}\]
Then $\delta F_{\mu\nu} = 0$ under (1.23), and therefore the photon kinetic energy term, will be gauge invariant if it is constructed out of $F_{\mu\nu}$:
\[\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.26}\]
The coefficient $-\frac{1}{4}$ is dictated by the requirement that the Euler-Lagrange equations result in Maxwell's equations with the conventional normalization of the electric charge $e$.

A photon mass term would have the form $-\frac{1}{2} m^2 A_{\mu} A^{\mu}$, which obviously violates local gauge invariance. The conclusion is that local gauge invariance is impossible unless the photon is massless.

It is supererogatory to observe that the photon was not discovered by requiring local gauge invariance. Rather, gauge transformations were discovered as a useful property of Maxwell's equations. However, in quantum electrodynamics, gauge invariance allows one to derive the Ward-Takahashi identities which in turn allow one to prove many theorems, including, most importantly, as we shall see, the theory's renormalizability.

The generalization of local gauge invariance to non-Abelian groups was first studied by Yang and Mills, for the case of isotopic spin, SU(2). It is elementary to generalize their idea to any internal symmetry group.

Let the group have generators $T_i$ as before:
\[\{T_i, T_j\} = i c_{ijk} T_k. \tag{1.27}\]

A collection of fields transforms according to
\[\phi(x) \rightarrow \phi'(x) = \exp\{-i L \cdot \Theta\} \phi(x) = U(\Theta) \phi(x) \tag{1.28}\]
where $\phi(x)$ is a column vector and $L^j$ is a matrix representation of the generators of the group.

The Lagrangian $\mathcal{L}$ is assumed to be invariant under transformation with constant $\theta^i$. The problem is to construct a theory which is invariant under local gauge transformations $\theta^i(x)$ as well, by introducing vector fields $A^i(x)$ in analogy with electrodynamics.

Under a local gauge transformation
\[\phi(x) \rightarrow U(\Theta) \phi(x) \tag{1.29}\]
and therefore
\[ \partial_\mu \phi(x) \to U(\theta) \partial_\mu \phi(x) + (\partial_\mu U(\theta)) \phi(x). \] (1.30)

The idea is to introduce a covariant derivative \( D_\mu \phi(x) \) which transforms like \( \phi(x) \):
\[ D_\mu \phi(x) \to U(\theta) D_\mu \phi(x). \] (1.31)

Then, if \( \partial_\mu \phi(x) \) appears in \( \mathcal{L} \) only as a part of \( D_\mu \phi(x) \), \( \mathcal{L} \) will be invariant under local gauge transformations.

The covariant derivative \( D_\mu \phi(x) \) is constructed by introducing a vector field \( A_\mu(x) \) for each dimension of the Lie algebra, and defining
\[ D_\mu \phi(x) = (\partial_\mu - igL \cdot A_\mu(x)) \phi(x). \] (1.32)

The coupling constant \( g \), analogous to \( e \), is arbitrary.

How do the \( A_\mu \) transform in order to ensure (1.31)? That is, \( A_\mu^\prime \) must be defined so that the quantity
\[ D_\mu \phi^\prime = \partial_\mu \phi^\prime - ig A_\mu^\prime \cdot L \]
\[ = (\partial_\mu U(\theta)) \phi(x) + U(\theta) \partial_\mu \phi - igA_\mu \cdot L U(\theta) \phi, \] (1.33)
is equal to
\[ U(\theta)(\partial_\mu - igA_\mu \cdot L) \phi. \] (1.34)
The solution is
\[ -igA_\mu \cdot L U(\theta) \phi = -igU(\theta) A_\mu \cdot L \phi - (\partial_\mu U(\theta)) \phi, \] (1.35)
or, since (1.35) must hold for all \( \phi \),
\[ A_\mu \cdot L = U(\theta) A_\mu \cdot L U^{-1}(\theta) - \frac{i}{g} (\partial_\mu U(\theta)) U^{-1}(\theta) \]
\[ = U(\theta) [A_\mu \cdot L - \frac{i}{g} U^{-1}(\theta) \partial_\mu U(\theta)] U^{-1}(\theta). \] (1.36)

We leave it as an exercise to show that the transformations form a group: in particular, if
\[ L \cdot A_\mu^\prime = U(\theta)[A_\mu \cdot L - \frac{i}{g} U^{-1(\theta)} \partial_\mu U(\theta)] U^{-1(\theta)} \]
and
\[ L \cdot A_\mu^\prime = U(\theta')[A_\mu \cdot L - \frac{i}{g} U^{-1(\theta')} \partial_\mu U(\theta') U^{-1}(\theta'), \]
then
\[ L \cdot A_\mu^\prime = U(\theta'')[A_\mu \cdot L - \frac{i}{g} U^{-1(\theta'')} \partial_\mu U(\theta'') U^{-1(\theta'')} \]

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where

\[ U(\theta'') = U(\theta')U(\theta). \]

This transformation rule appears to depend on the representation, but in fact depends only on the commutators \([L^i, L^j]\) whose form is representation-independent. This fact becomes apparent from the infinitesimal transformation:

\[
L^i \delta A^i_\mu = -\frac{1}{g} L^i \partial_\mu \theta^i + i L^i A^i_\mu \theta^i L^i - i \theta^i L^i A^i_\mu L^i \\
= -\frac{1}{g} L^i \partial_\mu \theta^i + i \theta^i A^i_\mu \{L^i, L^j\} = -\frac{1}{g} L^i \partial_\mu \theta^i - \theta^i A^i_\mu c_{ijk} L^k. \tag{1.37}
\]

Since the \(L^i\) are linearly independent,

\[
\delta A^i_\mu = -\frac{1}{g} \partial_\mu \theta^i + c_{ijk} \theta^j A^k_\mu. \tag{1.38}
\]

The transformation properties of \(A^i_\mu\) do not depend on the representation \(L^i\).

Next we must construct the analog of the kinetic energy term, i.e. the term \(\mathcal{L}_0\) which contains only the fields \(A^i_\mu\) and their derivatives. Because these fields do not all carry zero quantum numbers under all the \(T^i\) (unlike the photon, which is electrically neutral), \(\mathcal{L}_0\) cannot have the simple form it has in electrodynamics. In fact, from (1.38), it is easy to see that

\[
\delta [\partial_\mu A^i_\mu - \partial_v A^i_v] = c_{ijk} \theta^j (\partial_\mu A^k_\mu - \partial_v A^k_v) + c_{ijk} [(\partial_\mu \theta^j) A^k_\mu - (\partial_v \theta^j) A^k_v]. \tag{1.39}
\]

\(\mathcal{L}_0\) will be invariant if it is constructed out of a tensors \(F^i_{\mu\nu}\) according to

\[
\mathcal{L}_0 = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} \tag{1.40}
\]

provided the \(F^i_{\mu\nu}\) transform covariantly like a set of fields in the regular (adjoint) representation of \(G\). Therefore we must add something to \(\partial_\mu A^i_\mu - \partial_v A^i_v\) to cancel the unwanted terms in (1.39). Now from (1.38)

\[
c_{ijk} \delta [A^i_\mu A^k_\mu] = -\frac{c_{ijk}}{g} [((\partial_\mu \theta^j) A^k_\mu - (\partial_v \theta^j) A^k_v) + c_{ijk} c_{ilm} \theta^l A^m_\mu A^k_\mu + c_{ijk} c_{klm} \theta^l A^m_\mu A^k_\mu]. \tag{1.41}
\]

The first terms (times \(g\)) can just cancel the unwanted piece of (1.39). The last two terms can be rewritten, using the antisymmetry of the structure constants, as

\[
[c_{ilm} c_{klm} - c_{ijk} c_{kml}] \theta^l A^m_\mu A^k_\mu. \tag{1.42}
\]

Let \(T^i\) stand also for the regular representation matrices. Then \((T^i)^{jk} = -i c_{ijk}\), and the bracket in (1.42) is

\[
c_{ilm} c_{klj} - c_{ilm} c_{ljk} = [T^i, T^j]_{mj} = i c_{ijk} (T^k)_{mj} = c_{ilk} c_{kjm}.
\]

Therefore,
c_{ijk} \delta [A_{\mu j} A_{\nu k}] = -\frac{c_{ijk}}{g} \left[ (\partial_\mu \theta^l) A^k_\nu - (\partial_\nu \theta^l) A^k_\mu \right] + c_{ilk} \theta^l c_{kj m} A_{\mu j} A_{\nu m}.

So define

\[ F^i_{\mu \nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A_{\mu j} A_{\nu k} \]  \hfill (1.43)

then

\[ \delta F^i_{\mu \nu} = c_{ijk} \theta^l F^k_{\mu \nu} \]  \hfill (1.44)

and \( \mathcal{L}_o = -\frac{1}{4} F^i_{\mu \nu} F^{i \mu \nu} \) is invariant. Under finite gauge transformations, \( U(\theta) = \exp(-iL^i \theta^i) \), \( F^i_{\mu \nu} \) transforms as \( F^i_{\mu \nu} \cdot L \rightarrow U(\theta) F^i_{\mu \nu} \cdot L U^{-1}(\theta) \) so that \( \text{Tr}(F^i_{\mu \nu} \cdot L)^2 \sim F^i_{\mu \nu} \cdot F^{i \mu \nu} \) is invariant.

Again, a mass-term of the form \( \frac{1}{2} m^2 A^i_\mu A^i_\mu \) would violate the local gauge invariance.

We conclude by summarizing the construction of local gauge theories with non-Abelian symmetries. Start with a Lagrangian \( \mathcal{L}_i (\phi_i, \partial_\mu \phi_i) \) invariant under a Lie group \( G \) with generators \( T_i \) and structure constants \( c_{ijk} \). The fields transform according to some representation \( \exp(-iL^i \theta^i) \) of the group, with constant \( \theta^i \). Add to the theory a set of vector fields \( A^i_\mu \), one for each \( T_i \). The full Lagrangian is

\[ \mathcal{L} = \mathcal{L}_o + \mathcal{L}_i (\phi_i, (\partial_\mu - ig A^i_\mu \cdot L) \phi_i). \]  \hfill (1.45)

The first term is

\[ \mathcal{L}_o = -\frac{1}{4} F_{\mu \nu} \cdot F^{\mu \nu} \]  \hfill (1.46)

where

\[ F^i_{\mu \nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A^j_\mu A^k_\nu. \]  \hfill (1.47)

The transformation rule for the gauge bosons is

\[ L^'i \cdot A^i_\mu = U(\theta) L \cdot A^i_\mu U^{-1}(\theta) - \frac{1}{g} (\partial_\mu U(\theta)) U^{-1}(\theta). \]  \hfill (1.48)

where here \( \theta^i \) is a function of \( x \).

One final note. If \( G \) is a direct product of two or more subgroups, the coupling constants \( g \) associated with each subgroup need not be the same.

Bibliography

The standard references on non-Abelian gauge theory are:

The Ward-Takahashi identities were first discussed in:

We shall discuss the use of these identities in gauge theories extensively in Part II. In the generalized sense, these identities are the precise mathematical statements about the effects of gauge invariance (or other symmetries) of the Lagrangian on Green's functions.
2. Spontaneously broken symmetries

If my view is correct, the universe may have a kind of
domain structure. In one part of the universe you may
have one preferred direction of the axis; in another part,
the direction of the axis may be different.

Y. Nambu

Nature seems to possess useful symmetries which, unlike electric charge conservation, are not
exact symmetries of the S-matrix. Familiar examples are isospin, strangeness and SU(3). A traditional way of thinking about them is to imagine that the Lagrangian possesses a part which is
exactly symmetric and another, in some sense "small", term which violates the symmetries. This idea is behind our conventional picture of a "hierarchy" of interactions — strong, electromagnetics and weak — in which the stronger interactions possess more symmetry than the weaker ones. Another type of symmetry is PCAC, which even in the exact symmetry limit is not a symmetry of the physical spectrum, that is, particles do not occur in equal-mass multiplets which can be assigned to a representation of the group (in this case SU(2) × SU(2)). Nevertheless the Ward–Takahashi identities and current-algebra predictions of SU(2) × SU(2) symmetry are physically useful.

By now it is well-known that the second kind of symmetry can be obtained from an exactly symmetric Lagrangian, provided that the physical vacuum is not invariant under the symmetry group. Such a symmetry is popularly called a "spontaneously broken symmetry". The mechanics of how this works is the subject of this section. Then we will go on to see what wonderful things happen when the symmetry of the Lagrangian is made into a local gauge symmetry of the kind described in the first lecture.

It is instructive to begin by understanding how a field theory is like a collection of anharmonic oscillators. A simple Lagrangian density with only a single scalar field is

\[ \mathcal{L} = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi) - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4. \]  

(2.1)

For simplicity, let there be only one space dimension. Then the Lagrangian is

\[
L = \int_{-\infty}^{\infty} \mathcal{L}(x, t) \, dx
\]

\[ = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4 \right]. \]  

(2.2)

We may think of \( \phi(x, t) \) as being a canonical coordinate at each \( x \). Divide space into unit cells of length \( \epsilon \) labeled by the coordinate \( x_i; x_i - x_{i-1} = \epsilon \). Then we may replace the integral defining \( L \) by a discrete sum. The discrete coordinates are \( q_i(t) = \phi(x_i, t) \), and \( L \) becomes

\[
L = \sum_{i=-\infty}^{\infty} \left[ \frac{1}{2} \left( \frac{dq_i}{dt} \right)^2 - \frac{1}{2\epsilon^2} (q_i - q_{i-1})^2 - \frac{1}{2} \mu^2 q_i^2 - \frac{1}{4} \lambda q_i^4 \right]. \]  

(2.3)
The second term represents a coupling between coordinates at adjacent points, and the last term makes the potential anharmonic. The canonical momentum is

\[ p_i = \frac{dq_i}{dt} \]

and if we define

\[ V(z) = \frac{1}{2} \mu^2 z^2 + \frac{1}{4} \lambda z^4 \]

the Hamiltonian is

\[ H = \sum_{i=-\infty}^{\infty} \left[ \frac{1}{2} p_i^2 + \frac{1}{2e^2} (q_i - q_{i-1})^2 + V(q_i) \right]. \tag{2.4} \]

Field oscillations are bounded only if \( \lambda > 0 \), which we therefore require. In the usual case \( \mu^2 > 0 \) also. To do any kind of perturbation calculation, we must find the minimum of the potential,

\[ \sum_i \left[ \frac{1}{2e^2} (q_i - q_{i-1})^2 + V(q_i) \right] \]

and start with the unperturbed harmonic oscillator solutions as the zeroth approximation (these are the "free field" solutions of field theory). Whatever \( V \) is, we must have \( q_i = q_{i-1} \) at the minimum of the potential; i.e., all the \( q_i \) are equal. If \( \mu^2 > 0 \), the function \( V \) looks like fig. 2.1 and the minimum occurs at \( q_i = 0 \). On the other hand, if \( \mu^2 < 0 \), the potential looks like fig. 2.2. Now \( q = 0 \) is not a minimum. There are two symmetric minima at \( q = \pm \sqrt{-\mu^2/\lambda} \).

In field theory, the ground state is the vacuum. What we have shown in an admittedly heuristic manner is that if \( \mu^2 < 0 \) the vacuum expectation value of the field is not zero; rather, it is independent of \( x (q_i = q_{i-1}) \) and has the value \( \pm \sqrt{-\mu^2/\lambda} \) to zeroth order in perturbation theory.

Let \( \nu \) be the vacuum expectation value of the field:

\[ \langle \phi \rangle_0 = \nu = \pm \sqrt{-\mu^2/\lambda}. \tag{2.5} \]

Either value of \( \nu \) may be chosen, but not both. We may by convention choose the plus sign, since \( L \) is invariant under \( \phi \to -\phi \).

The only symmetry this simple Lagrangian possesses is reflection invariance: \( \phi \to -\phi \). Clearly the new vacuum is not an eigenstate of this operation, since \( \nu \neq -\nu \). In this way the symmetry is "spontaneously" broken. Define a new field \( \phi' \) by

\[ \phi' = \phi + \nu. \]

Fig. 2.1. The potential function for positive \( \mu^2 \).

Fig. 2.2. The potential function for negative \( \mu^2 \).
\( \phi' = \phi - v \)

then

\[ \langle \phi' \rangle_0 = 0 \]

so we can do ordinary perturbation theory in \( \phi' \). In terms of \( \phi' \) (up to a constant)

\[ \mathcal{L} = \frac{1}{2} (\partial^\mu \phi' \partial^\nu \phi') + \mu^2 \phi'^2 - \lambda \nu \phi'^3 - \frac{1}{4} \lambda \phi'^4. \] (2.6)

The bare states have (positive) mass \(-2\mu^2\), but do not exhibit the symmetry of the Lagrangian in an obvious way.

A slightly more complicated model has two fields, which we may call \( \sigma \) and \( \pi \):

\[ \mathcal{L} = \frac{1}{2} \left[ \partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi \right] - V(\sigma^2 + \pi^2) \] (2.7)

where

\[ V = \frac{1}{2} \mu^2 (\sigma^2 + \pi^2) + \frac{1}{4} \lambda (\sigma^2 + \pi^2)^2. \] (2.8)

\( \mathcal{L} \) is obviously invariant under \( O(2) \equiv U(1) \):

\[ \left( \begin{array}{c} \sigma' \\ \pi' \end{array} \right) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \sigma \\ \pi \end{array} \right). \] (2.9)

The minimum occurs when

\[ \frac{\partial V}{\partial \sigma} = 0 = \sigma [\mu^2 + \lambda (\sigma^2 + \pi^2)] \] (2.10a)

\[ \frac{\partial V}{\partial \pi} = 0 = \pi [\mu^2 + \lambda (\sigma^2 + \pi^2)]. \] (2.10b)

Clearly when \( \mu^2 < 0 \), the absolute minimum occurs on the circle \( \sqrt{\sigma^2 + \pi^2} = [-\mu^2/\lambda]^{1/2} \). We can always define the axes in the \( \sigma-\pi \) plane so that

\[ \langle \sigma \rangle_0 = [-\mu^2/\lambda]^{1/2}, \quad \langle \pi \rangle_0 = 0. \]

[Another approach is to add explicitly a small symmetry-breaking term \( c \sigma \) to \( V \), as in the \( \sigma \)-model of Gell-Mann and Lévy. Then the minimum occurs when

\[ \sigma [\mu^2 + \lambda (\sigma^2 + \pi^2)] = c, \quad \pi [\mu^2 + \lambda (\sigma^2 + \pi^2)] = 0. \]

The term \( c \sigma \) picks out the particular direction in \( (\sigma, \pi) \) space. There is no solution to these equations except \( \pi = 0 \), and \( \sigma [\mu^2 + \lambda \sigma^2] = c \); in the limit \( c \to 0 \), either \( \sigma = 0 \) or \( \sigma = [-\mu^2/\lambda]^{1/2} \). The first solution is a minimum when \( \mu^2 > 0 \), the second when \( \mu^2 < 0 \).]

As before, when \( \mu^2 < 0 \), define

\[ s = \sigma - \langle \sigma \rangle_0 \]

and rewrite \( \mathcal{L} \) in terms of \( s \) and \( \pi \) instead of \( \sigma \) and \( \pi \):

\[ \mathcal{L} = \frac{1}{2} \left[ \partial_\mu s \partial^\mu s + \partial_\mu \pi \partial^\mu \pi \right] + \mu^2 s^2 - \lambda \langle \sigma \rangle_0 s (s^2 + \pi^2) - \frac{1}{4} \lambda (s^2 + \pi^2)^2. \] (2.11)
Evidently, $s$ is the field of a particle with positive mass $-2\mu^2$ while the $\pi$-field is massless. This is our first example of Goldstone's theorem. If a theory has a symmetry of the Lagrangian which is not a symmetry of the vacuum, there must be a massless boson.

Here is a more general example. Let $\phi$ be an $n$-component real field, with Lagrange density

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi^i \partial^\mu \phi^i \right) - \frac{1}{2} \mu^2 \phi^i \phi^i - \frac{1}{4} \lambda (\phi^i \phi^i)^2. \quad (2.12)$$

$\mathcal{L}$ is obviously invariant under the orthogonal group in $n$ dimensions, $O(n)$. If $\mu^2 < 0$, the potential has a ring of minima at $v = [-\mu^2/\lambda]^{1/2}$ i.e. there is a minimum whenever $\phi^i \phi^i = -\mu^2/\lambda$. Let us choose the $n$th component of $\phi$ to be the one which develops a vacuum expectation value. That is to say, considering $\phi$ as an $n$-component column vector,

$$\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}$$

The original symmetry group, $O(n)$, has $\frac{1}{2} n(n-1)$ generators. The new feature in this example is that there is still a non-trivial group which leaves the vacuum invariant. This is the subgroup of $O(n)$ which does not mix up the $n$th component with the others; it is $O(n-1)$, with $\frac{1}{2} (n-1)(n-2)$ generators.

Let $L_{ij}$ be the $\frac{1}{2} n(n-1)$ independent matrices generating $O(n)$. Let $l_{ij}$ be the subset which form the surviving symmetry $O(n-1)$ [$l_{ij} = L_{ij}$ for $i, j \neq n$]. Then call the rest $k_i$ [$k_i = L_{in}$]. These are $n-1$ independent $k_i$. Instead of simply subtracting the vacuum-expectation value of the fields to define new fields as before, we can parametrize the $n$ field in a way which will be more useful later. Define $\eta$ and $\xi_i$, $1 \leq i \leq n-1$, by

$$\phi = \exp(i\xi_k k_i / v)$$

Since, in general, $(L_{ij})_{kl} = -i[\delta_{lk} \delta_{ji} - \delta_{li} \delta_{jk}]$, the $k_i$ have matrix elements

$$(k_i)_{kl} = (L_{kn})_{kl} = -i[\delta_{lk} \delta_{ni} - \delta_{li} \delta_{nk}]$$

and so $k_i$ operating on the column vector $v_i = v \delta_{in}$ is the vector
Thus, in lowest order this definition is equivalent to our previous procedure; up to terms quadratic
in the fields, \( \phi_i = \xi_i (i < n) \) and \( \phi_n = \nu + \eta \). [We will show in Part II that such a redefinition of the
fields leaves the renormalized S-matrix invariant, but not the Green's functions.]

In terms of the new fields \( \xi_i \) and \( \eta \), the Lagrangian is

\[
\mathcal{L} = \frac{1}{2} \left[ \partial_\mu \eta \partial^\mu \eta + \partial_\mu \xi_i \partial^\mu \xi_i \right] + \text{higher order terms with derivative couplings}
- \frac{1}{2} \mu^2 (\nu + \eta)^2 - \frac{1}{4} \lambda (\nu + \eta)^4.
\]

The \( \eta \) field has bare mass \(-2\mu^2 (>0)\), and the \( n-1 \xi_i \) fields are massless. Thus to each generator
of the original group which does not leave the vacuum invariant, there corresponds a massless
Goldstone boson.

The fact that the number of massless bosons is the same as the number of broken generators
seems to be an accident of our example, the \( n \)-dimensional representation of \( O(n) \). But it is in
fact general. Write any Lagrangian in terms of the \( n \) real scalar fields \( \phi_i \), which form an \( n \)-component vector \( \phi \) (a complex representation can always be turned into a real one by doubling the
number of basis vectors)

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - V(\phi).
\]

Of course, \( \mathcal{L} \) may contain other fields (e.g. spinor fields) which couple to each other and to \( \phi \),
but these terms are not relevant here. \( V(\phi) \) is a polynomial in \( \phi \) which is invariant under some
group \( G \) (and not under a larger group containing \( G \)). \( G \) has \( N \) generators \( T_\alpha \), and \( \phi \) transforms
according to an \( n \)-dimensional (in general reducible) representation \( L^\alpha \): \( \delta \phi = -i\theta^\alpha L^\alpha \phi \).

Because the representation is real, \( iL^\alpha \) must be a real matrix; so \( L^\alpha \) is an imaginary matrix, and
because it is Hermitean it is antisymmetric. Because \( V \) is invariant under \( G \), its response to an infinitiesimal group transformation (specified by \( \theta^\alpha \)) is

\[
0 = \delta V = \frac{\partial V}{\partial \phi_i} \delta \phi_i = -i \frac{\partial V}{\partial \phi_i} \theta^\alpha L^\alpha_{ij} \phi_j.
\]

Since \( \theta^\alpha \) are arbitrary, we obtain \( N \) equations

\[
\frac{\partial V}{\partial \phi_i} L^\alpha_{ij} \phi_j = 0
\]

for all \( \alpha \). Differentiating again, we get

\[
\frac{\partial^2 V}{\partial \phi_j \partial \phi_k} L^\alpha_{ij} \phi_j + \frac{\partial V}{\partial \phi_i} L^\alpha_{ik} = 0.
\]

Now evaluate (2.17) at \( \phi = \nu \), the value of \( \phi \) which minimizes \( V \):

\[
(\partial V/\partial \phi_j)_{\phi=\nu} = 0.
\]

The result is
\[
(\partial^2 V / \partial \phi_i \partial \phi_k)_{\phi = v} L^2_{ij} v_j = 0.
\]

If \( V \) is expanded about \( v \), there are no linear terms, and the constant term is irrelevant:

\[
V = \frac{1}{2} M^2_{ij} (\phi - v)_i (\phi - v)_j + \text{higher order terms.}
\]

Therefore \( \partial^2 V / \partial \phi_i \partial \phi_j \) evaluated at \( \phi = v \) is just \( M^2_{ij} \) where \( M^2_{ij} \) is the mass matrix, and so

\[
(M^2)_{ij} L^2_{jk} v_k = 0
\]

for each \( \alpha \).

Let \( S \) be the \( M \)-dimensional subgroup of \( G \) which remains a symmetry of the vacuum. If \( L^\alpha \) is a generator of \( S \), \( L^\alpha v = 0 \), and so (2.20) contains no information about \( M^2 \). For each of the \( N - M \) vectors \( L^\alpha v_k \) which are not zero, (2.20) says that \( M^2 \) has a zero eigenvalue. If the vectors \( L^\alpha v \) in fact span an \( N - M \) dimensional space, we have demonstrated that there are \( N - M \) massless (Goldstone) bosons in the theory.

This fact is almost obvious from our examples. To construct a formal proof, define

\[
A^{\alpha \beta} = (L^\alpha v, L^\beta v) \quad [(a, b) \text{ means } \sum_i a_i^* b_i, \text{ even though we have a real vector space}].
\]

Since \( L^\alpha \) is Hermitean, \( A^{\alpha \beta} = (v, L^\alpha L^\beta v) \). Then

\[
A^{\alpha \beta} - A^{\beta \alpha} = (v, [L^\alpha, L^\beta] v) = i \epsilon_{\alpha \beta \gamma} (v, L^\gamma v) = 0,
\]

the last equality following again because \( L \) is antisymmetric. Therefore let \( \tilde{A} \) be the \((N - M) \times (N - M)\) matrix obtained by restricting \( \alpha \) and \( \beta \) to those values for which \( L^\alpha v \neq 0 \). \( \tilde{A} \) is symmetric, and can be diagonalized. Then let \( O \) be the \((N - M) \times (N - M)\) orthogonal matrix which diagonalizes \( \tilde{A} \):

\[
\tilde{A}^{\alpha \beta} = (O \tilde{A} O^T)^{\alpha \beta} = (O^{\alpha \gamma} L^\gamma v, O^{\beta \delta} L^\delta v).
\]

Now \( O^{\alpha \gamma} L^\gamma \) cannot annihilate \( v \), since then it would be in \( S \), which it manifestly isn't. Thus \( O^{\alpha \gamma} L^\gamma \neq 0 \), and the diagonal elements of \( \tilde{A} \) are all positive, and the space not annihilated by the \( O^{\alpha \gamma} L^\gamma \), or equivalently the \( L^\alpha \), is \( N - M \) dimensional. The \( L^\alpha \) which do not annihilate \( v \) are independent, which completes the proof that \( M^2 \) has \( N - M \) zero eigenvalues. The matrix \( A^{\alpha \beta} \) will play a fundamental role in the next section.

Bibliography

The exploitation of the Ward–Takahashi identities to extract physical consequences of a spontaneously broken symmetry is typified by the case of spontaneously broken chiral \( \text{SU}(2) \times \text{SU}(2) \), in which case it is known as the current algebra and chiral dynamics:


For the discussions of the \( \sigma \)-model, see

and ref. [3] above.
The Goldstone theorem, and, in fact, the Goldstone mode of symmetry in quantum field theory (i.e., spontaneously broken symmetry) was first discussed in
and later elaborated by

The importance of the Goldstone theorem in a physical context was first expounded by

3. The Higgs mechanism

In this section we shall discuss Lagrangians with spontaneously broken symmetries which also possess the kind of local gauge invariance which we described in the first section. The combination leads to an exception to Goldstone’s theorem which provides the basis for a class of renormalizable models of the weak and electromagnetic interactions.

The simplest example is constructed from a single self-interacting charged field $\phi$ with Lagrangian

$$\mathcal{L} = (\partial_{\mu} \phi^* \partial^{\mu} \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2. \quad (3.1)$$

This Lagrangian is invariant under a U(1) group of transformations:

$$\phi \rightarrow \phi' = e^{-i\theta} \phi. \quad (3.2)$$

Next we introduce a gauge field $A_{\mu}$ and construct a Lagrangian invariant under local gauge transformations. Following the prescription derived in the first section, we obtain

$$\mathcal{L} = \left[ (\partial_{\mu} + ie A_{\mu}) \phi^* (\partial^{\mu} - ie A^{\mu}) \phi \right] - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \quad (3.3)$$

where $F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. Under local gauge transformations,

$$\phi(x) \rightarrow \phi'(x) = \exp \{-i\theta(x)\} \phi(x)$$

$$\phi^*(x) \rightarrow \phi^{*'}(x) = \exp \{i\theta(x)\} \phi(x)$$

$$A_{\mu} \rightarrow A'_{\mu} = A_{\mu} - \frac{1}{e} \partial_{\mu} \theta(x) \quad (3.4)$$

and $\mathcal{L}$ is invariant under the transformations (3.4).

If $\mu^2 > 0$, (3.3) is just the Lagrangian for charged scalar electrodynamics. If $\mu^2 < 0$, we must shift the fields to write $\mathcal{L}$ in terms of those with vanishing vacuum expectation values.

The Lagrangian (3.3) possesses the same O(2) symmetry as the $(\sigma, \pi)$ model discussed in eq. (2.7), transforming according to (2.9). The correspondence is $\sigma/\sqrt{2} \leftrightarrow \text{Re} \, \phi$, $\pi/\sqrt{2} \leftrightarrow \text{Im} \, \phi$. Just as $\sigma$ could always be chosen to develop a vacuum expectation value, we can assume, without loss of generality, that

$$\langle \phi \rangle_0 = v/\sqrt{2}$$

where $v$ is real.
Instead of shifting $\phi$ by subtracting $\langle \phi \rangle_0$ from it, we will parametrize $\phi$ exponentially, as we did with the real $n$-vectors in section 2: The new real fields are $\xi$ and $\eta$, defined by

$$\phi = \exp(i\xi/v)(v + \eta)/\sqrt{2} = \frac{1}{\sqrt{2}}[v + \eta + i\xi + \text{quadratic and higher order terms}].$$

(3.5)

The field $\xi$ is associated with the spontaneously broken U(1) symmetry. In the absence of the gauge field $A_\mu$, we could conclude that the $\xi$ field was massless, because when (3.3) is written in terms of $\xi$ and $\eta$ there is no term quadratic in $\xi$. This argument no longer works. Let us write (3.3) in terms of $\xi$ and $\eta$: 

$$\begin{align*}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial^\mu \eta \partial^\mu \eta + \frac{1}{2}\partial^\mu \xi \partial^\mu \xi \\
&+ \frac{1}{2}e^2v^2A_\mu A^\mu - evA_\mu \partial^\mu \xi + \mu^2 \eta^2 + \text{cubic and higher order terms}
\end{align*}$$

(3.6)

where we have the relation $v^2 = -\mu^2/\lambda$. The $\eta$ field has mass $-2\mu^2$, but the fields $A_\mu$ and $\xi$ have gotten mixed up in a way whose interpretation is not immediately apparent. Without the term $-evA_\mu \partial^\mu \xi$ in (3.6), we could have concluded that the vector field has mass-squared $\mu^2 = e^2v^2$ and that the $\xi$ field is massless. A correct procedure would be to compute the combined propagator for the $A_\mu$ and $\xi$ fields, find the Feynman rules, and examine the poles of the $S$-matrix. We'll do some of this in later lectures, but there is an easier way to discover the particle spectrum. Recall that the Lagrangian (3.3) is invariant under local gauge transformations (3.4). Choose the gauge function to be $\xi(x)/v$. Then

$$\begin{align*}
\phi &\rightarrow \phi' = \exp\{-i\xi(x)/v\}\phi = (v + \eta)/\sqrt{2} \\
A_\mu &\rightarrow A'_\mu = A_\mu - \frac{1}{ev}\partial_\mu \xi.
\end{align*}$$

(3.7a)

(3.7b)

Since $\mathcal{L}$ is invariant under these transformations,

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu + ieA'_\mu)(v + \eta)][(\partial^\mu - ieA'_\mu)(v + \eta)] - \frac{1}{2}\mu^2(v + \eta)^2 - \frac{1}{4}\lambda(v + \eta)^4 - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$$

(3.8)

where $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$.

Eq. (3.8) can be expanded as follows:

$$\begin{align*}
\mathcal{L} &= -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\partial_\mu \eta \partial^\mu \eta + \frac{1}{2}e^2v^2A'_\mu A'^\mu \\
&+ \frac{1}{2}e^2A'^2_\mu (2v + \eta) - \frac{1}{2}\eta^2(3\lambda v^2 + \mu^2) - \lambda \eta^3 - \frac{1}{4}\lambda \eta^4.
\end{align*}$$

(3.9)

In this gauge there are no terms coupling different particles, so that the (bare) spectrum can be simply read off the quadratic terms. There is a scalar $\eta$-meson, with mass $3\lambda v^2 + \mu^2$ (which in zeroth order is $-2\mu^2$), a massive vector meson $A'_\mu$, with mass $ev$, and no particle corresponding to $\xi$. In fact, the $\xi$-field has disappeared altogether! It has been “gauged away”.

Where has it gone? From eq. (3.7b), we can see that it is responsible for the longitudinal component of the vector field in the new gauge. It’s clear that there are the same number of actual particle states as there were before we redefined the fields in eq. (3.5). Originally, there were two
real scalar fields and a massless photon with two possible polarizations. For positive $\mu^2$, this is the correct collection of particles. When $\mu^2 < 0$, we have just seen that the theory describes one scalar particle (1 helicity state) and one massive vector particle (3 helicity states) so the total number of degrees of freedom — in the sense of particles with fixed polarizations — is the same in each case.

In the gauge (3.9) $\mathcal{L}$ looks like an ordinary field theory of particles, each decoupled from each other in second order, and therefore is manifestly unitary order by order in perturbation theory. This gauge is frequently called the unitary gauge, or U-gauge. (The U-gauge is “manifestly unitary” in the sense that the fictitious particles, whose Green’s functions have singularities that apparently violate unitarity, are manifestly absent. We do not mean to imply that the unitarity of the $S$-matrix, or even the correct Feynman rules, are obvious in this gauge.) However since it contains a massive vector meson, whose propagator for large $k$ grows as $k^2/m^2k^2$ instead of $-(\eta_\mu \eta_\nu - k_\mu k_\nu)/k^2$ characteristic of massless vector fields, this model is not obviously renormalizable in the U-gauge.

In the original Lagrangian (3.3), the fields admit gauge transformations (3.4) and it is necessary to choose a condition, such as $\partial^\mu A_\mu = 0$, which fixes the gauges. In Part II, we will show that the theory is renormalizable in such a gauge, which we shall call a renormalizable gauge, or R-gauge. In R-gauges, there are spurious poles in the vector and $\xi$ propagators, which must cancel in all $S$-matrix elements since they are absent in the U-gauge. The R-gauge formulation is not manifestly unitary.

For a non-Abelian example, we let the symmetry group be SU(2), and put the scalar mesons in the triplet representation. The fields transform according to

$$\delta \phi_i = -ie\mathcal{L}_{ik} \phi_k = e^{ij}k_k \phi_k.$$  

The part of the Lagrangian containing $\phi$ is

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi_i + ge^{ijk} A^{j}_\mu \phi_k) (\partial^\mu \phi_i + ge^{ijk} A^{j}_\mu \phi_k) - V(\phi^2)$$  

(3.10)

where $V$ is an SU(2) invariant quartic polynomial in $\phi$.

When $\phi = 0$ is a minimum of $V$, (3.10) is an ordinary, isospin conserving gauge invariant, Yang-Mills type theory. Our interest is in the spontaneous symmetry-breaking case: If $V$ has a non-zero minimum, we can always perform an isospin rotation so that it is the third component which acquires a vacuum-expectation value:

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}.$$  

The vacuum is no longer invariant under $T_1$ and $T_2$, but $T_3$ remains a good symmetry: there is one conserved quantum number, $T_3$ or electric charge.

We parametrize $\phi$ as in the previous lecture:

$$\phi = \exp\left\{ \frac{1}{\mu}(\xi_1 L^1 + \xi_2 L^2) \right\} \begin{pmatrix} 0 \\ 0 \\ v + \eta \end{pmatrix} = \langle \phi \rangle + \begin{pmatrix} \xi_2 \\ \xi_1 \\ \eta \end{pmatrix} + \text{higher orders}. \quad (3.11)$$
The fields $\xi_1$ and $\xi_2$ are the would-be Goldstone bosons associated with the two broken degrees of freedom. Since the Lagrangian (3.10) is invariant under local SU(2) transformations, we may make the following gauge transformation:

$$\phi' = \exp\left\{ -\frac{i}{v} (\xi_1 L^1 + \xi_2 L^2) \right\} \phi$$

$$L \cdot A_\mu' = \exp\left\{ -\frac{i}{v} (\xi_1 L^1 + \xi_2 L^2) \right\} L \cdot A \exp\left\{ \frac{i}{v} (\xi_1 L^1 + \xi_2 L^2) \right\} - \frac{1}{g} \left[ \partial_\mu \exp\left\{ -\frac{i}{v} (\xi_1 L^1 + \xi_2 L^2) \right\} \right] \times \exp\left\{ \frac{i}{v} (\xi_1 L^1 + \xi_2 L^2) \right\}. \quad (3.12)$$

Again, since $\phi' = \begin{bmatrix} 0 \\ v + \eta \end{bmatrix}$, the fields $\xi_1$ and $\xi_2$ completely disappear when the Lagrangian is written in the new gauge:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \eta \partial^\mu \eta + \frac{1}{2} g^2 v^2 e^{i\frac{\pi}{3}} \epsilon^{\mu \nu \rho} A_\mu' A_\nu' - V(\eta + \eta') + \text{higher order terms + terms independent of } \phi. \quad (3.13)$$

The term in (3.13) quadratic in the vector fields is

$$\frac{1}{2} M^2 [A_\mu^2 A_\nu^2 + A_\mu^4 A_\mu^4] \quad (3.14)$$

where $M^2 = g^2 v^2$. The vector mesons corresponding to the broken symmetry generators have acquired a mass $M = g v$. Since the $T_3$ symmetry survives, there remains one massless vector meson, $A_\mu^3$.

The general features of a spontaneously-broken gauge model should now be clear. We start out with a Lagrangian $\mathcal{L}$ invariant under local gauge transformations of some group $G$. There are $n$ scalar fields which transform under an $n$-dimensional representation. There are $N$ gauge mesons, $A^\alpha_\mu$. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F^\alpha_{\mu \nu} F^\alpha_{\mu \nu} + \frac{1}{4} \left( (\partial_\mu - ig^\alpha L^\alpha A_\mu^\alpha) \phi (\partial^\mu - ig^\beta L^\beta A^\mu) \phi \right)$$

$$- V(\phi) + \text{terms with other fields.} \quad (3.15)$$

Here, $F^\alpha_{\mu \nu}$ is given by (1.44); the $g^\alpha$ are independent of $\alpha$ within any simple subgroup of $G$; and $V(\phi)$ is a fourth-order $G$-invariant polynomial in $\phi$ which is minimized by setting $\phi = v$.

Now we suppose the symmetry-breaking leaves the vacuum invariant under an $M$-dimensional subgroup $S$ of $G$. There are $M$ generators $L^\alpha$ satisfying $L^\alpha v = 0$. There remain $(N - M)L^\alpha$ for which $L^\alpha v \neq 0$. We showed in section 2 that the $L^\alpha v$ span an $N - M$ dimensional space, and that in the absence of the gauge mesons there would be $N - M$ massless scalar particles.

We can parametrize $\phi$ by

$$\phi = \exp(\Sigma i \xi_\alpha L^\alpha/v)(v + \eta). \quad (3.16)$$

The sum is over those $(N - M)L^\alpha$ which do not annihilate $v$. The vector $\eta$ represents as many independent fields as there are dimensions in the part of the $n$-dimensional representation space orthogonal to all $L^\alpha v$.

Next we make the gauge transformation defined by
\[ \phi' = \exp(-i\xi^\alpha L^\alpha/v)\phi. \]

and

\[ A^\gamma L_\gamma = \exp\left(-\frac{i}{v}L^\alpha \xi^\alpha\right)\left(A^\gamma L_\gamma - \exp\left(\frac{i}{v}L^\beta \xi^\beta\right)i g^\alpha \partial_\mu \exp\left(-\frac{i}{v}L^\beta \xi^\beta\right)\right)\exp\left(\frac{i}{v}L^\alpha \xi^\alpha\right). \quad (3.17) \]

In the new gauge, \( \mathcal{L} \) depends only on the \( \eta_i \) and the gauge fields \( A_\mu^\alpha, N - M \) of which are now massive.

The term in \( \mathcal{L} \) responsible for the vector meson masses is

\[ \frac{1}{2}(g^\alpha L^\alpha v, g^\beta L^\beta v)A^\alpha_\mu A^\mu_\beta. \quad (3.18) \]

Since we may always restrict ourselves to real representations of \( G \), so that \( L \), being Hermitean, is antisymmetric, the vector meson mass matrix,

\[ (M^2)^{\alpha\beta} = g^\alpha g^\beta(v, L^\alpha L^\beta v) \quad \text{(no sum over} \alpha, \beta) \quad (3.19) \]

is symmetric and positive definite, with \( \alpha, \beta \) restricted to values for which \( L^\alpha v \neq 0 \). Except for the coupling constants \( g^\alpha \), \( (M^2)^{\alpha\beta} \) is just the matrix \( A^{\alpha\beta} \) we defined in section 2.

Thus, the \( N - M \) Goldstone bosons are not physical massless particles, but are absorbed into the longitudinal components of the \( N - M \) massive vector bosons: as can be seen from eq. (3.17),

\[ A_\mu^\alpha = A_\mu^\alpha - \frac{1}{g v} \partial_\mu \xi^\alpha + O(\xi^2). \]

The number of the independent degrees of freedom for a given momentum remains the same. The masses of the physical vector mesons are the eigenvalues of \((M^2)\). The remaining \( M \) vector mesons remain massless, corresponding to the surviving \( M \)-dimensional symmetry subgroup \( S \).

Weinberg has discovered an elegant proof that the unitary gauge always exists. In that gauge, \( \phi'(x) \) has no components in the subspace spanned by the Goldstone bosons, which we know is the space spanned by \( L^\alpha v \). Therefore

\[ (L^\alpha v, \phi'(x)) = 0 \quad (3.20) \]

defines the unitary gauge. (This definition is just as good as the more familiar definition of a gauge by imposing a condition on the vector fields.) Therefore, if \( \phi(x) \) is the scalar field in any gauge, there is a unitary gauge provided there exists a local gauge transformation

\[ O(x) = \exp\{ -i\xi^\alpha(x)L^\alpha \} \]

such that

\[ (L^\alpha v, O(x)\phi(x)) = 0 \quad (3.21) \]

for all \( \alpha \) and all \( x \). For any \( x \), \( O \) may be any element of the representation of \( G \) defined at \( x \). We have chosen only real representations, so \( O \) is orthogonal. Consider the scalar product

\[ (v, O\phi). \quad (3.22) \]

For fixed \( \phi \), the scalar product (3.22) is a real number which depends on \( O \). As long as the Lie group (of which \( O \) in an \( n \)-dimensional real representation) is compact, (3.22) maps the group into a compact portion of the real line, and therefore has a maximum and a minimum. Let \( O_\phi \) be
a matrix $O$ which is an extremum of (3.22). For any $O$, if we vary $O$ slightly
\[ \delta O = O - \exp\{-ie^{a}(x)L^{a}\}O = -ie^{a}(x)L^{a}O. \]
Since $O_{\phi}$ makes (3.22) take on an extreme value,
\[ 0 = \delta(v, O_{\phi})_{O=O_{\phi}} = (v, \delta O_{\phi})_{O=O_{\phi}} = -i e^{a}(v, L^{a}O_{\phi}) = -ie^{a}(L^{a}v, O_{\phi}). \]
(3.23)
Since $e^{a}$ is arbitrary,
\[ (L^{a}v, O_{\phi}) = 0 \]
for all $a$, so $\phi' = O_{\phi}$ satisfies the unitary gauge condition. Therefore the unitary gauge always exists: it can be obtained by making a gauge transformation $O$ from an arbitrary $\phi$ which extremizes $(v, O_{\phi})$ at each point $x$. If $G$ is simply connected, the real numbers $(v, O_{\phi})$ form a compact segment of the real line, and therefore have two extrema, a maximum and a minimum. Generally, if $(\phi', v)$ is one extremum, $(-\phi', v)$ is the other, and the physics of the two gauges are the same.

In nature, $M$ is apparently 1; the only conservation law associated with a massless vector meson is charge conservation. Nevertheless, it is instructive to consider the more general case.

In the next sections we will consider the application of these ideas to models of weak and electromagnetic interaction.

Bibliography

The Higgs mechanism was first discussed in the context of relativistic field theory in

The group theoretic ramifications of this phenomenon were first discussed in

The proof of the existence of the U-gauge follows closely the presentation of

4. Review of weak interaction phenomenology

In later sections we will discuss a class of models for weak and electromagnetic interactions which utilize the idea of spontaneously broken gauge symmetry. One constraint on these models is that they reproduce the known phenomenology of weak interaction. We will review some important features in this section.

Our notation will be that of the textbook by Bjorken and Drell. The Dirac $\gamma$-matrices are
\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The \( \gamma^\mu \) anticommute according to \{\( \gamma^\mu, \gamma^\nu \)\} = 2\varepsilon^{\mu\nu}, \) and \{\( \gamma^\mu, \gamma_5 \)\} = 0. The spin matrix is \( \sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \)

It is easy to see that \( \sigma^i = \frac{1}{2} i e^{ijk} \gamma^j \gamma^k \). The Lagrangian for a free Dirac spinor with mass \( m \) is

\[ \mathcal{L} = \bar{\psi}(x)(i\gamma \cdot \partial - m)\psi(x) \quad (4.1) \]

which leads to the equation of motion [see (1.3)]

\[ [i\gamma \cdot \partial - m] \psi = 0. \quad (4.2) \]

Most of our information about weak interactions comes from spontaneous decay processes, in which the energies and momenta transferred are small compared to the high energies available to study strong and electromagnetic interactions in particle accelerators. Therefore there is no reason to expect that a phenomenological description of known decays will be correct at higher energies, but nevertheless a more complete theory must agree with what we know at low energies.

The only known leptons are the muon, the electron and the neutrinos. All known experiments are consistent with lepton number conservation if we assign “lepton number” +1 to \( \mu^- \), \( e^- \), and \( \nu \), and −1 to \( \mu^+ \), \( e^+ \), and \( \bar{\nu} \). Furthermore, the decays \( \mu^- \rightarrow e^- + \gamma \) or \( \mu^- \rightarrow e^- + e^- + e^+ \) are not seen, even though they conserve lepton number. Apparently there is also a conserved “muon number” which forbids these processes. The neutrino associated with the muon is different from the neutrino associated with the electron. Experimentally, neutrinos produced in the decay \( \pi^- \rightarrow \mu^- + \bar{\nu} \) are not seen to produce electrons by inverse beta decay \( \nu + n \rightarrow p + e^- \). Therefore, we believe there are two doublets of leptons, \((\mu^-, \nu_\mu)\) and \((e^-, \nu_e)\), which are distinguished by a quantum number. It is possible that the muon quantum number is multiplicative (like parity), but there is at present no particular evidence for this unattractive idea.

The mass of the muon is 105.6594 ± 0.0004 MeV. It has a lifetime of \((2.994 \pm 0.0006) \times 10^{-6}\) seconds, decaying almost always into \( e^- + \nu_\mu + \bar{\nu}_e \). Other modes, if they exist, are very rare. The electron mass is \((0.511004 \pm 0.000002)\) MeV and it has a lifetime of at least \(6 \times 10^{28}\) seconds. As far as is known, the muon and the electron are identical in all properties except for their masses, the large difference between which is a major puzzle. Perhaps the empirical relation

\[ m_e = \frac{3}{2} am_\mu \quad (4.3) \]

which is accurate to better than one-percent, provides a clue.

The neutrinos appear to be massless although experimentally it is not possible to put such fantastic upper limits on their masses as are known for the photon. The electron-neutrino certainly weighs less than \(6 \times 10^{-5}\) MeV, but the muon neutrino may have a mass of an MeV or more. Nevertheless it is attractive — and consistent with experiment — to assume both are exactly massless, as we shall see below.

It is important in gauge theories that the photon be exactly massless. From the fact that the earth’s magnetic field has been detected tens of thousands of miles away, one concludes that the Compton wavelength of the photon must be of this order at least, corresponding to a mass less than \(10^{-21}\) MeV.

If the neutrino has a finite mass, it must occur in both helicity states, since a positive helicity state can be transformed into a negative one by Lorentz transformation. If the neutrino is exactly
massless, either helicity state provides a complete representation of the Poincaré group, and only parity conservation would require both to occur. Formally, it is easy to see that under the transformation \( \psi \rightarrow -\gamma_5\psi \), the kinetic energy term in (4.1) is invariant while the mass term is not. If the mass is zero, the free Lagrangian is invariant under this transformation. The interaction part of the Lagrangian will be invariant provided the neutrino field occurs only in the combination \((1 - \gamma_5)\psi\).

Let us introduce here a notation which will be useful later. On any spinor field, let \( P_L = \frac{1}{2}(1 - \gamma_5) \), and \( P_R = \frac{1}{2}(1 + \gamma_5) \). \( P_L \) and \( P_R \) are projection operators, in the sense that \( P_L^2 = P_L \), \( P_R^2 = P_R \), \( P_L P_R = P_R P_L = 0 \), and \( P_L + P_R = 1 \). Any spinor field \( \psi \) can be broken up using \( P_L \) and \( P_R \):

\[
\psi = \psi_L + \psi_R = P_L \psi + P_R \psi = \frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)\psi.
\] (4.4)

The free Lagrangian becomes

\[
\mathcal{L} = i\bar{\psi}_L \gamma \cdot \partial \psi_L + i\bar{\psi}_R \gamma \cdot \partial \psi_R - m[\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L].
\] (4.5)

If \( m \neq 0 \), the breakup (4.4) has no Lorentz invariant meaning. If \( m = 0 \), \( \psi_R \) is a solution to (4.2) with spin analog the direction of the momentum, \( \psi_L \) a solution with spin in the opposite direction. They have positive and negative helicities, respectively. These facts are easily obtained from the massless Dirac equation,

\[(\gamma^0 - \gamma \cdot \hat{n})\psi = 0,
\]

where \( \hat{n} \) is a unit vector in the direction of the neutrino's momentum, and from the definition of \( \gamma_5 \) and the spin operator \( \sigma \).

The decay spectra in those weak decays for which there are the most data, namely \( n \rightarrow p + e^- + \bar{\nu}_e \), \( \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \), and \( \pi^- \rightarrow \mu^- + \bar{\nu}_\mu \), plus a large collection of nuclear decays, show no sign of right-handed (positive helicity) neutrinos or left-handed (negative helicity) anti-neutrinos. We will assume that there exist only left-handed neutrinos, which is possible only if the neutrinos are exactly massless.

Finally, it is easy to show that if only the left-handed neutrino field, \( \psi_L(\nu) \), appears in the Lagrangian, the neutrino remains massless to all orders in perturbation theory. Under the operation \( \psi(x) \rightarrow -\gamma_5 \psi(x) \), \( \psi^\dagger(x) \rightarrow -\psi^\dagger(x) \gamma_5 \), and therefore \( \psi(x) \rightarrow \psi(x) \gamma_5 \). These rules hold for the interacting fields. Therefore the neutrino propagator (including interactions) is

\[
S'(p) = \int d^4x \exp(ip \cdot x)|T(\psi(x)\bar{\psi}(0))|\\
= -\int d^4x \exp(ip \cdot x)\gamma_5|T(\psi(x)\bar{\psi}(0))|\gamma_5.
\] (4.6)

Therefore

\[
S(p)\gamma_5 = -\gamma_5 S(p)
\]

and

\[
\gamma_5 S^{-1}(p) = -S^{-1}(p)\gamma_5.
\] (4.7)

In general, \( S^{-1}(p) = \gamma \cdot p + \delta m + O(\gamma \cdot p)^2 \). From (4.7), the term in \( \delta m \) is forbidden, and \( S^{-1}(0) = 0 \); the full propagator \( S(p) \) has a role at \( \gamma \cdot p = 0 \).
5. Weak interaction phenomenology (continued)

The idea that all known weak decays can be described by a local four-point interaction is due to Fermi, and such interactions are called Fermi couplings. Following the discovery that weak decays violate parity conservation, Feynman and Gell-Mann proposed that the correct form for the Fermi interaction is

\[ G [J_\mu(x)F^\nu_\mu(x)] / \sqrt{2}. \]  (5.1)

Here \( J_\mu(x) \) is a charged current, which has a lepton part and a hadron part:

\[ J_\mu(x) = l_\mu(x) + h_\mu(x). \]  (5.2)

The lepton part of the current is

\[ l_\mu(x) = \bar{\psi}_e(x)\gamma_\mu(1 - \gamma_5)\psi_{\nu_e}(x) + \bar{\psi}_\mu(1 - \gamma_5)\gamma_\mu(1 - \gamma_5)\psi_{\nu_\mu}(x). \]  (5.3)

From (5.2) and (5.3) the \( \mu^- \) decay spectrum can be calculated, and seems to be in agreement with experiment. The rate for \( \mu^- \) decay comes out to be

\[ \Gamma(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu) = G^2m_\mu^5/192\pi^3. \]  (5.4)

From (5.4) and the known rate of \( \mu^- \) decay, the Fermi coupling constant \( G \) which appears in (5.1) can be evaluated. An easy formula to remember is

\[ G = 1.01 \times 10^{-5} m_p^2 \]  (5.5)

where \( m_p \) is the proton mass.

The lepton current can be written

\[ l_\mu = 2\bar{\psi}_L\gamma_\mu\psi_L(\nu_e) + 2\bar{\psi}_L\gamma_\mu\psi_L(\nu_\mu). \]  (5.6)

It is entirely a left-handed current. We define a leptonic left-handed isospin by grouping \( \psi_e \) into a doublet \( \chi_L(e^-) \), and \( \psi_\mu \) into a doublet \( \chi_L(\mu^-) \). Then

\[ l_\mu = 2[\chi_L^\dagger(e^-)\gamma_\mu\tau^-\chi_L(\nu_e) + \chi_L^\dagger(\mu^-)\gamma_\mu\tau^-\chi_L(\nu_\mu)]. \]  (5.7)

where \( \tau^- = \frac{1}{2} \tau^1 - i\tau^2 = (0, 0). \)
We define a "left-handed isospin" current for leptons by
\[ j_{L}^{\mu}(x) = \frac{i}{2} [\gamma_{\mu}^{L}(e)\gamma^{\tau}X_{L}(e) + \gamma_{\mu}^{L}(\mu)\gamma^{\tau}X_{L}(\mu)] \]  
(5.8)
and the corresponding charges by
\[ T_{L}^{\mu} = \int j_{L}^{\mu}(x) d^{3}x. \]  
(5.9)
The \( T_{L}^{\mu} \) generate an SU(2)_{L} algebra:
\[ [T_{L}^{\mu}, T_{L}^{\nu}] = i e^{\frac{1}{2}} T_{L}^{\rho}. \]  
(5.10)
It is convenient to introduce
\[ T_{L}^{\pm} = [T_{L}^{\uparrow} \pm i T_{L}^{3}] \sqrt{2}. \]  
(5.11)
Then
\[ [T_{L}^{3}, T_{L}^{\pm}] = \pm T_{L}^{\pm}, \quad [T_{L}^{\pm}, T_{L}^{\mp}] = T_{L}^{3} \]  
(5.12)
and there is an analogous definition for \( j_{L}^{\pm} \).
Evidently,
\[ j_{L}^{\mu} = 2\sqrt{2} (j_{L}^{\pm})^{\mu}. \]
The leptonic part of the weak interactions in (5.1) is not invariant under SU(2)_{L}, since there is no term \( j_{\mu}^{3} j_{L}^{3} \) there. The existence and magnitude of a neutral leptonic current is an open experimental question.
The decay of the neutron \( n \rightarrow p + e^{-} + \nu_{e} \) is well described by assuming that the hadronic current \( h_{\mu} \) in (5.2) has a term
\[ \bar{\psi}(n)\gamma_{\mu}(g_{V} - g_{A} \gamma_{5})\psi(p). \]  
(5.13)
The vector coupling constant is strikingly close to 1, while \( g_{A} \) is about 1.24. An explanation of the fact that \( g_{V} \approx 1 \) was first suggested by Feynman and Gell-Mann, and by Gerschtein and Zel'dovich. Their hypothesis is that the strangeness conserving part of the \( h_{\mu} \) has the form
\[ (V_{\mu}^{1} - i V_{\mu}^{2}) - (A_{\mu}^{1} - i A_{\mu}^{2}) \]  
(5.14)
where \( V_{\mu}^{1} \) is a vector current and \( A_{\mu}^{1} \) is an axial vector current; and further that \( V_{\mu}^{1} \) and \( V_{\mu}^{2} \) are the first and second components of the isospin current. That is, that
\[ T^{\mu} = \int V_{\mu}(x) d^{3}x \]
are the isospin generators, conserved by the strong interactions. This rule is called the conserved vector current (CVC) hypothesis. Since the \( T^{\mu} \) form a Lie group, of which the proton and neutron form the basis of an irreducible representation, their matrix elements are fixed to be the Clebsch-Gordan coefficients, and \( g_{V} \) is predicted to be 1.
It is important to know whether \( g_{V} \) is really exactly one. The measured decay rates, both of the neutron and the muon, include electromagnetic corrections to the term obtained simply by replacing the fields in (5.13) with free wave functions. The radiative corrections to \( \mu \) decay were
calculated long ago, and turn out to be finite. The decay rate is corrected by about 4%. The radiative corrections to $\beta$-decay are logarithmically divergent. Putting in a cutoff of a few GeV, one can conclude that, even taking these corrections into account, there remains a discrepancy with the simple CVC prediction $g_v = 1$.

The calculation is unsatisfactory because the radiative corrections to neutron decay involve strong interaction corrections, and because of the difficulty in distinguishing the vector from the axial-vector part. The latter difficulty is overcome by considering $\beta$-decays of spin-zero particles, to which only the vector current contributes. The rate predicted for $\pi^+ \to \pi^0 + e^+ + \nu$ is in good agreement with CVC, but since the branching ratio of this mode to the principal mode, $\pi^+ \to \mu^+ + \nu$, is $10^{-8}$, the uncertainty is about 7%, which is too large for us to start worrying about radiative corrections. Decays of spin-zero heavy nuclei provide the best tests, because their rates can be accurately measured; but these calculations are plagued with nuclear physics complications. These have been estimated carefully for nine low-mass, spin zero nuclei. The result depends only on two parameters, the cutoff $\Lambda$ and a model dependent number $Q$ which depends on the underlying field theory. (In the quark model, $Q = \frac{1}{3}$.) For a wide range of $\Lambda$ and $Q$, $g_v$ is the same within experimental uncertainty for all nine nuclei. For $Q = \frac{1}{6}$ and $\Lambda = 30$ GeV, comparison with $\mu$-decay gives

$$g_v = 0.976 = \cos(0.22).$$

(5.15)

No reasonable values of the parameters give $g_v = 1$.

Thus, all our knowledge of non-strange $\beta$-decays is consistent with $h_\mu$ containing a term

$$g_v [(V^\mu_\mu - iV^\mu_\mu) - (A^\mu_\mu - iA^\mu_\mu)]$$

(5.16)

with $1 - g_v \approx 0.02$. Since $g_v \neq 1$, these vector currents alone do not generate a SU(2) group as the lepton currents do.

The decays of strange hadrons are consistent with the idea that $h_\mu$ contains a strangeness-changing vector and axial current term. From the observed absence of decays like $\Xi^0 \to p + e + \bar{\nu}$, we conclude that this term changes hypercharge by no more than one unit. From the absence of decays like $\Sigma^+ \to n + e^+ + \nu$ or $\Xi^0 \to \Sigma^- + e^+ + \nu$, one concludes that the strangeness-changing current changes the hypercharge (strangeness) and the electric charge by the same sign. This is known as the $\Delta S = \Delta Q$ rule. As a consequence, the change in $T^3$ is always $\pm \frac{1}{2}$, suggesting that this current has $T = \frac{1}{2}$.

Let us write $h_\mu$ as a sum of a $\Delta S = 0$ and a $\Delta S = 1$ part

$$h_\mu = g_\mu h^{(0)}_\mu + g_S h^{(1)}_\mu.$$  

(5.17)

$h^{(0)}_\mu$ has the form (5.14), and is the third component of an isotopic triplet. It is natural to extend this idea to SU(3), and assume that $h^{(1)}_\mu$ is the charged $\Delta S = 0$, $T = \frac{1}{2}$, member of an octet of currents (i.e., the one that transforms like $K^-$. By comparing a large number of decays, there is rather striking evidence that this is indeed the case. Therefore we can assume — since it is not in contradiction with experiment — that

$$h^{(1)}_\mu = (V^a_\mu - iV^a_\mu) - (A^a_\mu - iA^a_\mu)$$

(5.18)

where $F^l$
are the generators of SU(3) and the \( A^i_\mu(x) \) are an octet of axial currents.

Although SU(3) is not an exact symmetry, the matrix elements can still be estimated. The conclusion is that (5.18) does not disagree with experiment, but that \( g_s \) is nowhere near 1. The best fit is

\[
g_s/g_V \sim 0.25. \quad (5.20)
\]

In 1963 Cabibbo observed that within experimental error,

\[
g_s^2 + g_V^2 = 1. \quad (5.21)
\]

Therefore

\[
h_\mu = \cos \theta \, h_{\mu(0)}^{(0)} + \sin \theta \, h_{\mu}^{(1)}
= \exp(2i\theta F^7)h_\mu^{(0)}\exp(-2i\theta F^7),
\]

where \( F^7 \) is the 7th generator of SU(3).

In this way universality can be recovered, and the discrepancy between \( g_V \) and 1 understood. That is, if (5.22) is correct \( h_\mu \) is a correctly normalized component of a multiplet of currents which generate an SU(2) group. The angle \( \theta \) is called the Cabibbo angle, and is somewhere around \( 0.22 \sim 0.25 \). Its origin is unknown, and a plausible explanation would be very interesting.

Are there any neutral currents? We discussed leptonic neutral currents in the last section. The existence of the charged strangeness-conserving currents in (5.22) naturally suggests also neutral strangeness-conserving currents. Experimentally the existence of such currents is at this time an open question, which we shall return to in section 8.

By commuting \( h_\mu^{(1)} \) with \( T^+_1 \) (which is the charge associated with \( h^{(0)}_\mu \)), one obtains a neutral, strangeness-changing current, transforming under SU(3) like \( K^0 \). Experimentally, these currents do not seem to mediate leptonic weak interactions. Decays like \( \Sigma^+ \rightarrow p + e^+ + e^- \) are never seen. Furthermore, the upper limits for branching ratios of \( K^0 \rightarrow \mu^+ + \mu^- \) or \( K^+ \rightarrow \pi^+ + \nu + \bar{\nu} \) are of the order \( 10^{-8} \). Any model for weak decays must account for the absence or suppression of those currents.

Note that in writing (5.18) we tacitly assumed that \( h_\mu^{(1)} \) is a left-handed current like \( h_\mu^{(0)} \). If, in fact, it were right-handed—\( V + A \) instead of \( V - A \)—it would commute with \( T^+_1 \), and no strangeness-changing neutral current would exist. This idea has been occasionally suggested, but seems contradicted by experiments.

**Bibliography**

The ideas of CVC and the \( V - A \) interactions were proposed by

For a quick lesson in weak interactions, see, for example.

The radiative corrections to $\mu$-decay were discussed in

The radiative corrections to $\beta$-decay were discussed in

For the theory of the radiative corrections to $\beta$-decays of the pions and spin 0 nuclei, see

The Cabibbo theory was proposed in

For a recent review of the Cabibbo theory, both from theoretical and experimental viewpoints, see

6. Unitarity bounds, $W$-mesons, PCAC

We conclude our tour of the weak interactions with these topics: unitarity bounds, $W$-mesons, and PCAC.

Although equation (5.1) adequately describes decays, it cannot be a complete theory. When the interaction (5.1) is used to describe scattering, the Born approximation must fail at some energy, since the amplitude cannot be strictly real. Unlike electrodynamics, the Fermi coupling (5.1) does not lead to a renormalizable theory, so it is not possible to make these higher-order corrections.

For any leptonic scattering, the cross sections are not proportional to the lepton mass. The only other dimensional parameter available is $G$. Since the cross-sections are proportional to $G^2$, they are dimensionally constrained to grow like

$$\sigma \sim G^2 s$$  \hspace{1cm} (6.1)

neglecting the lepton masses. Because of the local form of (5.1), the cross-sections are restricted to a single partial wave, so there is a unitarity bound

$$\sigma \sim 1/s$$  \hspace{1cm} (6.2)

which is violated by (6.1) when $Gs$ is of the order 1.

For example, consider $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^-$. Ignoring the electron mass, the spin-averaged cross-section is

$$\bar{\sigma} = G^2 s / 3\pi.$$  \hspace{1cm} (6.3)

Since the electron which interacts with the neutrino is left-handed in this limit, and $\bar{\nu}_e$ is right-
handed, the total angular momentum along the direction of motion in the center of mass is 1, so the spin must be 1, not 0. Therefore scattering takes place in the spin-one state, if the electron mass, $m_e$, can be neglected. From the Jacob-Wick expansion for the scattering amplitude in the helicity representation, we have

$$
T_{\mu_4 \mu_3, \mu_2 \mu_1}(s, \theta) = \frac{1}{\pi} \sum_j (2j + 1) t_j^{\mu_4 \mu_3, \mu_2 \mu_1}(s) d_j^{\mu_1 - \mu_2, \mu_3 - \mu_4}(\theta)
$$

(6.4)

where $\mu_i$ are the helicities of the four particles, $d_j$ are the $j$-dimensional representations of rotations about the $y$ axis, and $t_j$ is the partial wave, normalized so that $\text{Im} t_j = (q/W)|t_j|^2$. ($q$ and $W$ are the c.m. momentum and $\sqrt{s}$, respectively; $W \approx 2q$.) Since (6.1) is a point interaction, there is no orbital angular momentum, and the spin is one, so only $j = 1$ contributes to the sum in (6.4). There is only one helicity state for each particle, so

$$
T_{1/2 - 1/2; 1/2 - 1/2} = \frac{3}{\pi} t_1^{1/2 - 1/2; 1/2 - 1/2} d_1(\theta) = \frac{3}{2\pi} t_1^{1/2 - 1/2; 1/2 - 1/2} (1 + \cos \theta).
$$

From unitarity, $|t_1|^2$ is bounded by 2. So in the forward direction

$$
|\text{Im} T(s, 0)| \leq 6/\pi
$$

and from the optical theorem

$$
\sigma = (4\pi^2/qW) |\text{Im} T(s, 0)| \leq 48\pi/s.
$$

The spin averaged cross-section includes both electron helicity states, so

$$
\bar{\sigma} = \frac{1}{2} \sigma \leq 24\pi/s.
$$

(6.5)

Comparing (6.3) with (6.5), we learn that (6.3) violates the unitarity bound when

$$
G^2 s^2 = 72\pi^2 \quad \text{or} \quad s = (\pi/G)\sqrt{72} = 2.7 \times 10^6 m_p = 2.5 \times 10^6 \text{ GeV}^2.
$$

(6.6)

The smallest such bound is obtained for the inelastic process $\nu_\mu + e^- \rightarrow \nu_e + \mu^-$. The $V - A$ spin wavefunction is antisymmetric, so that this process has only $j = 0$. Since $t^0$ for this amplitude is an off-diagonal matrix element of $(W/q)|\exp(2i\delta) - 1|/2i$, $|t^0| \leq 1$. The total $\nu_\mu + e^- \rightarrow \nu_e + \mu^-$ cross-section is

$$
\sigma = \frac{\pi^2}{s} \int |T|^2 \, d\Omega \leq \frac{4\pi}{s}
$$

(6.7)

and this spin-averaged cross-section is $\bar{\sigma} = 2\pi/s$. By direct calculation, $\bar{\sigma}$ is $G^2 s^2/\pi$ so that the Born approximation equals the unitarity bound when $s = \pi\sqrt{2}/G = 4.2 \times 10^4 \text{ GeV}^2$.

The upshot of all this is that the form (5.1) for leptonic weak decays violates the unitarity bound at about 700 GeV total center-of-mass energy.

A popular modification of (5.1) is obtained by recognizing the analogy between (5.1) and second order electromagnetic interactions. The amplitude for electron-electron scattering can be calculated from the Feynman graph of fig. 6.1.

The contribution of fig. 6.1 to the amplitude $T$ is
where the spinors are normalized so that \( u^\dagger u = E \), and \( k \) is the momentum transfer. The numerator has a current-current form, just like (5.1). We introduce a charged vector meson \( W_\mu \), interacting with the weak current (5.2) according to

\[
-L_1 = g_w [J_\mu W^\mu + \text{h.c.}]
\tag{6.9}
\]

\( W^\mu \) is negatively charged. Then \( \nu + e^- \rightarrow \nu + e^- \) is described by the graph in fig. 6.2

\[
\frac{i g_w^2}{2\pi^2} \frac{-ie^2}{2\pi^2} \bar{u}(p_3)\gamma^\mu(1 - \gamma_5)u(p_1)\bar{u}(p_4)\gamma^\nu(1 - \gamma_5)u(p_2) \left[ g_{\mu\nu} \frac{k_\mu k_\nu}{M_w^2} \right] \frac{1}{k^2 - M_w^2}.
\tag{6.10}
\]

In the Fermi theory (5.1), the amplitude for \( \nu + e^- \rightarrow \nu + e^- \) is

\[
-iG \frac{g_w^2}{2\sqrt{2}\pi^2} \frac{-ie^2}{2\pi^2} \bar{u}(p_3)\gamma^\mu(1 - \gamma_5)u(p_1)\bar{u}(p_4)\gamma^\nu(1 - \gamma_5)u(p_2)g_{\mu\nu}.
\tag{6.11}
\]

For low \( k \), (6.11) and (6.10) are indistinguishable provided

\[
g_w^2/M_w^2 = G/\sqrt{2}.
\tag{6.12}
\]

From the Dirac equation, \( \gamma \cdot k \) can be replaced by \( m_e \) in (6.10), so that the second term in the propagator does not grow faster than the first. The amplitude is damped by a factor \( 1/k^2 \) compared to Fermi point interaction, and doesn’t come into glaring conflict with unitarity. Nevertheless, the theory is not renormalizable, as can easily be seen by calculating the amplitude for \( \nu + \bar{\nu} \rightarrow W^+ + W^- \). In fact, if a renormalizable theory is constructed using \( W \) mesons coupled to charged currents, the theory must contain additional particles to cancel the divergences in graphs with \( W^\pm \) mesons alone.

All the models we are about to describe contain charged \( W \) mesons to moderate the weak interactions. From (6.12), the sign of \( G \) is determined to be positive. In principle, this sign can be measured by looking for the parity-violating interference between a weak and electromagnetic or strong term in e.g., \( p + p \rightarrow n + n \) or \( e^+ + e^- \rightarrow \mu^+ + \mu^- \).

The radiative corrections to both \( \mu \) and \( \beta \) decays in \( W \)-meson theories are ambiguous and depend on the method of computation. If one adopts the \( \xi \)-limiting procedure of Lee and Yang, the ratio of the rates for \( \mu \) and \( \beta \) decays is finite. For a \( W \) mass \( > 2 \) GeV, one obtains \( 1 - g_V > 0.024 \).

Finally we mention the success of the idea that the strong interactions are approximately in-
variant under SU(2) \_L \times SU(2) \_R. The generators are the $T^i_L$ discussed above, whose charged components are the weak currents, and the $T^i_R$, constructed like $T^i_L$, replacing $V - A$ by $V + A$. Thus for nucleons,

\[ T^i_L = \int d^3x \, \psi^\dagger(x) \frac{i}{2} \left( \frac{1 - \gamma_5}{2} \right) \psi(x) \]

\[ T^i_R = \int d^3x \, \psi^\dagger(x) \frac{i}{2} \left( \frac{1 + \gamma_5}{2} \right) \psi(x). \]  

(6.13)

From $T^i_L$ and $T^i_R$ we may construct

\[ T^i = T^i_L + T^i_R \]  

(6.14)

which are just the isotopic spin generators, and the axial charges

\[ T^i = T^i_L - T^i_R. \]  

(6.15)

The group algebra is

\[ [T^i_{L,R}, T^j_{L,R}] = i \epsilon^{ijk} T^k_{L,R} \]

\[ [T^i_L, T^j_R] = 0 \]  

(6.16)

or

\[ [T^i, T^j] = i \epsilon^{ijk} T^k \]

\[ [T^i, T^j] = i \epsilon^{ijk} T^k \]

\[ [T^i, T^j] = i \epsilon^{ijk} T^k. \]  

(6.17)

The charge $T^i$ is the space integral of the time component of the axial current.

The idea of PCAC (partially conserved axial current) is that SU(2) \_L \otimes SU(2) \_R is an approximate symmetry of the strong interactions, realized in the Goldstone mode and that the pions are the Goldstone bosons in the symmetry limit, their mass being a measure of the symmetry breaking. Thus the Lagrangian has the form

\[ \mathcal{L} = \mathcal{L}_{\text{SYM}} + \epsilon \mathcal{L}' \]  

(6.18)

where $\epsilon$ is "small" of the order $M_\pi^2 / m_\rho^2$ and $\mathcal{L}_{\text{SYM}}$ is invariant under the group.

The matrix element of the axial vector current $A^i_\mu (= j^i_R \mu - j^i_L \mu)$ between the vacuum and a one-pion state with momentum $p$ is

\[ \langle \pi^i(p) | A^i_\mu(x) \rangle = \frac{iF_\pi}{(2\pi)^{3/2}} \frac{p_\mu \delta_{ij} \exp(-ip \cdot x)}{\sqrt{2p^0}}. \]  

(6.19)

Except for the normalization constant $F_\pi$, the form of (6.19) is dictated by Lorentz invariance. The value of $F_\pi$ can be determined from the decay $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. From equations (5.1), (5.14) and (6.19) we calculate the total rate for $\pi^- \rightarrow \mu$ decay to be
\[ \Gamma(\pi \rightarrow \mu + \nu) = G^2 M_\mu^2 F_\pi^2 (m_\mu^2 - m_\pi^2)^2 / 4 \pi M_\pi^3. \] (6.20)

From the measured pion life time \(2.60 \times 10^{-8}\) sec, and from the value of \(G\) obtained from \(\mu \rightarrow e + \nu + \bar{\nu}\), the value of \(F_\pi\) is determined to be

\[ F_\pi = 93 \text{ MeV}. \] (6.21)

As a consequence of (6.19)

\[ \langle \partial^\mu A_\mu^i(x) | \pi^I(p) \rangle = \frac{i F_\pi m_\pi}{(2\pi)^{3/2} \sqrt{2 E_\pi}} \delta_{ij} \exp(-ip \cdot x) = F_\pi m_\pi^2 \langle \phi^i(x) | \pi^I(p) \rangle \] (6.22)

where \(\phi^i(x)\) is the renormalized pion field. In the symmetry limit \((\epsilon \rightarrow 0)\), \(m^2 = 0\) and \(\partial^\mu A_\mu^i = 0\).

In this limit, any matrix element of \(A_\mu^i(x)\),

\[ M_{ab} = \langle b | A_\mu^i(0) | a \rangle \] (6.23)

has a pole at \(q^2 = 0\) \((q = p_b - p_a)\) of the form

\[ M_{ab} = \frac{i F_\pi q_\mu}{q^2} \langle b | j_{\pi}^i(0) | a \rangle \] (6.24)

where \(j_{\pi}^i(x) = \Box \phi^i(x)\) is the source of the pion field.

Low energy theorems in the unphysical world with an exactly conserved axial current can be obtained from (6.25) and its generalizations. The content of the PCAC assumption is that these are approximately true in the real world. Here are some examples.

Let \(a\) and \(b\) be nucleons. Then the most general form of (6.23) is

\[ \frac{1}{(2\pi)^3} i\bar{u}(p_0) [g_A(q^2) \gamma_\mu \gamma_5 - q_\mu \gamma_5 h(q^2)] \frac{\tau^I}{2} u(p_a). \] (6.25)

From the conservation of the axial current, we know that (6.25) multiplied by \(q^\mu\) is zero, therefore

\[ 2Mg_A(q^2) = q^2 h(q^2). \] (6.26)

From (6.24) and the fact that

\[ \langle \bar{N}_b | j_{\pi}^i(0) | N_a \rangle = \frac{g}{(2\pi)^3} \bar{u}(p_b) \tau^I \gamma_5 u(p_a) \] (6.27)

(where \(g\) is the pion-nucleon coupling) we obtain the Goldberger-Treiman relation

\[ F_\pi = Mg_A(0)/g \] (6.28)

where \(g_A(0) = g_A\).

Experimentally, \(g^2/4\pi \approx 14.6\), so \(Mg_A/g \approx 83\text{ MeV}\); comparing with the value 93 MeV obtained from \(\pi\)-decay, one gets an idea of the accuracy of PCAC.

Many other soft-pion theorems can be obtained from (6.24). The full power of the method be-
comes apparent when two or more soft pions are considered simultaneously, for then the commutators (6.17) of the SU(2) × SU(2) group enter the calculation.

For example, let

\[ T^{\mu \nu}(q) = \int \langle b | T(A^\mu_\pi(x) A^\nu_j(0)) | a \rangle \exp(-i q \cdot x) d^4x \]  

(6.29)

where \( a \) and \( b \) are nucleon states with momentum \( p \) and isospin indices \( a \) and \( b \). \( T \) has a double pole at \( q^2 = 0 \), whose residue is proportional to the forward \( \pi N \) scattering amplitude \( T_{\pi N}(q) \):

\[ T^{\mu \nu} = F_\pi^2 \frac{q^\mu q^\nu}{(q^2)^2} \int \langle b | T(j^\mu_\pi(x) j^\nu_j(0)) | a \rangle \exp(-i q \cdot x) d^4x + \text{less singular terms} \]

\[ = \frac{F_\pi^2}{2 \pi ip^0} \frac{q^\mu q^\nu}{(q^2)^2} T_{\pi N}(q) + \text{less singular terms at } q \to 0 \]  

(6.30)

where \( T_{\pi N} \) is normalized as in (6.4). Next we contract (6.30) by multiplying \( T^{\mu \nu} \) with \( q_\mu \),

\[ q_\mu T^{\mu \nu} = \frac{F_\pi^2}{2 \pi ip^0} \frac{q^\nu}{q^2} T_{\pi N}, \quad \text{as } q \to 0. \]  

(6.31)

On the other hand, from (6.29)

\[ q_\mu T^{\mu \nu} = i \int \frac{\partial}{\partial x^\mu} \langle b | T(A^\mu_\pi(x) A^\nu_j(0)) | a \rangle \exp(i q \cdot x) d^4x. \]

Because \( \partial_\mu A^\mu_\pi = 0 \), only the equal time commutator remains.

\[ q_\mu T^{\mu \nu} = \frac{F_\pi^2}{2 \pi ip^0} \frac{q^\nu}{q^2} T_{\pi N}, \quad \text{as } q \to 0. \]

(6.32)

where we have used the local form of the second of equations (6.17),

\[ [A^\nu_j(x), A^\nu_j(0)] \delta(x^0) = i \epsilon^{ijk} V_k(0) \delta^4(x). \]  

(6.33)

Since the vector currents are conserved,

\[ \langle b | V^\nu_k(0) | a \rangle = \frac{1}{(2 \pi)^3} p^\nu \frac{(\tau_k)_{ab}}{2}. \]  

(6.34)

Combining (6.33) with (6.31), we obtain

\[ T_{\pi N} = \frac{1}{8 \pi^2 F_\pi} p \cdot q T_{\pi} \cdot \tau_{ab} \]

(6.34)

where \( (T^{\dagger}_{\pi} \cdot \tau_{jk} = -i \epsilon_{ijk} \) are the pion isospin matrices. Equation (6.34) is a threshold theorem for \( \pi N \) scattering in the symmetry limit. We can apply eq. (6.34) to the real world at the real threshold \( (v = p \cdot q = Mm_\pi) \), since the nucleon pole terms, being P-wave, do not contribute there. Important corrections to eq. (6.34) for the real world are symmetric in \( i \) and \( j \), so we shall deal only
with the antisymmetric part. The result is a formula for the difference between the \( I = \frac{1}{2} \) and \( I = \frac{3}{2} \) scattering lengths. Using the Goldberger-Treiman relation (6.28) for \( F_\pi \), this difference is predicted to be

\[
a_{I=\frac{3}{2}} - a_{I=\frac{1}{2}} = \frac{3g^2}{8\pi} \frac{1}{M^2g_A^2} \frac{Mm_\omega}{M + m_\pi}.
\]

(6.35)

This is an equation for \( G_A \) in terms of measurable scattering lengths, as well as a relation between \( a_{1/2} \) and \( a_{3/2} \). Both are well-satisfied experimentally.

Assuming that \( T_{\pi N}(q) \) satisfies an unsubtracted dispersion relation, one obtains a sum rule for \( g_A \), ignoring terms of order \( \frac{m_\pi^2}{M^2} \):

\[
1 - \frac{1}{g_A^2} = \frac{2M^2}{g^2} \frac{1}{\pi} \int_0^\infty \frac{ds}{s^2 - M^2} [\sigma^+(s) - \sigma^-(s)]
\]

(6.36)

which is the original form obtained by Adler and Weisberger. In eq. (6.36), \( s = (p + q)^2 \), and \( \sigma^\pm \) is the total \( \pi^\pm \to p \) cross section.

Many other soft-pion theorems can be found using similar methods. The reader is referred to the book by Adler and Dashen for a more complete treatment.

Finally we mention an example of a class of theorems which aren't true. Let

\[
e^{(1)}_\mu e^{(2)}_\nu T^{\mu\nu\lambda} = \langle \gamma(e^{(1)}, k_1) \gamma(e^{(2)}, k_2) | A^\lambda(0) \rangle
\]

(6.37)

be the matrix element of the neutral axial current between two photons and the vacuum. Eq. (6.37) should contain a pole of the form

\[
F_\pi q^\lambda q^2 \langle \gamma\gamma|p_\pi^\lambda(0)\rangle
\]

(6.38)

where \( q = k_1 + k_2 \) and \( \langle \gamma\gamma|p_\pi^\lambda(0)\rangle \) is proportional to the \( \pi^o \to \gamma\gamma \) amplitude:

\[
T(\pi^o\gamma\gamma) = -\frac{2\pi}{s^2} \sqrt{k_1^2 k_2^2} \langle \gamma\gamma|p_\pi^\lambda(0)\rangle.
\]

(6.39)

Kinematically, \( T(\pi^o \to \gamma\gamma) \) must have the form

\[
T = e^{(1)}_\mu e^{(2)}_\nu \epsilon^{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta f(q^2).
\]

(6.40)

Physically, \( f(m_\pi^2) \) determines the \( \pi^o \) lifetime. We assume \( f(m_\pi^2) \approx f(0) \) to relate physical quantities to the predictions of PCAC.

The non-pole term in \( T^{\mu\nu\lambda} \) must be a three-index pseudotensor. The only term first order in the momenta one can construct which is symmetric in the two photons is

\[
e^{\mu\nu\lambda\alpha}(k_1^\alpha - k_2^\alpha).
\]

But this term violates electromagnetic gauge invariance, which requires \( k_{1\mu} T^{\mu\nu\lambda} = k_{2\nu} T^{\mu\nu\lambda} = 0 \). We conclude that

\[
e^{(1)}_\mu e^{(2)}_\nu T^{\mu\nu\lambda} = -\frac{iF_\pi q^\lambda}{q^2} \frac{T(\pi^o \to \gamma\gamma)}{(2\pi)^{s/2} \sqrt{k_1^2 k_2^2}} + e^{(1)}_\mu e^{(2)}_\nu T^{\mu\nu\lambda}
\]

(6.41)
when $T^\mu{}_{\nu}\lambda$ is at least second order in the momenta. We multiply (6.41) by $q^\lambda$ and use the conservation of $A^\lambda$ to obtain

$$0 = \frac{-iF_\pi}{(2\pi)^{d/2} \sqrt{k_1^0 k_2^0}} \epsilon^{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma f(q^2) + q^\lambda T^\mu{}_{\nu}\lambda.$$  \hspace{1cm} (6.42)

Since $q^\lambda T^\mu{}_{\nu}\lambda$ is at least third order in the momenta, it follows that $f(0) = 0$.

It has been shown that $f(0) = 0$ cannot in general be maintained in perturbation theory because of the singularities of the theory invalidate the formal arguments. Experimentally, $f(m^2_\pi) = O(m^2_\pi/M^2)$ predicts far too small a pion decay rate. Correct expressions in perturbation theory can be obtained if we set

$$\partial_\mu A^\mu = \frac{2\overline{Q}\alpha}{4\pi} F^{-\mu\nu}(x)f^\nu(x)$$  \hspace{1cm} (6.43)

where $F$ is the electromagnetic field tensor and $\overline{Q}$ is usually the same model-dependent number which entered our discussion of radiative corrections to $\beta$-decay. The value $\overline{Q} = \frac{1}{2}$, characteristic of a simple theory with one elementary charged fermion, like the proton, is in good agreement with experiment. The original Ward identity $\partial_\mu A^\mu = 0$, is recovered in models with an equal number of positive and negative fermion fields. Identities based on (6.43), which are correct in perturbation theory, are called anomalous Ward identities.

Bibliography

The unitarity bounds on weak processes are based on unpublished notes of one of us (E.S.A). See also

For the discussion of radiative corrections to $\mu$- and $\beta$-decays in intermediate vector meson theory
   T.D. Lee and C.S. Wu, op. cit.

For the Goldberger-Treiman relation, and the Adler-Weisberger relation, see
These subjects are excellently reviewed in Adler and Dashen, op. cit.

For the subject of anomalous Ward identities, see

This subject is reviewed in

A non-Abelian generalization of eq. (6.43) is given by
7. The Weinberg-Salam model

In this section we will describe the first model, which was proposed about five years ago by
Weinberg and Salam and which combines the weak and electromagnetic interaction through the
use of the Higgs mechanism.

The idea is to put the SU(2) L group discussed in section 5 together with electromagnetic gauge


bosons. There remains a heavy neutral vector meson, the photon, and one Higgs scalar. When
no confusion can arise, we will use the name of a particle to stand for its field. In general, we fol-


low Weinberg's notation.


In the simplest version, the only leptons are the electron e and its neutrino \( \nu \) (we omit the sub-
script in \( \nu_e \) for the moment). These may be grouped into a left-handed SU(2) L doublet

\[
L = \begin{pmatrix} \nu_L \\ e_L^* \end{pmatrix} \tag{7.1}
\]

where \( e_L = \frac{1}{2} (1 - \gamma_5)e \), and an SU(2) L singlet, \( R = e_R = \frac{1}{2} (1 + \gamma_5)\bar{e} \). We assign to the doublet a


"hypercharge" \( Y = -1 \) and to the singlet \( e_R \) a "hypercharge" \( Y = -2 \), so that the rule

\[
Q = T^I_L + \frac{1}{2} Y \tag{7.2}
\]

holds for all particles. Since all members of each irreducible multiplet of SU(2) L have the same


hypercharge,

\[
[T^I_L, Y] = 0. \tag{7.3}
\]

The group generated by \( T^I_L \) and \( Y \) is SU(2) \( \oplus \) U(1). We make this into the gauge symmetry of the
model, introducing three gauge mesons \( A^I_{\mu} \) associated with SU(2) L and a fourth \( B_{\mu} \) associated


with the U(1) subgroup. So far the model contains two pieces:

\[
\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{leptons}} \tag{7.4}
\]

where, according to the prescription of the first lecture,

\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^{I}_{\mu \nu} F^{I}_{\mu \nu} - \frac{1}{2} B_{\mu \nu} B^{\mu \nu}. \tag{7.5}
\]

In (7.5)

\[
F^{I}_{\mu \nu} = \partial_\mu A^{I}_\nu - \partial_\nu A^{I}_\mu + g e^{IJK} A^{J}_\mu A^{K}_\nu,
B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{7.6}
\]

The lepton part of \( \mathcal{L} \) is

\[
\mathcal{L}_{\text{leptons}} = \bar{R} i \gamma^\mu (\partial_\mu + ig' B_\mu) R + \bar{L} i \gamma^\mu \left( \partial_\mu + \frac{1}{2} g' B_\mu - ig \frac{T^I}{2} A^I_\mu \right) L. \tag{7.7}
\]

Recall that if the symmetry group is a direct product, the coupling constants may differ for each
factor. We take \( g \) to be associated with SU(2) L, and \( \frac{1}{2} g' \) with U(1). Notice that the SU(2) L in-
variance prohibits an electron mass term from appearing in (7.7).
We want to end up with three of the four vector mesons acquiring masses, since the final theory should have only one conserved quantity, the electric charge $Q$, and one massless meson, the photon.

To this end we introduce a doublet of (complex) Higgs scalars

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \quad (7.8)$$

The doublet $\phi$ transforms like $L$ of eq. (7.1) under SU(2)$_L$, and has $Y = +1$ in order to maintain (7.2). It contributes a term to the Lagrangian

$$\mathcal{L}_{\text{scalars}} = \left( \partial_\mu \phi^\dagger + \frac{i g'}{2} B_\mu \phi^\dagger + \frac{g}{2} r^i A^i_\mu \phi^\dagger \right) \left( \partial^\mu \phi - \frac{i g'}{2} B_\mu \phi - \frac{i g}{2} r^i A^i_\mu \phi \right) - V(\phi^\dagger \phi). \quad (7.9)$$

The most general form for $V$ is

$$V = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2. \quad (7.10)$$

There may also be an interaction term

$$\mathcal{L}_{\text{inter}} = -G_e [\bar{R} \phi^\dagger L + \bar{L} \phi R] \quad (7.11)$$

which is symmetric under the whole group as well as being Lorentz invariant.

Next, we let $\mu^2$ be negative so that one component, which we choose to be the neutral component of $\phi$, develops a vacuum-expectation value,

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} / \sqrt{2}. \quad (7.12)$$

Notice that this breaks both the SU(2)$_L$ and the hypercharge U(1) symmetry. The surviving symmetry operator is the combination $Q$ [eq. (7.2)]. We choose $v$ to be real, as in the example in section 3. From (7.10),

$$v = [ -\mu^2 / \lambda ]^{1/2}. \quad (7.13)$$

Next, we redefine the scalar fields, associating a new field with each broken generator. Actually, it is not necessary to find the generators orthogonal to $Q$; any three independent ones satisfying

$$T \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0$$

will do. Therefore, we define

$$U(\xi) = \exp(-i \xi \cdot \tau / 2v)$$

and write

$$\phi = U^{-1}(\xi) \begin{pmatrix} 0 \\ (v + \eta) / \sqrt{2} \end{pmatrix} \quad (7.14)$$

replacing the four real components of $\phi$ by $\eta$ and $\xi^i$. 

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Next we make a gauge transformation to the U-gauge, so that the particle content of the model becomes manifest:

$$\phi \rightarrow \phi' = U(\xi)\phi = \left( \begin{array}{c} 0 \\ \nu + \eta \end{array} \right) / \sqrt{2}$$

$$L \rightarrow L' = U(\xi)L$$

$$A_{\mu} \rightarrow A'_{\mu}$$

where

$$\tau \cdot A'_{\mu} = U(\xi) \left[ \tau \cdot A_{\mu} - \frac{1}{g} U^{-1}(\xi) \partial_{\mu} U(\xi) \right] U^{-1}(\xi)$$

(7.15)

and $B_{\mu}$ and $R$ are unchanged. We will drop the primes on $L'$ and $A'_{\mu}$. The new fields are just as good as the old ones, since the gauge transformation is not singular.

Now there are new terms quadratic in the new fields in both $\mathcal{L}_{\text{inter}}$ and $\mathcal{L}_{\text{scalars}}$. Eq. (7.11) becomes

$$\mathcal{L}_{\text{inter}} = -\frac{G_{e}v}{\sqrt{2}} [RL + LR] + \text{cubic and higher order terms} = -\frac{G_{e}v}{\sqrt{2}} \bar{e}e + ....$$

(7.17)

The electron has acquired a mass:

$$m_{e} = G_{e}v/\sqrt{2}.$$  (7.18)

The neutrino remains massless because there still are no right-handed neutrino fields. The part of the Lagrangian describing the $\phi$ field, eq. (7.9), has become

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} \partial^{\mu} \eta \partial_{\mu} \eta + \frac{(\nu + \eta)^{2}}{8} \chi_{-}^{\dagger} (g'B_{\mu} + g\tau^{i}A^{i}_{\mu})(g'B_{\mu} + g\tau^{i}A^{i}_{\mu}) \chi_{-} - V \left( \frac{(\nu + \eta)^{2}}{2} \right)$$

(7.19)

where $\chi_{-} = (\eta^{0})$.

The remaining scalar field $\eta$ has a mass-squared $-2\mu^{2}$. The quadratic term in the vector meson fields is

$$\frac{1}{2} \bar{\eta}^{2} [(g'B_{\mu} - gA^{3}_{\mu})(g'B_{\mu} - gA^{3}_{\mu}) + g^{2}(A^{1}_{\mu})^{2} + (A^{2}_{\mu})^{2})].$$

(7.20)

Define

$$W_{\mu}^{\pm} = (A_{\mu}^{1} \mp iA_{\mu}^{2})/\sqrt{2}.$$  (7.21)

Evidently the charged fields $W_{\mu}^{\pm}$ have mass

$$M_{w} = \frac{1}{2} gv.$$  (7.22)

Define two neutral fields

$$Z_{\mu} = \frac{-gA^{3}_{\mu} + g'B_{\mu}}{\sqrt{g^{2} + g'^{2}}} , \quad A_{\mu} = \frac{gB_{\mu} + g'A^{3}_{\mu}}{\sqrt{g^{2} + g'^{2}}}.$$  (7.23)
$Z_\mu$ and $A_\mu$ are eigenstates of the mass matrix, with masses

$$M_Z = \frac{1}{2} \sqrt{g^2 + g'^2}, \quad M_A = 0. \quad (7.24)$$

The single massless vector meson is the photon, corresponding to the surviving $U(2)$ symmetry $\exp(-i\theta Q)$.

It's instructive to rewrite $L_{\text{leptons}} - \text{eq. (7.7)}$ in terms of the $W^\pm, Z$, and photon. From (7.21)

$$A_\mu^{(1)} = (W^-_\mu + W^+_\mu)/\sqrt{2}, \quad A_\mu^{(2)} = (W^-_\mu - W^+_\mu)/i\sqrt{2}. \quad (7.21')$$

Therefore the term in (7.7) containing $W^\pm$ is

$$\frac{g}{2} \bar{\ell} \gamma^\mu (\tau^1 A_\mu^1 + \tau^2 A_\mu^2) \ell = \frac{g}{2} \left( \bar{\ell}_L \gamma^\mu e_L + \bar{\ell}_L \gamma^\mu \nu_L \right) \frac{(W^-_\mu + W^+_\mu)}{\sqrt{2}} - i(\bar{\ell}_L \gamma^\mu e_L - \bar{\ell}_L \gamma^\mu \nu_L) \frac{(W^-_\mu - W^+_\mu)}{i\sqrt{2}}$$

$$= \frac{g}{\sqrt{2}} \left[ \bar{\ell}_L \gamma^\mu e_L W^+_\mu + \bar{\ell}_L \gamma^\mu \nu_L W^-_\mu \right]. \quad (7.25)$$

Comparing with eq. (6.12) we obtain

$$G/\sqrt{2} = g^2/8M_W^2 = 1/2v^2. \quad (7.26)$$

Next we examine the terms in (7.7) containing $A_\mu^3$ and $B_\mu$. Define an angle $\theta_w$ by

$$g'/g = \tan \theta_w. \quad (7.27)$$

Then from (7.23)

$$A_\mu = \cos \theta_w B_\mu + \sin \theta_w A_\mu^3 \quad Z_\mu = \sin \theta_w B_\mu - \cos \theta_w A_\mu^3. \quad (7.28)$$

Inverting, we get

$$B_\mu = \cos \theta_w A_\mu + \sin \theta_w Z_\mu, \quad A_\mu^3 = \sin \theta_w A_\mu - \cos \theta_w Z_\mu. \quad (7.29)$$

The terms in $L_{\text{lepton}}$ coupling $A_\mu^3$ and $B_\mu$ to the leptons are

$$-\frac{g'}{2} \left[ 2\bar{e}_R \gamma^\mu e_R + \bar{\ell}_L \gamma^\mu e_L + \bar{\ell}_L \gamma^\mu \nu_L \right] \left[ \cos \theta_w A_\mu + \sin \theta_w Z_\mu \right] - \frac{g}{2} \left[ \bar{e}_L \gamma^\mu e_L - \bar{\ell}_L \gamma^\mu \nu_L \right] \left[ \sin \theta_w A_\mu - \cos \theta_w Z_\mu \right]$$

$$= \frac{-Z_\mu}{2\sqrt{g^2 + g'^2}} \left[ g'^2 (2\bar{e}_R \gamma^\mu e_R + \bar{\ell}_L \gamma^\mu e_L + \bar{\ell}_L \gamma^\mu \nu_L) - g^2 (\bar{e}_L \gamma^\mu e_L - \bar{\ell}_L \gamma^\mu \nu_L) \right] - \frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu \left[ \bar{e}_R \gamma^\mu e_R + \bar{\ell}_L \gamma^\mu e_L \right]. \quad (7.30)$$

Thus the massless vector meson $A_\mu$ does couple to the electric current $\bar{e} \gamma^\mu e$, and we can identify the electron's charge $-e$:

$$e = gg'/\sqrt{g^2 + g'^2}. \quad (7.31)$$

Finally, we verify that local gauge invariance still holds for the local $U(1)$ group corresponding to $Q$, with the photon field $A_\mu$ being the gauge meson. Under an infinitesimal transformation generated by $Q = \frac{1}{2} Y + T^B_L$,
\[ \delta A_3 = \frac{1}{g} \partial_\mu \epsilon(x) \]
\[ \delta B_\mu = \frac{2}{g'} \partial_\mu e(x) = \frac{1}{g'} \partial_\mu \epsilon(x) \]

so, from (7.28)
\[ \delta Z_\mu = \left( \frac{1}{g} \sin \theta_w - \frac{1}{g} \cos \theta_w \right) \partial_\mu \epsilon(x) = 0 \]
\[ \delta A_\mu = \left[ \frac{\cos \theta_w}{g'} + \frac{\sin \theta_w}{g} \right] \partial_\mu \epsilon(x) \]
\[ = \frac{1}{gg'} \left[ g \cos \theta_w + g' \sin \theta_w \right] \partial_\mu \epsilon(x) = \frac{\sqrt{g^2 + g'^2}}{gg'} \partial_\mu \epsilon(x) = \frac{1}{e} \partial_\mu \epsilon(x). \quad (7.32) \]

Bibliography

The model described in this section was proposed by

A model based on the same gauge group was proposed by

8. Phenomenology of the model. Incorporation of hadrons

Since both \( g/[g^2 + g'^2]^{1/2} \) and \( g'/[g^2 + g'^2]^{1/2} \) are less than 1, we can conclude from (7.31) that
\[ g \sin \theta_w = e, \quad g' \cos \theta_w = e, \quad (8.1) \]
so both \( g \) and \( g' \) are greater than \( e \).

From (7.22), the mass of the W is given by \( M_w^2 = \frac{1}{4} g^2 v^2 \). From (7.26), \( v^2 = 1/G \sqrt{2} \). Therefore
\[ M_w^2 = \frac{g^2 v^2}{4} = \frac{e^2}{\sin^2 \theta_w} \frac{1}{4 \sqrt{2} G}. \quad (8.2) \]

The W mass must be quite large,
\[ M_w = \left[ \frac{\pi \alpha}{\sqrt{2} G} \right]^{1/2} \frac{1}{\sin \theta_w} \approx \frac{38}{\sin \theta_w} \text{ GeV.} \quad (8.3) \]

Evidently, in this model, the minimum value of \( M_w \) is too large to be produced in present-day accelerators; nevertheless, it is not nearly as large as the unitarity bound, which is of the order of hundreds of GeV.

The Z meson is even heavier. From eq. (7.24)
\[ M_Z = \frac{\sqrt{g}}{2 \cos \theta_W} = \frac{M_W}{\cos \theta_W} = \frac{38 \text{ GeV}}{\frac{1}{2} - \sin^2 \theta_W}. \]

(8.4)

Since \( g' = 0 \) is not allowed, \( \cos \theta_W < 1 \), and

\[ M_Z > M_W, \quad M_Z > 76 \text{ GeV}. \]

(8.5)

The value of the dimensionless \( e^-e^-\eta \) coupling constant \( G_e \) can be obtained from (7.18) and (7.26)

\[ G_e = \frac{\sqrt{2} m_e/v = \sqrt{2} m_e \cdot \sqrt{2} \sqrt{G} \sim 2 \times 10^{-6}} {2 \times 10^{-6}} \]

(8.6)

which is small, indicating that graphs with \( \eta \) vertices can often be ignored compared to graphs with photon or \( Z \) vertices.

What is the effect of the \( W \) on the spectrum for \( \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \)? The \( (\mu^-, \nu_\mu) \) doublet is easily incorporated into the model in exact analogy to the \( (e^-, \nu_e) \) doublet, and the \( \mu \)-mass generated by the \( \mu - \nu_\mu - \phi \) coupling. The coupling constant \( G_\mu \) must have the value

\[ G_\mu = (m_\mu/m_e) G_e \]

which is larger than (8.6) but still very small. The amplitude for \( \mu^- \)-decay is

\[ \frac{-ig^2}{16\pi^2} \bar{u}(\nu_\mu)\gamma^\mu(1 - \gamma_5)u(\mu)\bar{u}(e)\gamma^\mu(1 - \gamma_5)v(\nu_e) \frac{[\sigma_{\mu\nu} - k.k'/M_W^2]}{k^2 - M_W^2} \]

(8.7)

where \( k = p(\mu) - p(\nu_\mu) = p(e) + p(\bar{\nu}_e) \). In (8.7), the \( g_{\mu\nu} \) term reproduces the point interaction spectrum up to terms of the order \( k^2/M_W^2 \). The second term is of the order \( m_e m_\mu/M_W^2 \). So the effect on the spectrum is very small.

The most accessible test of the model seems to be \( \bar{\nu} - e^- \) elastic scattering. The \( W \) contribution comes from fig. 8.1(a).

At low energies, the contribution of fig. 8.1(a) is indistinguishable from the Fermi theory:

\[ T^{(a)} = \frac{iG}{2\sqrt{2}\pi^2} v(\bar{\nu})\gamma^\mu(1 - \gamma_5)u(\bar{\nu})\bar{u}(e^-)\gamma^\mu(1 - \gamma_5)u(e) \]

(8.8)

where we have applied a Fierz transformation to the \((V - A)(V - A)\) coupling. The sign in (8.8) is the product of a minus sign from Fermi statistics and a minus sign from the Fierz transformation. The \( Z \)-exchange contribution can be obtained from (7.30). At low energies it is
\[
\frac{iG}{2\sqrt{2}\pi^2} v(\nu')\gamma^\mu (1 - \gamma_5) u(\nu)\bar{u} (v) [2\sin^2\theta_W - \frac{1}{2} + \frac{1}{2}\gamma_5] u(e). \tag{8.9}
\]

In general, we may write the amplitude for \(\nu + e \rightarrow \nu + e^-\) as
\[
\frac{iG}{2\sqrt{2}\pi^2} v(\nu')\gamma^\mu (1 - \gamma_5) u(\nu)\bar{u} (e^-) [C_V - \gamma_5 C_A] u(e^-). \tag{8.10}
\]

Then \(W\) exchange alone (or Fermi coupling) predicts
\[
C_V = C_A = 1 \tag{8.11}
\]

while the present model predicts
\[
C_V = 2\sin^2\theta_W + \frac{1}{2}, \quad C_A = \frac{1}{2}. \tag{8.12}
\]

From (8.10), the spin-averaged differential cross section can be calculated. The cross section into solid angle \(d\Omega\) (in the center of mass frame) is
\[
\frac{d\sigma}{d\Omega} = \frac{G^2}{4\pi^2 s} \left[ (C_V - C_A)^2 (p \cdot q)^2 + (C_V + C_A)^2 (p' \cdot q')^2 - m_e^2 (C_V^2 - C_A^2) (p \cdot p') \right] \tag{8.13}
\]

where \(p\) and \(p'\) are the initial and final neutrino momenta, \(q\) and \(q'\) the initial and final electron momenta, and \(s = (p + q)^2\). In terms of the lab-frame electron recoil energy, \(T\), we obtain from (8.13)
\[
\frac{d\sigma}{dT} = \frac{G^2 m_e}{2\pi} \left[ (C_V - C_A)^2 + (C_V + C_A)^2 \left[ 1 - \frac{T}{\omega} \right] - (C_V^2 - C_A^2) \frac{m_e T}{\omega^2} \right] \tag{8.14}
\]

where \(\omega\) is the neutrino energy in the initial electron's rest frame (the lab frame). The last term is small for \(\omega \gg m_e\). In the \(V - A\) model \((C_V = C_A = 1)\), \(d\sigma/dT\) decreases, for fixed \(T\), like \(1/\omega\). Otherwise, there is a constant term. If \(C_V = -C_A\) (not possible in the W.-S. model), \(d\sigma/dT\) would be entirely independent of \(\omega\).

Gurr, Reines and Sobel have looked for \(\nu e\) events from anti-neutrinos produced by a Savannah River Plant reactor. What they measure is the rate given by (8.14) integrated from a minimum to a maximum value of \(T\), folded into the neutrino spectrum. \(T_{\text{max}}\) is just the neutrino energy \(\omega\),

![Fig. 8.2. Region of values of \(C_V\) and \(C_A\) in agreement with the experiment of Gurr, Reines and Sobel.](image)
and $T_{\text{min}}$ is determined by the experimental conditions. They have established that the cross section is less than twice that predicted by $C_V = C_A = 1$. Fig. 8.2 is from their paper. It is a map of the $C_V - C_A$ space, the shaded region being the value allowed by their experiment. The $V - A$ theory is not excluded, and the W.-S. model is acceptable for $\sin^2 \theta_W \leq 0.35$, corresponding to a $W$ mass greater than 60 GeV.

The amplitude for $\nu_{\mu} + e \rightarrow \nu_{\mu} + e$ can be parametrized in a similar way. It is a particularly interesting process because it is forbidden if only charged currents exist, since $\nu_{\mu} - e$ does not couple to $W$. If there is a neutral $Z$, elastic $\nu_{\mu} e$ scattering will be mediated by $Z$ exchange, as in fig. 8.3.

The effective interaction is

\[
-i \frac{G}{\sqrt{2}} \bar{\nu}_\mu \gamma^\mu (1 - \gamma_5) \nu_\mu [\bar{e} \gamma_\mu (C_V' - C_A' \gamma_5) e].
\]

(8.17)

In the W.-S. model,

\[
C_A = \frac{1}{2}, \quad C_V = \frac{1}{2} - 2 \sin^2 \theta_W.
\]

(8.18)

In pure $V - A$ theory, $C_A' = C_V' = 0$.

Recent experiments at CERN have put bounds on both the $\nu_{\mu} e$ and $\nu_{\mu} e$ elastic cross sections. Like $\nu_e + e$, both grow linearly with the (anti) neutrino energy $\omega$ for $\omega \ll M_Z$. If $\omega$ is measured in GeV, the cross sections are less than $0.7 \times 10^{-41} \omega \text{ cm}^2$ and $1.1 \times 10^{-41} \omega \text{ cm}^2$, respectively. A formula like (8.14) describes these cross sections also in terms of $\theta_W$. The experimental bounds restrict $\sin^2 \theta_W$ to be less than about 0.6, which so far is less restrictive than the bounds obtained from elastic $e^+ e^-$ scattering.

There have been many attempts to include hadrons in a W.-S. type model. One of the principal difficulties is that a realistic theory must have $\Delta S = 1, \Delta Q = 1$ currents, but no neutral strangeness-changing currents. The question of hadronic neutral non-strange currents is still open experimentally.

A straightforward way to add hadrons to the model without changing its basic structure is to add three fundamental "quark" fields, which we shall call $p, n, \lambda$. We will not worry about approximate SU(3) symmetry here, but assume that the Lagrangian contains some very strong, symmetric term, like a vector gluon interaction, which does not affect the rest of the discussion.

Next we group the left-handed quarks into an SU(2)$_L$ doublet

\[
N_L = \begin{pmatrix}
  p_L \\
n_L \cos \theta + \lambda_L \sin \theta
\end{pmatrix} = \begin{pmatrix}
p_L \\
n_{\text{el}}
\end{pmatrix}
\]

(8.19)

where $\theta$ is the Cabibbo angle introduced in the eq. (5.22). The remaining singlets are the right-
handed quarks, $n_R$, $p_R$ and $\lambda_R$, and the combination orthogonal to the bottom line in (8.19), namely
\begin{equation}
\lambda_c = (\lambda_L \cos \theta - N_L \sin \theta).
\end{equation}

Since we are not interested here in SU(3) transformations, we assign $Y = 1$ to $n_L$, $Y = 2$ to $p_R$, and $Y = 0$ to the rest. Then $\rho$ has unit positive charge, and $\lambda$ and $n$ are neutral. (The conventional quark charges can be obtained by shifting the $Y$ assignments.)

The Lagrangian $\mathcal{L}$ must be symmetric under SU(2) $\times$ U(1). The lepton and gauge field pieces already are, and so can be the very strong vector gluon coupling term. A quark mass term, of the form
\begin{equation}
m_p \{ \bar{\rho}_R \rho_L + \bar{\rho}_L \rho_R \}
\end{equation}
is forbidden by the symmetry, so cannot appear in $\mathcal{L}$. The quark masses arise from interaction with the scalar doublet $\phi$.

To write the most general interaction, we need
\begin{equation}
\tilde{\phi} = i \alpha_2 \phi^* = \begin{pmatrix} \phi^o \\ -\phi^* \end{pmatrix}
\end{equation}
which also transforms like a doublet under SU(2), but has $Y = -1$. The general quark-scalar interaction has the form
\begin{equation}
G_1 [N_L \tilde{\phi} p_R + \text{h.c.}] + G_2 [\bar{N}_L \phi p_R + \text{h.c.}] + G_3 [\bar{N}_L \phi \lambda_R + \text{h.c.}] + G_4 [n_R \lambda_c] + G_5 [\lambda_R \lambda_c].
\end{equation}
The quark mass matrix is obtained by replacing $\phi$ by its vacuum expectation value:
\begin{align*}
\langle \phi^* \rangle &= 0, \\
\langle \phi^o \rangle &= v/\sqrt{2}.
\end{align*}
The term quadratic in the fermion fields in eq. (8.22) becomes
\begin{equation}
\frac{v}{\sqrt{2}} [G_1 \bar{\rho} p + G_2 (\bar{n} \cos \theta + \bar{\lambda} \sin \theta)n + G_3 (\bar{n} \cos \theta + \bar{\lambda} \sin \theta)\lambda \\
+ G_4 \bar{n} (\lambda \cos \theta - n \sin \theta) + G_5 \bar{\lambda} (\lambda \cos \theta - n \sin \theta)].
\end{equation}
Evidently, $G_1, \ldots, G_5$ must be adjusted so that the mass of the $p$-quark is $m_p$, etc., and the physical $n$- and $\lambda$-quarks are mass eigenstates. This determines the couplings $G_1, \ldots, G_5$ completely.

In terms of the quark masses and the Cabibbo angle $\gamma$, (8.22) may be written
\begin{equation}
\frac{\sqrt{2}}{v} \left[ m_p (N_L \tilde{\phi} p_R + \text{h.c.}) + m_n [\bar{n}_R (\phi^* N_L \cos \theta - \frac{v}{\sqrt{2}} \lambda_c \sin \theta) + \text{h.c.}] \\
+ m_\lambda [\bar{\lambda}_R (\phi^* N_L \sin \theta + \frac{v}{\sqrt{2}} \lambda_c \cos \theta) + \text{h.c.}] \right].
\end{equation}
Since $v$ is determined by eq. (7.26) in terms of the Fermi constant $G$, we conclude that the coupling constants $G_i$ in (8.23) are quite small, of the order 1%, and therefore the Higgs scalar couples to the quarks weakly.
Let us examine the neutral quark currents. The coupling to $B_\mu$ and $A_\mu^3$ is

$$g'B_\mu[\vec{p}_R\gamma^\mu p_R + \frac{1}{2}(\vec{n}_L\gamma^\mu n_L \cos^2\theta + \vec{\lambda}_L\gamma^\mu \lambda_L \sin^2\theta + \cos\theta \sin\theta(\vec{n}_L\gamma^\mu \lambda_L + \vec{\lambda}_L\gamma^\mu n_L))] + gA_\mu^3\gamma^\mu (8.25)$$

where

$$j^{(3)}_\mu = [\vec{p}_L\gamma^\mu p_L - \vec{n}_L\gamma^\mu n_L \cos^2\theta - \vec{\lambda}_L\gamma^\mu \lambda_L \sin^2\theta - \cos\theta \sin\theta(\vec{n}_L\gamma^\mu \lambda_L + \vec{\lambda}_L\gamma^\mu n_L)]. \quad (8.26)$$

The terms proportional to $\cos\theta \sin\theta$ are the strangeness-changing neutral currents. In terms of $A_\mu$ and $Z_\mu$, the neutral vector eigenstates given by (7.23) in terms of $A_3^\mu$ and $B_\mu$, the interaction (8.25) can be written

$$\frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu^\mu j^{\text{(em)}} (8.27)$$

where $j^{(3)}_\mu$ is given by (8.26) and

$$j^{\text{(em)}} = \vec{p}\gamma^\mu p.$$

For other charge assignments eq. (8.27) still holds, with appropriate $j^{\text{(em)}}$. Since $j^{\text{(em)}}$ contains no terms with $\lambda$ or $n$, $Z_\mu$ does couple to a strangeness-changing neutral current.

This result is impossible to avoid with only those quarks, whatever their charge assignments. Because of the limits placed experimentally on such currents by the absence of $K^+ \to \pi^+ + e^+ + e^-$ or $K^0_L \to \mu^+ + \mu^-$, it is desirable to eliminate them. A model which does this has been suggested by Glashow, Iliopoulos and Maiani. They add a fourth quark, $q^+$, and group the quarks into two SU(2)$_L$ doublets:

$$\begin{pmatrix} p \\ n_c \end{pmatrix}_L \quad \text{and} \quad \begin{pmatrix} q^+ \\ \lambda_c \end{pmatrix}_L. \quad (8.28)$$

If the mass of the $q^+$ is very high, no unwanted effects will appear. Instead of (8.26), the neutral current is now

$$j^{(3)}_\mu = \frac{1}{2} [\vec{p}_L\gamma^\mu p_L - \vec{n}_c\gamma^\mu n_c + \vec{\lambda}_c\gamma^\mu \lambda_c - \vec{\lambda}_c\gamma^\mu n_c + \vec{\lambda}_c\gamma^\mu \lambda_c]. \quad (8.29)$$

Because $(n_c, \lambda_c)$ is obtained from $(n, \lambda)$ by making a unitary transformation (5.22), the combination $\vec{n}_c\gamma^\mu n_c + \vec{\lambda}_c\gamma^\mu \lambda_c$ in (8.29) is just $\vec{n}_L\gamma^\mu n_L + \vec{\lambda}_L\gamma^\mu \lambda_L$. The cross terms proportional to $\cos\theta \times \sin\theta$ cancel, and the unwanted currents are eliminated.

In this model, the $Z$-hadron coupling is still given by (8.27), with $j^{(3)}_\mu$ given by (8.29), and the $Z$-lepton coupling is unchanged [see eq. (7.30) or (8.9)]. Specifically, the $Z$ coupling to hadrons and neutrinos takes the form

$$\sqrt{g^2 + g'^2} Z^\mu [j^{(3)}_\mu - \sin^2\theta_W j^{\text{(em)}}_\mu + \frac{1}{2} \bar{\nu}\gamma^\mu \frac{1}{2}(1 - \gamma_5)\nu].$$

For low energies, the amplitude for $\nu + a \to \nu' + b$, where $a$ and $b$ are hadron states, is proportional to

$$\frac{\bar{\nu} j^{(3)}_\mu}{4M^2_Z} \langle b | j^{(3)}_\mu - \sin^2\theta_W j^{\text{(em)}}_\mu | a \rangle \bar{\nu}'\gamma^\mu(1 - \gamma_5)\nu. \quad (8.30)$$
Using (7.24) for $M_z$, we obtain

$$\frac{(g^2 + g'^2)}{4M_z} = \frac{1}{v^2} = \sqrt{2} G. \quad (8.31)$$

The rates are therefore independent of the $Z$ mass at low energies, when the $Z$-propagator can be approximated by $-g_{\mu\nu}M_z^2$.

For example, the amplitude for elastic $\nu p$ scattering has been measured to be $(0.12 \pm 0.06)$ times the rate for $\nu + n \rightarrow \ell^- + p$. To make a theoretical prediction, the matrix element from (8.30) can be obtained as follows: The matrix elements of $j^{(em)}$ are well known from electromagnetic form factors. The current $j^{(3)}_\mu$ is the neutral component of a triplet whose charged member is just what is measured in $\nu + n \rightarrow \mu^- + p$. Thus the amplitude from (8.30) is known experimentally. Pais and Treiman predict the branching ratio to be

$$0.15 \leq \frac{\sigma(\nu + p \rightarrow \nu + p)}{\sigma(\nu + n \rightarrow \mu^- + p)} \leq 0.25 \quad (8.32)$$

provided $\theta_w < 0.35$, as required by the $e^-\nu$ elastic scattering experiments.

Even more stringent bounds can be obtained from experiments looking for weak pion production. We will say only a few words and refer you to the literature for details. Consider the process $\nu + p \rightarrow \nu + p + \pi^0$. We need the matrix element

$$\langle \pi^0 | j^{(3)}_\mu | p \rangle \quad (8.33)$$

The electromagnetic current can be measured in $\pi^0$ electroproduction. The charged version of $j^{(3)}_\mu$, $\langle \pi^0 | j^{(3)}_\mu | n \rangle$ can be measured in $\nu + n \rightarrow \nu + \pi^0 + p$ experiments. Actually this matrix element is not simply related to $\langle \pi^0 | j^{(3)}_\mu | p \rangle$ by isospin, because $p\pi^0$ can have either $I = \frac{1}{2}$ or $I = \frac{3}{2}$. However, inequalities can be deduced, $\pi^+$ and $\pi^-$ amplitudes may be averaged, isospin zero nuclei may be used for targets, or events may be selected where the $3-3$ resonance is known to dominate.

There are experimental bounds on many branching ratios for neutral-to-charged neutrino-induced pion production processes. One of the most stringent is

$$R = \frac{\sigma(\nu + p \rightarrow \nu + p + \pi^0) + \sigma(\nu + n \rightarrow \nu + n + \pi^0)}{2\sigma(\nu + n \rightarrow \mu^- + p + \pi^0)} \leq 0.14. \quad (8.34)$$

Theoretical arguments, with inputs from other experiments, predict $R > 0.2$. Although these numbers are subject to considerable theoretical and experimental uncertainties, it is beginning to look as if there may not be any neutral hadron currents which couple to neutrinos. However, only more detailed measurements can settle this point.

**Bibliography**

The $\bar{\nu}-e$ scattering experiments at the Savannah River reactor are described in


The analysis of these experiments is by


The hadron model without strangeness-changing neutral currents is due to

The inelastic $\nu-p$ experiments are reported by

They have been analyzed by

9. Models with heavy leptons

In this section we shall describe models without neutral vector mesons coupling to neutrinos. In these models, the rates for all neutrino processes described in the last section vanish to order $G^2$.

All these sections involve heavy leptons. The reason is simply that the graph in fig. 9.1 for $\nu + \bar{\nu} \rightarrow W^+ + W^-$ exists in all models.

The amplitude calculated from this graph grows linearly with $s$, and therefore violates the unitarity bound. This behavior leads to a non-renormalizable theory, because the box graph occurring in the fourth order $\nu + \bar{\nu}$ elastic amplitude is quadratically divergent. In the Weinberg–Salam theory, the leading asymptotic behavior of the graph in fig. 9.1 is cancelled by the graph in fig. 9.2. The skeptical reader should calculate the ZWW vertex and verify this cancellation.

If $\nu\nu Z$ vertices are to be banned, the linear growth of the graph in fig. 9.1 must be cancelled somehow. The only other alternative is more leptons, as in fig. 9.3.

The linear term in fig. 9.3 has the opposite sign to the linear term in fig. 9.1, and therefore they can cancel with appropriate coupling constants, leading to a theory which may be renormalizable. The hypothetical $E^*$ is a “heavy” lepton, because if it were lighter than the $K^+$ meson, it would already have been seen in $K^+ \rightarrow E^* + \bar{\nu}$.

Heavy leptons can be introduced in the context of an SU(2) × U(1) model, where one of their functions is to eliminate the $Z\nu\nu$ coupling. For example, we may introduce a left-handed triplet

$$\begin{pmatrix} E^* \\ \nu \\ e^- \end{pmatrix}$$

(9.1)

In addition the model contains right-handed SU(2)$_L$ singlets, $e^-_R$ and $E^*_R$. The triplet can be assigned $Y = 0$. The electron and $E^*$ have $Y = -2$ and $Y = +2$ respectively. Then the neutral current is

$$j^{(3)}_\mu = \bar{E}^*_L \gamma_\mu E^*_L - \bar{e}^-_L \gamma_\mu e^-_L$$

(9.2)

Fig. 9.1. Electron exchange graph for $\nu + \bar{\nu} \rightarrow W^+ + W^-$. 

55
Fig. 9.2. Z annihilation graph for \( \nu + \nu \rightarrow W^+ + W^- \).

Fig. 9.3. Heavy lepton exchange graph for \( \nu + \nu \rightarrow W^+ + W^- \).

which contains no \( \bar{\nu} \gamma \mu \nu \) term. Neither \( A_\mu^{(3)} \) nor \( B_\mu \) couple to the neutrinos, so neither do the linear combinations \( A_\mu \) or \( Z_\mu \).

Another possibility is to add a neutral \( E^0 \) to the scheme just described, and group the leptons into two doublets

\[
\begin{pmatrix}
(\nu + E^0)/\sqrt{2} \\
e^-
\end{pmatrix}_L \\
\begin{pmatrix}
E^+ \\
(\nu - E^0)/\sqrt{2}
\end{pmatrix}_L
\]

(9.3)

with \( Y = -1 \) and \( Y = +1 \) respectively. \( E^0_R \) has \( Y = 0 \). The hypercharge current

\[
\frac{1}{2}(\bar{\nu}_L + \bar{E}^0_L)\gamma^\mu(\nu_L + E^0_L) - \frac{1}{2}(\bar{\nu}_L - \bar{E}^0_L)\gamma^\mu(\nu_L - E^0_L)
\]

contains no term in \( \bar{\nu} \gamma^\mu \nu \) and neither does \( j_\mu^{(3)} \). In such a model one would expect \( \nu + e^- \rightarrow E^o + e^- \) at sufficiently high energy, but no elastic \( \nu + e \) scattering. The former model is known as the LPZ model; the latter as the PZ II model.

A rather different idea has been suggested by Georgi and Glashow. Instead of SU(2) \( \times \) U(1), let the basic gauge group be O(3). Then there will be only one neutral current, and it must be just \( j^{(em)} \). In this model there is no other neutral current at all, so that there is no parity violation predicted in electromagnetic processes like \( e^- + e^- \rightarrow e^- + e^- \) or \( e^- + p \rightarrow e^- + p \).

The simplest way to realize this idea is to add a neutral lepton \( E^o \) and group it together with \( E^+, \nu, \) and \( e^- \) into a triplet:

\[
L = \begin{pmatrix}
E^+ \\
\nu \sin \beta + E^0 \cos \beta \\
e^-
\end{pmatrix}_L
\]

(9.4)

\( E^o \) must have a mass, so we can form a right-handed triplet also

\[
R = \begin{pmatrix}
E^+ \\
E^0 \\
e^-
\end{pmatrix}_R
\]

(9.5)

There remains a left-handed singlet:

\( (E^0 \sin \beta - \nu \cos \beta)_L \).

The interaction of the leptons with the gauge fields \( A^I_\mu \) is, according to the general prescription
\[ g A^\mu \mu i = g(A^{(0)}_\mu j^{(0)} + A^+\mu j^{+} + A^-\mu j^-) \]  

(9.6)

where

\[ A^\mu = \frac{A^{(1)} + i A^{(2)}}{\sqrt{2}} \]

\[ j^\mu = \frac{j^{(1)} + i j^{(2)}}{\sqrt{2}} \]  

(9.7)

and therefore

\[ j^+_\mu = L_\alpha \gamma_\mu T^+_\alpha R_\beta + R_\alpha \gamma_\mu T^-_\alpha R_\beta \]

\[ j^0_\mu = L_\alpha \gamma_\mu T^0_\alpha R_\beta + R_\alpha \gamma_\mu T^0_\alpha R_\beta. \]  

(9.8)

In the spherical representation

\[ T^+ = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

[The phase is chosen so that for a neutral triplet (\( \phi^*, \phi^0, \phi^- \)) we have \( \phi^* = (\phi^-)^\dagger \).]

We identify \( A^\mu_+ \) with \( W^\mu_+ \), and \( A^{(0)} \) with the photon, \( A_\mu \).

The neutral term in eq. (9.6) is

\[ g A_\mu [\bar{\nu}_{\mu} \gamma_\mu e^- - \bar{E^\nu} \gamma_\mu E^\nu] \]  

(9.10)

and does not violate parity. Therefore we can identify

\[ g = e. \]  

(9.11)

The charged term is

\[ e W_\mu^\nu [(\bar{\nu}_{L} \sin \beta + E_{L} \cos \beta) \gamma_\mu e_{L} - \bar{E}_{L} \gamma^\nu (\nu_{L} \sin \beta + E_{L} \cos \beta) + \bar{E}_{R} \gamma_\mu e_{R} - \bar{E}_{R} \gamma^\nu E_{R}] + \text{H.C.} \]  

(9.12)

The term in (9.12) which couples electrons to neutrinos is

\[ e \sin \beta W^* \bar{\nu}_{L} \gamma_\mu e^- + \text{H.C.} = \frac{1}{2} e \sin \beta W^* \bar{\nu} \gamma_\mu (1 - \gamma_5)e^- + \text{H.C.} \]  

(9.13)

Therefore

\[ G/\sqrt{2} = e^2 \sin^2 \beta/4M_W^2. \]  

(9.14)

In the W.-S. model — compare eq. (8.2) — we had

\[ G/\sqrt{2} = e^2/(8M_W^2 \sin^2 \theta_W). \]  

(9.15)

The Georgi–Glashow model therefore has an upper bound for the W mass

\[ M_W < \sqrt{2} (38 \text{ GeV}) = 53 \text{ GeV}. \]
Obviously, the muon and its neutrino can be introduced analogously, at the cost of two new heavy muons.

The Higgs mechanism for this model can be constructed from a triplet of scalar fields \( \phi^i \). The gauge invariant kinetic energy term for the \( \phi^i \) is given by (3.10), and we saw in the third section that if

\[
\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0, \quad \langle \phi_3 \rangle = v,
\]

(9.16)

then the W meson acquires a mass \( ev \).

It will be useful later to write the Lagrangian in terms of the charge eigenstates \( \phi^o = \phi_3 \) and \( s^\pm = (\phi_1 \mp i\phi_2)/\sqrt{2} \). Then \( (s^+, \phi^o, s^-) \) form a basis for the representation (9.9) of \( O(3) \). The Lagrangian term for the Higgs scalars is

\[
\mathcal{L}_{\text{scalars}} = \frac{1}{2} (\partial_\mu \phi^i - i e A_\mu \cdot (T)_{ik} \phi_k) (\partial_\mu \phi^i - i e A^\mu \cdot (T)_{ik} \phi_k)
\]

(9.17)

which becomes, in terms of \( s^\pm \) and \( \phi^o \),

\[
\mathcal{L}_{\text{scalars}} = \frac{1}{2} [\partial_\mu \phi^o + i e(W^- s^+ - W^+ s^-)] [\partial^\mu \phi^o - i e(W^+ s^- - W^- s^+)]
\]

\[
+ (\partial_\mu s^+ - i e A_\mu s^+ + i e \phi^o W^+) (\partial^\mu s^- + i e A^\mu s^- - i e \phi^o W^-).
\]

(9.18)

Write \( \phi^o = v + \psi \). Then it is evident from (9.18) that the photon remains massless, while \( W^\pm \) acquires a mass \( \mu \):

\[
\mu = ev.
\]

(9.19)

We see that there is a direct \( s^- W^\mu A_\mu \) coupling term

\[
-e \mu g^\mu \nu
\]

(9.20)

in addition to those explicit in (9.18). Of course, it is possible to eliminate the \( s^\pm \)-fields by writing

\[
\begin{pmatrix}
\phi^o \\
s^+ \\
s^-
\end{pmatrix} = \exp \{i(T_+ \xi_+ + T_- \xi_-)/v\}
\begin{pmatrix}
0 \\
u + \psi \\
0
\end{pmatrix}
\]

and performing the gauge transformation \( U = \exp(-i)[T_+ \xi_+ + T_- \xi_-] \) on the scalar, vector and fermion fields. This is the U-gauge discussed previously. However in a later section we will need the Feynman rules in other gauges, and for this reason we have written the Lagrangian in terms of the \( (s^+, \phi^o, s^-) \) fields, without eliminating the fictitious components.

Finally there may be fermion mass terms and fermion-scalar couplings. An invariant fermion mass term has the form

\[
\mathcal{L}_{\text{mass}} = -m_o [\bar{L} R + \bar{R} L]
\]

\[
= -m_o [\bar{E}^o E^o + \cos \beta \bar{E}^0 E^0 + \frac{1}{2} \sin \beta (\bar{\nu}(1 + \gamma_5) \bar{E}^0 + \bar{E}^0 (1 - \gamma_5) \nu) + \bar{\nu} e^-].
\]

(9.21)

There are two possible invariant coupling terms
\[ L_{\text{coupling}} = G_1 \left[ \bar{L}(T \cdot \phi)R + \text{H.C.} \right] + G_2 \left[ E_L^0 \sin \beta - \bar{\nu}_L \cos \beta \phi \cdot R + \text{H.C.} \right] \]  

(9.22)

where \( \phi \cdot R \) means \( s^e \bar{e}_R^* + s^e \bar{E}_R^* + \phi^0 E_R^0 \). Replacing \( \phi^0 \) with \( \nu \), we see that (9.22) contributes another fermion mass term to the Lagrangian. It is

\[ G_1 \nu \left[ \bar{E}^* E^* - \bar{R}^* \nu^* \right] \sin \beta \quad G_2 \nu \left[ \bar{E}^0 E^0 - \frac{1}{2} \cos \beta \right] \]

(9.23)

From (9.21) and (9.23), we obtain a fermion mass matrix, which we should diagonalize, and then impose the condition that the field we denoted by \( \nu \) is indeed massless. (Since there is one more neutral left-handed fermion than right-handed fermion, there is bound to be a massless left-handed field.) This condition gives, from eqs. (9.21) and (9.23)

\[ m_\nu \sin \beta + G_2 \nu \cos \beta = 0. \]  

(9.24)

The heavy leptons which occur in the models we have discussed may actually be reasonably light, and, if they exist, may be discovered long before the heavy vector mesons. All we really know is that they are all heavier than the K meson. They can probably be produced most easily in colliding e^+e^- beam, which can set lower limits on their masses close to the beam energy. Reactions like \( \nu + p \to E^+ + \text{hadrons} \) have also been studied, and appear to be feasible experiments at NAL energies. Decay modes like \( E^+ \to e^+ + \nu_e \), \( E^+ \to \nu_e + \mu^+ + \nu_\mu \), \( E^+ \to E^0 + \text{hadrons} \), or \( E^+ \to \nu_e + \text{hadrons} \), should all be easy to identify because of the apparent violation of momentum conservation. We have listed some recent references in the bibliography.

The masses of the fermions can be expressed in terms of \( m_\nu \), \( G_1 \), \( G_2 \) and \( \nu \):

\[ m_{E^+} = m_\nu - G_1 \nu \]

\[ m_{E^0} = \cos \beta \, m_\nu - \sin \beta \, G_2 \nu \]

\[ m_{\nu e} = m_\nu + G_1 \nu. \]  

(9.25)

From the first and third equations in (9.25), we obtain

\[ m_\nu = \frac{1}{2} (m_{E^+} + m_{\nu e}) \]  

(9.26)

and from the remaining relation and (9.24), we obtain

\[ m_{E^+} + m_{\nu e} = 2 \cos \beta \, m_{E^0} \]  

(9.27)

which is a general constraint on the masses of the leptons in this model. Then from (9.19), (9.25), (9.26) and (9.27),

\[ G_2 = \frac{e}{\mu} \sin \beta \, m_{E^0} \]  

(9.28)

and

\[ G_1 = \frac{e}{\mu} \frac{M_{\nu e} - M_{E^0}}{2} = \frac{e}{\mu} \left( m_{e^-} - \cos \beta \, m_{E^0} \right). \]  

(9.29)

Thus all the scalar-fermion couplings are fixed in terms of \( \beta \) and the \( e^- \) and \( E^0 \) masses. Alternatively \( \beta \) can be expressed through (9.27) in terms of the three masses. We shall use these results in Part II to calculate the anomalous magnetic moment of the muon of this model.
Bibliography

The models described in this section were introduced by
These papers also discuss the incorporation of hadrons in these models.

For a review of the phenomenology of heavy leptons, including calculations of production cross sections and branching ratios in various models, see

10. More on model building

A copy of the universe is not what is required of art; one of the damned thing is ample.

Rebecca West

In this section we shall try to describe various ramifications of gauge models of weak and electromagnetic interactions based on O(3) or U(2), their defects, and possible other avenues in model building. We will not dwell upon any one idea in detail, but rather try to present a panoramic overview on these developments. Instead of presenting a long list of recent articles and preprints exhaustively, we will cite representative works that have been at least partly digested by us.

We have seen a few examples of models based on SU(2) or U(2) gauge symmetries. The basic strategy of model building may be stated as follows:

A. Choose a gauge group.
B. Choose the representation of the Higgs scalar fields and their charge assignments.
C. Choose the representations of the spin $\frac{1}{2}$ chiral fermions.
D. Couple the gauge fields invariantly to the Higgs scalars and the fermions.
E. Couple the Higgs fields to themselves invariantly and renormalizably, so that the potential of the Higgs fields attains the minimum when neutral Higgs fields acquire non-vanishing vacuum-expected values.
F. Couple the Higgs fields invariantly to the fermions.

When these steps are taken,

a. Some gauge bosons acquire masses:
\[ \frac{1}{2} \left( \partial_{\mu} \phi + g W_\mu \phi \right)^2 \rightarrow \frac{1}{2} g^2 \left( \phi \right)^2 W_\mu^2. \]
b. Some fermions acquire masses:
\[ f(\bar{\psi}_R \psi_L \phi + \text{h.c.}) \rightarrow f(\phi) \bar{\psi} \psi. \]
c. At least one vector boson remains massless, because electric charge conservation is unbroken.
d. Some of the Higgs fields undergo a transmutation: they turn into the longitudinal components of the massive vector bosons.
In this strategy, the left-handed lepton ($e_L$ or $\mu_L$) and its neutrino are placed in a multiplet of SU(2), the right-handed component to another multiplet, by inventing heavy leptons as they are needed. If the multiplets chosen are such that $Q = T_3$, a neutral massive vector boson is not needed, and the unification can be achieved in an O(3) framework. Otherwise we need an SU(2) × U(1) scheme. Bjorken and Llewellyn-Smith have considered many schemes of this type:

1. J.D. Bjorken and C.H. Llewellyn-Smith, Phys. Rev. D7 (1973) 887, Appendix A. So far, we have closed our eyes to the CP-violation in weak interactions.

The latter scheme is based on the O(4) gauge group, which deserves attention on its own right.

Quite apart from this line of development, the Higgs mechanism provides us with a means of constructing renormalizable models of strong interactions based on the notion of "field algebra":


The field algebra is the field theoretic expression for vector dominance, by equating the hadronic currents with massive gauge bosons. In the past, the mass term for the gauge bosons was put in "by hand" — such a procedure breaks the renormalizability of the theory. The Higgs mechanism allows endowing the gauge bosons with masses. This was first noticed by 't Hooft;


and has since been generalized and elaborated on:


These are a number of applications of this idea to hadron physics. For example 't Hooft discussed the $\pi^+ - \pi^0$ mass difference from this point of view. For other applications, see

8. K. Bardakci, to be published.

There have been many attempts to incorporate three triplets of hadronic building blocks (such as the Han-Nambu, or three-color-quark schemes) which seem better suited to correlate various facets of hadron physics. See


The defect of the models discussed in previous sections is their inability to accommodate hadrons in a realistic, and "natural" manner. Let us illustrate this remark in terms of the scheme discussed in section 8, in which the quartet of spin $\frac{1}{2}$ fundamental hadronic building blocks is incorporated in the Weinberg-Salam model. The necessity of including four, rather than three, such objects arose from the absence of the $\Delta S = 1$ neutral current, and this fact should not be considered as a defect. Rather, it must be considered as heralding, possibly, a new dimension in
hadron spectroscopy, with a new quantum number associated with the “fourth quark”. The defect lies in that the approximate hadronic symmetries such as SU(2), SU(3) or chiral SU(2) × SU(2) are purely accidental in this scheme. For example, the hadronic isospin symmetry SU(2) has to be explained in this scheme as a consequence of an approximate equality of $m_p$ and $m_n$, which is not demanded by the gauge or other symmetries of the Lagrangian. It has long been the conviction (prejudice?) of particle physicists that the proton-neutron mass difference is due to electromagnetism and possibly also due to weak interaction, so that in an ultimate theory the mass difference should be computable. In the model under discussion, this mass difference is not zero even in lowest order, but is a free parameter.

The following papers discuss various conditions and circumstances under which intramultiplet mass differences are computable, as well as the definition of computability:


The central idea underlying these discussions is that any relationship which is true in lowest order in the presence of all gauge invariant, renormalization counterterms is also true in high orders with a finite computable correction.

Thus, if the mass difference within a hadronic multiplet is to be computable, the underlying hadron symmetry must not be broken by any renormalization counterterms in the Lagrangian. Future developments in model building ought to lie in the construction of models in which hadronic symmetries are accounted for naturally. There have been two important developments in this direction.

The first is the works of Bars, Halpern and Yoshimura and of de Wit:


The models proposed by these authors treat the hadronic and leptonic worlds as separate up to a point, each having its own set of gauge bosons; the two worlds communicate to one another through the intermediary of a new kind of Higgs mesons which carry both leptonic and hadronic quantum numbers and whose vacuum expectation values are responsible for the coupling of the two kinds of gauge bosons, in much the same way as in the field algebra. The following work is very similar to the above two in this respect:


The second is perhaps more profound in its concept. Weinberg notes that under certain circumstances the potential of the Higgs scalar fields cannot help but have a symmetry $\bar{G}$ larger than the gauge symmetry of weak and electromagnetic interaction. If the symmetry $\bar{G}$ is spontaneously broken so that the vacuum expectation value of the scalar fields, determined by minimizing the potential, leaves the subgroup $S$, $S \subset \bar{G}$ unbroken, then, in lowest order, there are Goldstone bosons corresponding to the generators of the cosets $\bar{G}/S$. Presumably in a realistic theory, the intersection $G \cap S$ is just the U(1) corresponding to the electric charge conservation. The Goldstone bosons corresponding to the remaining generators of the gauge group $G$ are the unphysical Higgs scalars which become the longitudinal components of the massive vector bosons. The remaining Goldstone bosons which do not correspond to any generators of the group $G$ of
Fig. 10.1. Diagrammatic representation of Lie algebras $G$, $\bar{G}$ and $S$, and their correspondence to massive and massless gauge bosons and pseudo-Goldstone bosons.

The entire Lagrangian then acquire computable masses in higher order due to the fact that the pseudosymmetry $G$ is broken down by weak and electromagnetic interactions, and are called pseudo-Goldstone bosons. See fig. 10.1.


The idea here is that $\bar{G}$ includes some approximate hadronic symmetry, and the pseudo-Goldstone bosons discussed here are the would-be Goldstone bosons (such as pions) seen in nature. This view has many very profound implications on the nature of hadronic symmetries and their breaking. So far no realistic model has been written down which realizes this view.
PART II

QUANTIZATION AND RENORMALIZATION OF GAUGE THEORIES

11. Path integral quantization

One feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of the calculus.

R.P. Feynman

In this section we develop the quantization procedure based on the notion of path integration. The first hint of this procedure appeared in a paper by Dirac in 1933; the method was perfected by Feynman in 1948. We shall first consider a quantum mechanical system with one degree of freedom, and generalize to quantum field theory in the next section.

Let \( |q, t\rangle_H \) be the Heisenberg picture state vector describing a state which at time \( t \) is an eigenstate of the coordinate \( Q_H \) with eigenvalue \( q \):

\[
Q_H(t) |q, t\rangle_H = q |q, t\rangle_H,
\]

\[
Q_H(t) = e^{iHt} Q_S e^{-iHt},
\]

where \( Q_S \) is the time-independent position operator in the Schrödinger picture, and \( H \) in the exponent is the Hamiltonian. The state

\[
|q\rangle = e^{-iHt} |q, t\rangle_H
\]

is an eigenstate of \( Q_S \) with eigenvalue \( q \):

\[
Q_S |q\rangle = q |q\rangle
\]

and

\[
|q, t\rangle_H = e^{iHt} |q\rangle.
\]

The transformation matrix element

\[
F(q', t'; q, t) = \langle q', t' |q, t\rangle_H = \langle q'| \exp \{-iH(t' - t)\} |q\rangle
\]

plays a fundamental role in quantum mechanics. We are going to express \( F(q', t'; q, t) \) as a path integral. We shall subdivide the time interval into \( n+1 \) equal segments, and define

\[
t_i = \epsilon + t, \quad t' = (n+1)\epsilon + t.
\]

We make use of the completeness of the state vectors \( |q_i, t_i\rangle \) to write

\[
F(q', t'; q, t) = \int dq_1(t_1) \int dq_2(t_2) \cdots \int dq_n(t_n) |q', t' |q_n, t_n\rangle |q_{n-1}, t_{n-1}\rangle \cdots |q_1, t_1 |q, t\rangle.
\]

Here and in the following, we shall drop the subscript-H and understand the state \( |q, t\rangle \) to mean that in the Heisenberg picture. For sufficiently large \( n \), the time interval \( t_i - t_{i-1} \) can be made as
small as one likes, and we may write

\[ \langle q', e|q, 0 \rangle = \langle q'|e^{-ieH}|q \rangle = \delta(q - q') - ie \langle q'|H|q \rangle + O(e^2) \]  

(11.6)

where the first equality follows from (11.3).

The Hamiltonian \( H = H(P, Q) \) is a function of the operators \( P \) and \( Q \). Consider the case when \( H \) is of the form

\[ H = \frac{1}{2}P^2 + V(Q). \]  

(11.7)

In this case

\[ \langle q'|H(P, Q)|q \rangle = \int \frac{dp}{2\pi} \exp \{ip(q' - q)\} \frac{1}{2}p^2 + V(q) \]  

(11.8)

where \( H(p, q) \) is the classical Hamiltonian. We can write eq. (11.6) correct up to first order in \( \epsilon \), as

\[ \langle q'|H(P, Q)|q \rangle \approx \int \frac{dp}{2\pi} \exp \{ip(q' - q)\} + V(q) \]  

(11.9)

Substituting (11.9) into (11.5), we obtain for the amplitude to find \( q' \) at time \( t' \) from a state which was an eigenstate of the coordinate with eigenvalue \( q \) at an earlier time \( t \),

\[ F(q', t'; q, t) = \lim_{n \to \infty} \int \prod_{i=1}^{n} dq_i \prod_{i=1}^{n+1} dp_i \exp \left\{ i \sum_{j=1}^{n+1} \left[ p_j (q_j - q_{j-1}) - H(p_j, \frac{1}{2} (q_j + q_{j-1}))) (t_j - t_{j-1}) \right]\right\} \]  

(11.10)

with \( q_0 = q \) and \( q_{n+1} = q' \).

We shall streamline our notation a little bit. We write (11.10) as

\[ F(q', t'; q, t) = \int \frac{dqd\!p}{2\pi\hbar} \exp \left\{ i \int_{t}^{t'} (p\dot{q} - H(p, q))d\tau \right\} \]  

(11.11)

which is a suggestive shorthand notation for the operation implied by the right-hand side of eq. (11.10). In eq. (11.11)

\[ \int \frac{dqd\!p}{2\pi\hbar} = \int \prod_{\tau} \frac{dq(\tau)d\!p(\tau)}{2\pi\hbar}. \]  

(11.12)

We have restored briefly \( \hbar = 1 \) to indicate that the functional integration is over all phase space volume \( \int (\Delta q \Delta p/\hbar) \) for all times between \( t \) and \( t' \).

When the Hamiltonian has the form of eq. (11.7), the \( p \)-integration on the right-hand-side of eq. (11.10) can be performed explicitly by making use of the formula

\[ \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp \{ie(p\dot{q} - \frac{1}{2}p^2)\} = [2\pi ie]^{-1/2} \exp(\frac{1}{2}ieq^2). \]  

(11.13)

The result is
where $L$ is the Lagrangian,

$$L = \frac{1}{2} \dot{q}^2 - V(q)$$

and $q_0 = q(t)$ and $q_{n+1} = q'(t_{n+1})$.

The quantity

$$S = \int L(q, \dot{q}) dt$$

is the action which generates the temporal development of the quantum mechanical system described by the Lagrangian (11.15).

We derived eq. (11.14) from the usual formalism of quantum mechanics. Alternatively, one can start from eq. (11.14) and derive the Schrödinger equation. All this and many other related matters were discussed in Feynman’s original paper. In a few simple cases, the functional integrations in eq. (11.14) can be carried out explicitly.

When the Hamiltonian is not in the form of eq. (11.7), we must be careful about specifying the ordering of the operators $P$ and $Q$. We shall assume that there is a way of ordering the operators in the quantum mechanical Hamiltonian $H(P, Q)$ so that the transformation matrix $F(q', t'; q, t)$ is correctly given by eq. (11.10) for this Hamiltonian, with the understanding that whenever there is an ambiguity, the integrals over $p_i$ are to be performed before the $q$-integrations. When the Hamiltonian is not of the form of eq. (11.7), we must use eq. (11.10) to find the “effective action”, $S_{\text{eff}}$, i.e., the quantity which, after the $p_i$-integrations are performed, replaces the action in eq. (11.14). In general, $S_{\text{eff}}$ is not given by (11.16).

As an illustration of this prescription, we apply it to the non-linear Lagrangian

$$L = \frac{1}{2} \dot{q}^2 f(q)$$

where $f(q)$ is a non-singular function of $q$. Eq. (11.17) describes a particular class of systems with velocity-dependent potentials. The momentum $p$ canonically conjugate to $q$ is

$$p = \partial L/\partial q = \dot{q} f(q)$$

and the Hamiltonian is

$$H(p, q) = p \dot{q} - L = \frac{1}{2} p^2 [f(q)]^{-1}.$$
The $p$-integrations can be performed as before, and we obtain

$$F(q', t'; q, t) = \prod_{i=1}^{n} \frac{dq_i}{[2\pi i\epsilon]^{1/2}} \delta(q_1 - q)\delta(q_n - q')$$

\[ \times \left[ \exp \left( i \sum_{i=1}^{n} \epsilon \left( \frac{q_i - q_{i-1}}{\epsilon} \right)^2 f \left( \frac{q_i + q_{i-1}}{2} \right) \right) \right] \prod_{i=2}^{n} \left[ f \left( \frac{q_i + q_{i-1}}{2} \right) \right]^{1/2}. \]  

(11.19)

The last factor can be written as

\[ \prod_{i} \left[ f \left( \frac{q_i - q_{i-1}}{2} \right) \right]^{1/2} = \exp \left( \frac{1}{2} \sum_{i} \ln f \left( \frac{q_i + q_{i-1}}{2} \right) \right) \]

\[ = \exp \left( \frac{1}{2\epsilon} \sum_{i} \epsilon \log f \left( \frac{q_i + q_{i-1}}{2} \right) \right) \to \exp \frac{1}{2} \delta(0) \int dt \ln f(q) \]  

(11.20)

where we have used the limits

\[ \sum_{i} \epsilon \to \int dt, \quad \frac{1}{\epsilon} \delta_{ij} \to \delta(t_i - t_j). \]  

(11.21)

Finally, therefore, we can write eq. (11.16) as

$$F(q', t'; q, t) = \lim_{n \to \infty} \int \prod_{i=1}^{n} \frac{dq_i}{[2\pi i\epsilon]^{1/2}} \delta(q_1 - q)\delta(q_n - q')$$

\[ \times \exp \left[ i \sum_{i=1}^{n} \epsilon \left( \frac{q_i - q_{i-1}}{\epsilon} \right)^2 f \left( \frac{q_i + q_{i-1}}{2} \right) - \frac{i}{2\epsilon} \ln f \left( \frac{q_i + q_{i-1}}{2} \right) \right] \]

\[ = \int \left[ \frac{dq}{[2\pi i\epsilon]^{1/2}} \right] \exp(\text{i}S_{\text{eff}}) \]  

(11.22)

where

$$S_{\text{eff}} = \int dt [L(q, \dot{q}) - i/2 \delta(0) \ln f(q)] = \int dt L_{\text{eff}}(q, \dot{q}).$$  

(11.23)

This result was first obtained by Lee and Yang.

If $S_{\text{eff}}$ is used to calculate transformation function $F(q', t'; q, t)$ or the scattering matrix for a particle with this Lagrangian $L$, an infinite term will appear to cancel the explicit term we have symbolically written $\delta(0)$. To do the calculation, one may go back to the explicit form in (11.22) before the limit $n \to \infty$ is taken, do the $q_i$ integrations, then take the limit $n \to \infty$.

The advantage, or even the rationale, of following the prescription which led to eq. (11.23) is that the result written in the form

$$F(QT; qt) = \int \left[ \frac{f^{1/2}(q) dq}{\sqrt{2\pi i\epsilon}} \right] \exp \{ \text{i}S(q, \dot{q}) \}$$  

(11.24)
is manifestly invariant under point transformations of the coordinate. In general, writing the transformation function as a path-integral enables us to express quantum-mechanical quantities in terms of the classical Lagrangian, so that we can study the effects on quantum-mechanical quantities of various symmetries present in the classical Lagrangian.

We develop a few properties of path integrals which will be useful in a generalization of the method to quantum field theory.

First of all, the generalization of eq. (11.11) to systems with more than one degree of freedom is straightforward. If there are \( N \) degrees of freedom, eq. (11.11) becomes

\[
\langle q'_1, q'_2, \ldots, q'_N, t'|q_1, q_2, \ldots, q_n, t \rangle = \int \prod_{n=1}^{N} \frac{dq_n dp_n}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} \int_{t}^{t'} \left[ \sum_{n=1}^{N} p_n \dot{q}_n - H(p, q) \right] dt \right\} \tag{11.24}
\]

with \( q_n(t) = q_n, q_n(t') = q'_n \).

For the rest of this section we restrict ourselves to \( N = 1 \); we shall use eq. (11.24) in the development of field theory.

Next, instead of the simple transformation function \( \langle q', t'|q, t \rangle \), let us consider the matrix element of the co-ordinate operator \( Q \) evaluated at time \( t_0 \), between \( \langle q', t' \mid q, t \rangle \). We restrict \( t_0 \) to lie in the interval

\[
t' > t_0 > t.
\]

Now let us write \( \langle q', t'|Q(t_0)q, t \rangle \) as in eq. (11.5), selecting \( t_0 \) to be one of the \( t_i \), say \( t_{i_0} \). Thus

\[
\langle q', t'|Q(t_0)q, t \rangle = \int \prod_{i} dq_i \langle q', t'|q_n, t_n \rangle \langle q_n, t_n \mid q_{n-1}, t_{n-1} \rangle \ldots
\]

\[
\langle q_{i_0+1}, t_{i_0+1} \mid q_{i_0}, t_{i_0} \rangle \langle q_{i_0}, t_{i_0} \mid Q(t_0) \mid q_{i_0-1}, t_{i_0-1} \rangle \ldots \langle q_1, t_1 \mid q, t \rangle.
\]

In eq. (11.24), we have placed the operator \( Q(t_0) \) next to one of its eigenstates, so

\[
\langle q_{i_0}, t_{i_0} \mid Q(t_0) \mid q_{i_0-1}, t_{i_0-1} \rangle \]

becomes \( q_{i_0} \langle q_{i_0}, t_{i_0} \mid q_{i_0-1}, t_{i_0-1} \rangle \). The argument leading to eq. (11.10) can now proceed; nothing is changed except that an extra factor of \( q_{i_0} \) will appear under the integral on the right-hand side. Instead of eq. (11.11) we now obtain

\[
\langle q', t'|Q(t_0)q, t \rangle = \int \left[ \frac{dq dp}{2\pi} \right] q(t_0) \exp \left\{ \int_{t}^{t'} \left[ \dot{p}q - H(p, q) \right] dt \right\} \tag{11.25}
\]

Next, suppose we want to express

\[
\langle q', t'|Q(t_1)Q(t_2)q, t \rangle
\]

as a path integral. We proceed as above, choosing \( t_1 \) and \( t_2 \) to be two of the times which bound the small intervals into which the interval \( t' - t \) is broken. If \( t_1 > t_2 \), we can write

\[
\langle q', t'|Q(t_1)Q(t_2)q, t \rangle = \int \pi dq_1 \langle q', t'|q_n, t_n \rangle \langle q_n, t_n \mid q_{n-1}, t_{n-1} \rangle \ldots
\]

\[
\langle q_{i_1}, t_{i_1} \mid Q(t_1) \mid q_{i_1-1}, t_{i_1-1} \rangle \ldots \langle q_{i_2}, t_{i_2} \mid Q(t_2) \mid q_{i_2-1}, t_{i_2-1} \rangle \ldots \langle q_1, t_1 \mid q, t \rangle.
\]

\[
\tag{11.26}
\]
After going through a series of steps analogous to those which led to eq. (11.25), we obtain

\[
\langle q', t' | Q(t_1)Q(t_2) | q, t \rangle = \int \left[ \frac{dp \, dq}{2\pi} \right] q(t_1)q(t_2) \exp \left\{ i \int_{t}^{t'} [p \dot{q} - H(p, q)] \, dt \right\}.
\]

Eq. (11.27) holds only if \( t_1 > t_2 \). If \( t_2 > t_1 \), we could not have derived eqs. (11.26) and (11.27) the way we did. In fact, it is easy to see that if \( t_2 > t_1 \), the right-hand side of eq. (11.27) is equal to

\[
\langle q', t' | Q(t_2)Q(t_1) | q, t \rangle.
\]

Therefore the path integral in eq. (11.27) is the matrix element of the time-ordered product

\[
T[Q(t_1)Q(t_2)].
\]

The result generalizes immediately to the product of any number of \( Q \)'s

\[
\langle q', t' | T[Q(t_1)Q(t_2) \ldots Q(t_N)] | q, t \rangle = \int \left[ \frac{dp \, dq}{2\pi} \right] q(t_1)q(t_2) \ldots q(t_N) \exp \left\{ i \int_{t}^{t'} [p \dot{q} - H(p, q)] \, dt \right\}.
\]

Next we want to demonstrate a crucial theorem. Let \( L \) be a Lagrangian which does not depend explicitly on time, and let \( \phi_n(q) = \langle q | n \rangle \) be the wave function of the energy eigenstate \( | n \rangle \). In particular, let \( \phi_0(q) \) be the ground state. If the system is in the ground state at a time \( T \) in the distant past, we want to calculate the amplitude for it to be in that state at a time \( T' \) in the distant future, when an arbitrary external source term \( J(t)q(t) \) is added to \( L \) between \( T \) and \( T' \).

To do this, consider

\[
\langle Q', T' | Q, T \rangle = \int \left[ \frac{dp \, dq}{2\pi} \right] \exp \left\{ i \int_{T}^{T'} [p \dot{q} - H(p, q) + Jq] \, dt \right\}
\]

where \( J \) is an arbitrary function of \( t \), except that we restrict it to be non-vanishing only between \( t \) and \( t' \), where \( T' > t' > t > T \). We can write eq. (11.29) as

\[
\langle Q', T' | Q, T \rangle = \int dq' \int dq \, \langle Q', T' | q', t' \rangle \langle q', t' | q, t \rangle \langle q, t | Q, T \rangle.
\]

Now \( \langle q, t | Q, T \rangle \) and \( \langle Q', T' | q', t' \rangle \) are given by formulae like (11.29) without the \( J(t)q(t) \) term.

Let us insert a complete set of energy eigenstates in \( \langle q, t | Q, T \rangle \):

\[
\langle q, t | Q, T \rangle = \langle q | \exp \{-iH(t - T)\} | Q \rangle = \sum_n \phi_n(q) \phi_n^*(Q) \exp \{-iE_n(t - T)\}
\]

The \( T \)-dependence in (11.31) is known explicitly because we have required \( J(t) = 0 \) between \( T \) and \( t \). Therefore, we can continue \( T \) along the positive imaginary axis. In that limit, all the terms with \( n > 0 \) drop out, as \( T \to i\infty \), and

\[
\lim_{T \to i\infty} \exp(-iE_0T) \langle q, t | Q, T \rangle = \phi_0(q, t) \phi_0^*(Q),
\]

\[
\phi_0(q, t) = \phi_0(q) \exp(-iE_0t).
\]
We can do the same analysis for \( \langle Q', T' | q, t' \rangle \). Therefore, provided \( Q \) and \( Q' \) approach some constants in the limit, we have

\[
\lim_{T' \to -i\infty} \lim_{T \to i\infty} \exp\left\{ -iE_0(T' - T) \right\} \phi_o^*(Q') \phi_o(Q) = \int dq \int dq' \phi_o^*(q', t') \phi_o(q, t)
\]

\[
\langle Q', T' | Q, T \rangle = \frac{\langle Q', T' | q, t' \rangle}{\phi^*_o(q', t') \phi_o(q, t)}
\]

(11.33)

which is the theorem we set out to prove. The right-hand side of (11.33) is just the ground state to ground state amplitude of interest, since \( t' \) and \(-t\) can be taken as large as one pleases. Let us denote it, symbolically, as \( W[J] \). Then eq. (11.33) tells us how to calculate \( W[J] \).

Why is \( W[J] \) of interest? In (11.33), \( \langle q', t' | q, t \rangle \) is given by a form like eq. (11.29), with \( t \) and \( t' \) replacing \( T \) and \( T' \). The effect of varying \( W \) with respect to \( J(t_0) \) is to bring down a factor \( i\phi(t_0) \) in front of the exponential. Let us do this \( n \) times, and then set \( J = 0 \).

\[
\lim_{J \to 0} \frac{\delta^n W[J]}{\delta J(t_1) \delta J(t_2) \ldots \delta J(t_n)} = i^n \int dq \int dq' \phi^*_o(q', t) \phi_o(q, t) \int \left[ \frac{dp dq}{2\pi} \right] \exp \left\{ i \int_{t'}^{T} \left[ pq - H(p, q) \right] d\tau \right\}
\]

\[\times q(t_1)q(t_2)\ldots q(t_n), \quad t' > t_1, t_2, \ldots, t_n > t. \quad (11.34)\]

Comparing with eq. (11.28), we see that this expression is just the matrix element of the time-ordered product \( T(Q(t_1)Q(t_2)\ldots Q(t_n)) \) between the ground state at \( t \) and the ground state at \( t' \). Therefore the expression (11.34) is the ground state expectation value of a time-ordered product of co-ordinates. In field theory, these will become the Green's functions.

We shall indicate how \( W[J] \) can be evaluated from eq. (11.33). To within a multiplicative factor independent of \( J \)

\[
W[J] \sim \int [dq] \exp \left\{ i \int_{t'}^{T} \left[ L_{\text{eff}}(q, \dot{q}) + J(t)q(t) \right] dt \right\}. \quad (11.35)
\]

In field theoretic applications, the multiplicative factors independent of \( J \) never matters, and we are allowed to be cavalier about it.

From eq. (11.34) and the remark following it, we have

\[
\langle T(Q(t_1)\ldots Q(t_n)) \rangle_0 \sim \lim_{T' \to -i\infty} \lim_{T \to i\infty} \int dq_1 \ldots dq_n \langle Q', T'| q_1, t_1 \rangle \langle q_1, t_1 | q_2, t_2 \rangle \langle q_2, t_2 | \ldots \langle q_n, t_n | Q, T \rangle,
\]

where \( t_1 > t_2 \ldots > t_n \), and \( \langle \rangle_0 \) denotes the ground state expectation value. Let us consider continuing \( \langle T(Q(t_1)\ldots Q(t_n)) \rangle \) in \( t_i \) analytically, from real to imaginary values \( t_i = -i\tau_i \). Since

\[
\langle q, t | q', t' \rangle = \lim_{n \to \infty} \int [dq] \left( \frac{d_i}{\sqrt{2\pi i e}} \right) \exp \left\{ i \sum_{i} \left( \frac{q_{i} + q_{i-1}}{2}, \frac{q_{i} - q_{i-1}}{e} \right) \right\}
\]

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depends on \( t - t' \) only through \( \epsilon \):

\[
\epsilon = \frac{(t - t')}{(n + 1)},
\]

the analytic continuation is effected by writing

\[
\langle q, t | q', t' \rangle \bigg|_{t = -i \tau, t' = -i \tau'} = \lim_{n \to \infty} \int \prod_{i} \frac{dq_i}{\sqrt{2 \pi i \epsilon}} \exp \left( \sum \epsilon' L_{\text{eff}} \left( \frac{q_i + q_{i-1}}{2}, \frac{q_i - q_{i-1}}{-i \epsilon'} \right) \right)
\]

where

\[
\epsilon' = \frac{(\tau - \tau')}{(n + 1)}.
\]

Thus the analytic continuation of \( \langle T(Q(t_1) \ldots Q(t_n)) \rangle_0 \) may be written as

\[
\langle T(Q(t_1) \ldots Q(t_n)) \rangle_0 \bigg|_{t_i = -i \tau_i} \sim \lim_{\tau_{f \to \infty}} \int [dq] q(\tau_1)q(\tau_2)\ldots q(\tau_n) \exp \left( \int_{\tau_i}^{\tau_f} L_{\text{eff}} \left( q, i \frac{dq}{d\tau} \right) \right).
\]

This suggests going over to an imaginary time, or Euclidean, formulation and defining

\[
W_{E}[J] = \int [dq] \exp \left( \int_{-\infty}^{\infty} d\tau \left[ L_{\text{eff}} \left( q, i \frac{dq}{d\tau} \right) + J(\tau)q(\tau) \right] \right). \tag{11.36}
\]

The boundary condition to be imposed on (11.36) is that \( q \) approaches some constants as \( \tau \to \pm \infty \). It is convenient, but not necessary, to take these constants to be zero. The connection between \( W[J] \) and \( W_{E}[J] \) is that

\[
\frac{1}{W[J]} \frac{\delta^n W[J]}{\delta J(t_1) \ldots \delta J(t_n)} \bigg|_{J = 0} = \left( \frac{1}{W_{E}[J]} \frac{\delta^n W_{E}[J]}{\delta J(\tau_1) \ldots \delta J(\tau_n)} \bigg|_{J = 0, \tau_i = it_i} \right)^n \tag{11.37}
\]

where analytic continuation is implied on the right-hand side. Equation (11.37) is manifestly independent of the overall normalizations of \( W[J] \) and \( W_{E}[J] \) which are independent of \( J \).

Finally, in order to illustrate the formal discussion, and especially the Euclidean formulation, we discuss a simple example. Consider a simple harmonic oscillator in one dimension, whose Lagrangian is

\[
L(q, \dot{q}) = \frac{1}{2}(q^2 - \omega^2 \dot{q}). \tag{11.38}
\]

The transformation matrix in the presence of external source \( J \) can be computed from eq. (11.29):

\[
\langle q', t' | q, t \rangle' = \lim_{n \to \infty} \int \prod_{i=1}^{n} \frac{dq_i(t_i)}{\sqrt{2 \pi i \epsilon}} \exp \left( \int_{t_i}^{t'} i d\tau [L(q(\tau), \dot{q}(\tau)) + J(\tau)q(\tau)] \right) \tag{11.39}
\]

with the boundary condition \( q(t') = q' \), \( q(t) = q \). The integral can be worked out explicitly. The calculation is posed as a problem, with enough hints, in Feynman and Hibbs, "Quantum Mechanics..."
and Path Integrals\textsuperscript{39}, p. 64. The answer is
\begin{equation}
\langle q', t' | q, t \rangle = [\omega/2\pi i \sin \omega T]^{1/2} \exp \{iQ(q', t', q, t)\}
\end{equation}
(11.40)
where
\[ T = t' - t \]
and
\[ Q(q', t', q, t) = \frac{\omega}{2 \sin \omega T} \left[ (q^2 + q'^2) \cos \omega T - 2qq' \right] 
+ \frac{q'}{\sin \omega T} \int_t^{t'} J(\tau) \sin \omega(\tau - t) d\tau 
+ \frac{q}{\sin \omega T} \int_t^{t'} J(\tau) \sin \omega(t' - \tau) d\tau 
- \frac{1}{\omega \sin \omega T} \int_t^{t'} d\sigma \int_t^\sigma J(\sigma) J(\tau) \sin \omega(t' - \sigma) \sin \omega(\tau - t) d\tau.
\]
(11.41)
\begin{equation}
W[J] = \exp \left\{ i \int_t^{t'} d\tau \int_t^\sigma J(\sigma) \left[ \frac{i}{2\omega} \exp \{-i\omega(\sigma - \tau)\} \right] J(\tau) \right\}.
\end{equation}
(11.44)

We leave the derivation of eq. (11.41) as an exercise.

The quantity $W[J]$ defined in the remark following (11.33) is
\begin{equation}
W[J] = \langle 0, t' | 0, t \rangle \phi_0(q', t') \phi_0(q, t) \langle q', t' | q, t \rangle 
\end{equation}
(11.42)
where "0" in (11.42) means the ground state, not the state with eigenvalue 0 for the coordinate; $\phi_0$ is the ground state wave function of the simple harmonic oscillator:
\begin{equation}
\phi_0(q, \tau) = (\omega/\pi)^{1/4} \exp(-\frac{1}{2} \omega q^2) \exp(-i \frac{1}{2} \omega \tau)
\end{equation}
(11.43)
so that the integrals over $q$ and $q'$ are just Gaussian integrals. The result is
\begin{equation}
\langle 0, t' | 0, t \rangle = \exp \left\{ i \int_t^{t'} d\tau \int_t^\sigma J(\sigma) \left[ \frac{i}{2\omega} \exp \{-i\omega(\sigma - \tau)\} \right] J(\tau) \right\}.
\end{equation}
(11.44)

We will make the result more general by extending the limits on the integrals from $-\infty$ to $+\infty$. Thus, if we are interested in the effect on the oscillator of the force term for just the period $t$ to $t'$, we may restrict $J(\tau)$ to vanish outside of this interval. Finally, we shall write eq. (11.44) as
\begin{equation}
W[J] = \exp \left\{ \frac{-i}{2} \int_t^{t'} d\tau \int_t^\sigma J(\tau) D_\ast(\tau - \sigma) J(\sigma) \right\}
\end{equation}
(11.45)
where
\begin{equation}
D_\ast(t) = \frac{1}{2i\omega} [\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}].
\end{equation}
(11.46)
Notice that
\[ D_s(t - t') = \left. \frac{\delta^2 W[J]}{\delta J(t) \delta J(t')} \right|_{J = 0} = \frac{i\delta^2}{\delta J(t) \delta J(t')} \ln W[J] \bigg|_{J = 0} \] (11.47)

In the Euclidean formulation (11.36), we have
\[ W_E[J] = \int [dq] \exp \{-S_E[J]\} \quad (11.48) \]
where
\[ S_E[J] = \int_{-\infty}^{\infty} d\tau \, L_E(\tau), \]
\[ L_E = \frac{1}{2}(dq/d\tau)^2 + \frac{1}{2} \omega^2 q^2 - J(\tau)q(\tau). \] (11.49)

We expand \( q(\tau) \) around \( q(\tau_0) \), writing \( q(\tau) = q(\tau_0) + y(\tau) \), and then expand \( S_E \) in powers of \( y \),
\[ S_E(q) = S_E(q_0) + \int_{-\infty}^{\infty} \left( \frac{\delta L_E}{\delta q} \frac{dq}{d\tau} + \frac{\delta L_E}{\delta q} y \right) d\tau + \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{dy}{d\tau} \right)^2 + \frac{\omega^2}{2} y^2 \right) d\tau. \] (11.50)

We wish to choose \( q_0(\tau) \) so that the term in (11.50) linear in \( y \) vanishes. If the boundary condition is taken to be \( q(\tau = \pm\infty) = 0 \), we require \( q_0(\tau = \pm\infty) = 0 \) also. Then the surface terms vanish when the term \( (\delta L_E/\delta q)(dy/d\tau) \) is integrated by parts, and we require that \( q_0(\tau) \) satisfy the classical equations of motion:
\[ \left\{ \frac{d}{d\tau} \frac{\delta L_E}{\delta q(\tau)} - \frac{\delta L_E}{\delta q(\tau)} \right\}_{q=q_0(\tau)} = 0. \] (11.51)

[If \( q \) is allowed to approach non-zero constants \( q_{\pm} \) in the limits \( \tau \to \pm\infty \), we may require \( q_0 \to 0 \) and \( y \to q_{\pm} \). Then there will be surface terms equal to \( q_0(\pm\infty)q_{\pm} \) in (11.50). However, from the general solution below it is evident that \( q_0(\pm\infty) = 0 \) if \( q_0(\pm\infty) \) vanishes.]

Now we insert eq. (11.50) into eq. (11.48) and perform the integral over paths. The term linear in \( y \) has disappeared, so we write
\[ W_E[J] \sim \exp \{-S_E(q_0)\} \int \prod_i dy(\tau_i) \exp \left\{ - \int_{-\infty}^{\infty} \left[ \dot{y}^2 + \frac{\omega^2 y^2}{2} \right] \right\} d\tau. \]

The integration over \( y(\tau_i) \) is just a number, independent of \( J \), so we are left with
\[ W_E[J] \sim \exp \{-S_E(q_0)\}. \quad (11.52) \]

Let us evaluate \( q_0(J) \). From eq. (11.51)
\[ \left[ \frac{d^2}{d\tau^2} - \omega^2 \right] q_0(\tau) = -J(\tau). \] (11.53)
Define a Euclidean Green's function $D_E(\tau)$ by

$$\left[\frac{d^2}{d\tau^2} - \omega^2\right] D_E(\tau) = \delta(\tau)$$  \hspace{1cm} (11.54)

with the boundary condition $\lim_{\tau \to \pm \infty} D_E(\tau) = 0$. The solution is

$$D_E(\tau) = -\int_{-\infty}^{\infty} d\nu \frac{e^{-\nu \tau}}{2\pi \nu^2 + \omega^2} = -\frac{e^{\omega |\tau|}}{2\omega}$$  \hspace{1cm} (11.55)

and therefore

$$q_0(\tau) = -\int_{-\infty}^{\infty} D_E(\tau - \sigma) J(\sigma) \, d\sigma.$$  \hspace{1cm} (11.56)

[With other boundary conditions, the most general solution is (11.56) plus the general solution to the homogeneous equation, namely, $A e^{-\omega t} + B e^{\omega t}$. If $q_0$ approaches a constant at both $+\infty$ and $-\infty$, $A = B = 0$, and it follows from (11.56) that $q_0(\pm \infty) = 0$ also.]

Now in the definition (11.49) of $S_E$, we substitute $\omega^2 q_0(\tau)$ from eq. (11.53), integrate by parts, to obtain, using eq. (11.56)

$$S_E(q_0) = -\frac{1}{2} \int_{-\infty}^{\infty} J(\tau) q_0(\tau) \, d\tau = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma J(\tau) D_E(\tau - \sigma) J(\sigma)$$  \hspace{1cm} (11.57)

so that, from (11.52)

$$W_E[J] \sim \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} d\tau d\sigma J(\tau) D_E(\tau - \sigma) J(\sigma) \right\}$$  \hspace{1cm} (11.58)

or

$$D_E(\tau - \tau') = \left. \frac{1}{W_E[J]} \frac{-\delta^2 W_E[J]}{\delta J(\tau) \delta J(\tau')} \right|_{J = 0}$$

We can get the propagator $D_+(t)$ by analytic continuation in $\tau$, by rotating counter-clockwise from real $\tau$ to imaginary $\tau$:

$$D_+(t) = i D_E(it)$$  \hspace{1cm} (11.59)

which yields eq. (11.46) immediately.

Note that the functional integral in (11.36) is a well behaved Gaussian (or more precisely, Wiener-Hopf) integral. Our notation may be simplified by writing the real time, ground-state to ground-state amplitude (11.49) as

$$W[J] \sim \int \{ dq(t) \} \exp\left\{ i \int_{-\infty}^{\infty} \left[ \frac{1}{2} \left( \frac{dq(t)}{dt} \right)^2 - \frac{1}{2} (\omega^2 - i\epsilon) q^2 + J(t) q(t) dt \right] \right\}.$$  \hspace{1cm} (11.60)
Then we can repeat the imaginary time analysis using (11.60) directly, to obtain eqs. (11.45) and (11.46) the $\text{i}\epsilon$ in (11.60) serving to select the correct boundary condition on the propagator:

$$D_s(t) = \int_0^\infty \frac{d\nu}{2\pi} \frac{\exp(-\text{i}\nu t)}{\nu^2 - \omega^2 + \text{i}\epsilon}.$$ 

Bibliography

For the path integral formulation of quantum theory, the basic papers are
1. P.A.M. Dirac, Physik. Z. Sowjetunion 3 (1933) 64.
These are reprinted in
A textbook is available on this subject:
For the path integral formulation of eq. (11.11), see
See also

The effective action for the velocity-dependent potential was first obtained by

Canonical transformations are discussed in any good book on classical mechanics. For example

The problem of operator ordering of a velocity-dependent potential has been discussed in

H. Kamo and T. Kawai, to be published; and references cited therein.
See also
for a derivation of Schrödinger's equation for a velocity-dependent potential, which amounts to deriving the operator-ordered Hamiltonian which corresponds to the path integral formulation.

This section incorporates several useful remarks of S. Coleman, D. Gross and S.B. Treiman.

12. Path integral formulation of field theory

Physics – Where the Action Is.

Anonymous

We have remarked that the generalization of the considerations in section 11 to many degrees of freedom is immediate. The transformation function is given by (11.24), which is a shorthand for

$$\lim_{n \to \infty} \prod_{\alpha = 1}^N \prod_{i = 1}^n dq_\alpha(t_i) \prod_{i = 1}^{n+1} dp_\alpha(t_i)$$

$$\times \exp \left[ i \sum_{j=1}^n \left( \sum_{\alpha=1}^N p_\alpha(t_j) [q_\alpha(t_j) - q_\alpha(t_{j-1})] - \epsilon H \left( p(t_j), \frac{q(t_j) + q(t_{j-1})}{2} \right) \right) \right]$$

(12.1)
Eq. (12.1) can be applied to field theory. Consider a neutral scalar field \( \phi(x) \). Let us subdivide space into cubes of dimension \( \varepsilon^3 \) and label them by an integer \( \alpha \). We define the \( \alpha \)th coordinate \( q_\alpha(t) = \phi_\alpha(t) \) by
\[
\phi_\alpha(t) = \frac{1}{\varepsilon^3} \int d^3x \phi(x, t),
\]
where the integration is over the \( \alpha \)th cell of dimension \( \varepsilon^3 \). We can also rewrite the Lagrangian as
\[
L = \int d^3x \mathcal{L} \rightarrow \sum_\alpha \varepsilon^3 \mathcal{L}_\alpha(\phi_\alpha(t), \phi_\alpha(t), \phi_{\alpha\pm s}(t))
\]
where \( \phi_\alpha(t) \) is the average of \( \partial \phi(x, t)/\partial t \) over the \( \alpha \)th cell and \( \phi_{\alpha\pm s} \) is the average value of the field in the neighboring cell \( \alpha \pm s \). The canonical momenta \( p_\alpha \) conjugate to \( \phi_\alpha \) are
\[
p_\alpha(t) = \frac{\partial L}{\partial \dot{\phi}_\alpha(t)} = \varepsilon^3 \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\phi}_\alpha(t)} = \varepsilon^3 \pi_\alpha(t).
\]
The Hamiltonian is
\[
H = \sum_\alpha p_\alpha \dot{\phi}_\alpha - L = \sum_\alpha \varepsilon^3 \mathcal{H}_\alpha,
\]
\[
\mathcal{H}_\alpha = \pi_\alpha \dot{\phi}_\alpha - \mathcal{L}_\alpha = \mathcal{H}_\alpha(\pi_\alpha, \phi_\alpha, \phi_{\alpha\pm s}).
\]
We may now write the expression (12.1) as
\[
\lim_{\varepsilon \to 0} \int \prod_{\alpha} \prod_{i=1}^{n+1} d\phi_\alpha(t_i) \prod_{i=1}^{n+1} \frac{\varepsilon^3}{2\pi} d\pi_\alpha(t_i)
\]
\[
\times \exp \left[ \frac{i}{\varepsilon} \sum_{j=1}^{n+1} \sum_{\alpha} \varepsilon^3 \left( \pi_\alpha(t_j) \frac{\phi_\alpha(t_j) - \phi_\alpha(t_{j-1})}{\varepsilon} - \mathcal{H}_\alpha \left( \pi_\alpha(t_j), \frac{\phi_\alpha(t_j) + \phi_\alpha(t_{j-1})}{2}, \frac{\phi_{\alpha\pm s}(t_j) + \phi_{\alpha\pm s}(t_{j-1})}{2} \right) \right) \right]
\]
\[
\equiv \int [d\phi] \left[ \frac{\varepsilon^3}{2\pi} d\pi \right] \exp \left[ i \int d\tau \int d^3x \left( \pi(x, \tau) \frac{\partial \phi(x, t)}{\partial \tau} - \mathcal{H}(x, \tau) \right) \right] \quad (12.2)
\]
where we defined the momentum density conjugate to \( \phi(x, t) \) by
\[
\pi(x, t) = \partial \mathcal{L}/\partial \dot{\phi}(x, t).
\]
Its cell average is just the \( \pi_\alpha(t) \) defined above.

In field theory, all physical quantities are derivable from the vacuum-to-vacuum transition amplitude in the presence of external sources. The physical vacuum is the ground state, and plays the same role as the state whose wavefunction is \( \phi_0(q) \) in eq. (11.33).

This amplitude, which we shall call \( W[J] \), can be calculated from eq. (12.2) with a term \( \int d^3x J(x, t) \phi(x, t) \) added to the Lagrangian, in the limit \( t' \to \infty, t \to -\infty \). That is
\[ W[J] = \int [d\phi] \left( \frac{e^2}{2\pi} \ d\pi \right) \exp \left[ i \int d^4x \ (\pi(x)\dot{\phi}(x) - \mathcal{V}(x) + \frac{1}{2} i e \phi^2 + J(x)\phi(x)) \right]. \] (12.3)

The extra term \( \frac{1}{2} i e \phi^2 \) is simply a symbolic way of indicating how to rotate the time-integration contour to pick out the correct limit as indicated on the left-hand side of eq. (11.33). More on this later.

Now it follows from eq. (11.34) and the discussion following it that

\[ \delta^n W[J] \bigg|_{J=0} = i^n \langle T(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \rangle = i^n G(x_1\ldots x_n) \] (12.4)

where \( G \) is the \( n \)-point Green's function, the vacuum expectation value of the time-ordered product of \( n \) fields. The fact that the Green's functions may be defined by (12.4) was first discovered by Schwinger, and does not depend on the path-integral formula (12.3) for \( W[J] \). However, eq. (12.3) provides not only a simple proof of (12.4), but an explicit formula for computing \( W[J] \).

Eq. (12.4) gives the complete Green's functions. In general, these include some contributions from disconnected vacuum to vacuum diagram, which are simply products of lower order Green's functions.

The connected graphs are given by

\[ G_{c}(x_1, x_2\ldots x_n) = \frac{(-i)^n}{W[J]} \frac{\delta^n W[J]}{\delta J_1(x_1)\ldots \delta J_n(x)} \] (12.5)

or, writing

\[ W[J] = \exp \{ i Z[J] \}, \] (12.6)

\[ G_{c}(x_1\ldots x_n) = (-i)^{n-1} \frac{\delta^n Z[J]}{\delta J(x_1)\ldots \delta J(x_n)}. \] (12.7)

The proof that the connected parts of the \( n \)-point function is given by eq. (12.7) is left as an exercise.

When the Hamiltonian density takes the form

\[ \mathcal{H}(x) = \frac{1}{2} \pi^2(x) + f[\phi(x), \nabla \phi(x)]. \] (12.8)

The \( \pi \)-integrations can be carried out explicitly, and we obtain

\[ W[J] \sim \int [d\phi] \exp \{ i \int [\mathcal{L}(x) + J(x)\phi(x)] d^4x \} \] (12.9)

where \( \mathcal{L}(x) \) is the Lagrangian density

\[ \mathcal{L}(x) = \frac{1}{2} (\partial_0 \phi)^2 - f[\phi(x), \nabla \phi]. \]

When we discuss vector meson theories, the form (12.9) will be inadequate and we shall have to use the original form (12.3). As an example, however, let us first consider a case where (12.9) is applicable.
Let us concentrate, for definiteness, the case in which the Lagrangian is of the form
\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \quad \mathcal{L}_0 = \frac{1}{2} [(\partial_\mu \phi)^2 - \mu^2 \phi^2] \tag{12.10} \]
and
\[ \mathcal{L}_1 = \mathcal{L}_1(\phi). \]

The functional \( W[J] \) of eq. (12.9) is in general an ill-defined integral even in the "lattice" approximation. Recent advances in axiomatic field theory indicate that if one can construct a well-behaved field theory in the Euclidean space \((x, t)\), obeying certain appropriate axioms, then there is a corresponding field theory in the Minkowski space \((x_\alpha, x)\) as the analytic continuation of the former as \( t = i x_\alpha \), which obeys the Wightman axioms. Thus any ambiguities should be resolved by appealing to the Euclidianity Postulate, namely that the Green's functions (12.5) are the analytic continuation of those defined by the well-defined functional integral in the Euclidean field theory:
\[
W_E[J] = \int [d\phi] \exp \left\{ -\int d^3x d\tau \left[ \frac{\partial^2}{\partial t^2} \right] + (\nabla \phi)^2 + \mu^2 \phi^2 - \mathcal{L}_1(\phi) - J\phi \right\}.
\]

Note that since \( -\mathcal{L}_1 \) is bounded from below the quantity in the square bracket in the exponent above is also bounded from below. As we anticipate, the Euclidianity postulate determines the boundary conditions to be imposed on propagators. For the present problem it means that we may provide a damping factor for the functional integration by adding a term in \( \mathcal{L}_0 \):
\[
\mathcal{L}_0 \to \frac{1}{2} [(\partial_\mu \phi)^2 - \mu^2 \phi^2 + i \epsilon \phi^2]
\]
as we did in eq. (12.3).

First, consider the free field case:
\[
W_o[J] = \int [d\phi] \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \mu^2 \phi^2 + \frac{1}{2} i \epsilon \phi^2 + J\phi \right] \right\}
= \lim_{\epsilon \to 0} \int \prod_\alpha d\phi_\alpha \exp \left\{ i \left( \sum_\alpha e^4 \sum_\beta e^4 \frac{1}{2} \phi_\alpha K_{\alpha \beta} \phi_\beta + \sum_\alpha e^4 J_\alpha \phi_\alpha \right) \right\}. \tag{12.11}
\]
Here, \( \alpha \) labels space-time cells of dimension \( e^4 \), and the matrix \( K_{\alpha \beta} \) is such that
\[
\lim_{\epsilon \to 0} K_{\alpha \beta} = (-\partial^2 - \mu^2 + i \epsilon) \delta^4(x - y)
\]
where \( \alpha \to x \) and \( \beta \to y \) as \( \epsilon \to 0 \). The \( \phi \)-integrations in eq. (12.11) can be performed explicitly. We obtain
\[
W_o[J] = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\det K_{\alpha \beta}}} \prod_\alpha \sqrt{\frac{2\pi}{ie^8}} \exp \left\{ -\frac{1}{2} i \pi \sum_\alpha e^4 \sum_\beta e^4 J_\alpha \frac{1}{e^8} (K^{-1})_{\alpha \beta} J_\beta \right\}
\]
where, of course, \( K^{-1} \) is the inverse of \( K \):
\[
\sum_\gamma K_{\alpha \gamma} (K^{-1})_{\gamma \beta} = \delta_{\alpha, \beta}
\]
or,

\[ \sum_{\gamma} \epsilon^4 K_{\alpha\gamma} \left( \frac{1}{\epsilon^8} K^{-1} \right)_{\gamma\beta} = \frac{1}{\epsilon^4} \delta_{\alpha,\beta}. \] (12.12)

As \( \epsilon \to 0 \), we have

\[ \frac{1}{\epsilon^4} \delta_{\alpha,\beta} \to \delta^4(x - y), \quad \sum_{\alpha} \epsilon^4 \to \int d^4x, \]

so, with the definition

\[ \frac{1}{\epsilon^8} (K^{-1})_{\alpha\beta} \to \Delta_F(x - y), \]

eq. (12.12) may be written, in the continuum limit \( \epsilon \to 0 \), as

\[ (-\partial^2 - \mu^2 + i\epsilon)\Delta_F(x - y) = \delta^4(x - y). \] (12.13)

Therefore, neglecting an inessential multiplicative factor, we can write

\[ W_0[J] = \exp(-\frac{1}{2}i) \int d^4x \int d^4y J(x) \Delta_F(x - y) J(y) \] (12.14)

where

\[ \Delta_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp(-ik(x - y))}{k^2 - \mu^2 + i\epsilon} \] (12.15)

is the Feynman propagator.

Now we are ready to discuss the interacting case. Returning to eqs. (12.10) and (12.11), we write

\[ W[J] \sim \int [d\phi] \exp \left\{ i \int d^4x \left[ \mathcal{L}_0 + \mathcal{L}_1(\phi) + J\phi \right] \right\} \]

\[ = \exp \left[ i \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] \int [d\phi] \exp \left\{ i \int d^4x \left[ \mathcal{L}_0 + J\phi \right] \right\} \]

\[ \sim \exp \left[ i \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] \exp(-\frac{1}{2}i) \int d^4x \int d^4y J(x) \Delta_F(x - y) J(y). \] (12.16)

Equation (12.16) is the basis for the Feynman-Dyson expansion of the Green's functions of this theory, and when it is substituted in eq. (12.4), we obtain a formula which generates Green's functions. \( W[J] \) can be expanded in powers of \( \mathcal{L}_1 \), for example, by simply expanding the exponential factor

\[ \exp \left( i \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J} \right) \right) = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \left[ \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J} \right) \right]^n. \]

What corresponds to Wick's theorem is simply the rule for functional differentiation:
\[
\frac{\delta}{\delta J(x)} J(y) = \delta^4(x - y).
\]

The student should convince himself the rules outlined here are in fact the Feynman rules discussed in the second volume of Bjorken and Drell. In fact, collateral reading of the first six sections of Chapter 17 of this book is urged.

In order to quantize fermion fields by the method of path integrations, it is necessary to introduce the concept of anticommuting \(c\)-numbers. We shall forgo this though, because the incorporation of fermion fields presents no special problem in quantizing a gauge theory.

In general, \(\mathcal{L}_f\) is a function of \(\phi\) as well as \(\phi\), and eq. (12.4) is inadequate. Just as in the one-dimensional example discussed in the preceding lecture, we shall see, the action of eq. (12.16) must then be replaced by an "effective action", which contains a correction to the integral over the Lagrangian. In that case, the correct Feynman rules are modified, and cannot be directly read off the Lagrangian.

Bibliography

For the Euclidian Field Theory and its connection to the Minkowsky field theory, see
3. E. Nelson, Construction of Quantum Fields from Markov Fields (to be published); The Free Markov Fields (to be published).

The view of taking (12.1) as the basis of quantization rather than (12.9) was first expounded by

The idea that the Green's function can be obtained from variation of the vacuum-to-vacuum amplitude in the presence of an external source term is due to Schwinger. See, for example

13. The Yang-Mills field in the Coulomb gauge

We wish to apply these path-integral methods to theories with gauge vector mesons. Indeed, it is in this case that the method becomes a powerful tool both to discover the correct Feynman rules and to study renormalization, while the canonical Wick theorem methods become awkward.

We shall study the three-component Yang-Mills field, although the generalization to other compact non-Abelian groups is immediate. In this section, we work out the canonical formalism in the Coulomb gauge, and construct the \(\mathcal{W}[J]\) function, starting from the basic equation (12.3). In later sections we shall study gauge-invariance and work out the Feynman rules in a more manifestly covariant gauge.

It is convenient to write out the Yang-Mills Lagrangian in the first-order formulation, in which \(A_\mu\) and \(F_{\mu\nu}\) are treated as independent co-ordinates:

\[
\mathcal{L}_F = \frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - \frac{1}{2} F_{\mu\nu} \cdot (\partial^\mu A^\nu - \partial^\nu A^\mu + g A^\mu \times A^\nu).
\]  

(Bold-face symbols, dots and crosses all refer to isovectors and operations among them; we write out the space-time vector indices explicitly.)
The Lagrangian (13.1) is invariant under infinitesimal gauge transformations

\[ A_\mu(x) \rightarrow A_\mu(x) + u(x) \times A_\mu(x) - \frac{1}{g} \partial_\mu u(x) \]

\[ F_{\mu\nu} \rightarrow F_{\mu\nu} + u \times F_{\mu\nu}. \quad (13.2) \]

The Euler-Lagrange equations,

\[ \frac{\delta L}{\delta F_{\mu\nu}^a} = 0, \quad \partial_\mu \left( \frac{\delta L}{\delta (\partial_\mu A_\nu^a)} \right) = \frac{\delta L}{\delta A_\nu^a}, \]

give

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu \quad (13.3) \]

and

\[ \partial_\mu F_{\mu\nu} + g A_\mu \times F_{\mu\nu} = 0. \quad (13.4) \]

Equations (13.3) and (13.4) together are equivalent to the Euler-Lagrange equations of the second-order formulation, in which \( L_s \) is written in terms of \( A_\mu \) and \( \partial^\mu A_\nu \) only.

In classical field theory, one is given an initial configuration of fields in a space-like hyperplane and then one tries to determine the fields at later times. Equations (13.3) and (13.4) can be separated into two classes: those which specify the temporal evolutions of the fields are called equations of motion; the others are constraint equations. From (13.3) and (13.4), the equations of motion for \( A_i \) and \( F_{oi} \) are

\[ \partial_\xi A_i = F_{oi} + (\delta_l + g A_i \times) A_o \quad (13.5) \]

\[ \partial_\xi F_{oi} = (\partial_l + g A_i \times) F_{ji} - g A_o \times F_{oi}. \quad (13.6) \]

Next, let us determine the independent variables. Since

\[ \delta L/\delta (\partial_\xi A^a_\mu) = -F^{o\mu} \]

\( F_{oi} = -F^{oi} \) are the momenta canonically conjugate to \( A_i \). Since \( L \) is independent of \( \partial_\xi A^o \), \( A^o \) does not have a conjugate momentum, and must be treated as a dependent variable.

The constraint equations are

\[ F_{ij} = \partial_i A_j - \partial_j A_i + g A_i \times A_j \quad (13.8) \]

which defines \( F_{ij} \) in terms of \( A_i \) at equal times, and

\[ (\delta_k + g A_k \times) F_{ko} = 0 \quad (13.9) \]

which tells us that not all the conjugate momenta \( F_{ok} \) are independent (eq. (13.9) is analogous to \( \nabla \cdot E = 0 \) in ordinary electrodynamics). It follows that not all the \( A_k \) can be treated as independent, and we are forced to impose a gauge condition. We choose the Coulomb gauge:

\[ \nabla_k A_k = 0. \quad (13.10) \]

This is always possible because of the gauge invariance of the second kind of the Lagrangian.
Eq. (13.10) means that $A$ must be transverse. Therefore, the longitudinal component $F_{oij}^L$ of the canonical momentum $F_{oij}$ is not independent, but depends on the other degrees of freedom through the constraint equation (13.9). $F_{oij}^L$ and the transverse component $F_{oij}^T$ can be defined

$$F_{oij} = F_{oij}^T + F_{oij}^L, \quad \nabla_i F_{oij} = \nabla_i F_{oij}^L, \quad \epsilon^{ijk} \nabla_j F_{oik}^L = 0. \tag{13.11}$$

Our task is now to express $A_\alpha$ and $F_{oij}^L$ in terms of the independent variables and construct the Hamiltonian. Let us write

$$F_{oij}^L = -\nabla_i f, \quad F_{oij}^T = E_i, \quad \nabla_i F_{oij} = -\nabla^2 f \tag{13.12}$$

where $E_i$ is purely transverse. Therefore $E_i$ and the transverse components of $A_i$ are the independent variables conjugate to one-another. From the constraint eq. (13.9), we find that

$$(\nabla^2 + g A_k \times \nabla_k) f = g A_i \times E_i. \tag{13.13}$$

Equation (13.13) can be formally solved by introducing a Green’s function $\mathcal{D}_c$, defined as the solution to

$$(\nabla^2 \delta^{ab} + g e^{acb} A_k^c \nabla_k) \mathcal{D}_{cd}^{bd}(x, y; A) = \delta^{ad}\delta^b(x - y). \tag{13.14}$$

Then $f$ is a solution of (13.13) if

$$f^a(x, t) = g \int d^3y \mathcal{D}^{ab}(x, y; A) e^{bcbd} A_k^c(y, t) E_k^d(y, t). \tag{13.15}$$

Considering $\mathcal{D}_c$ to be an integral operator, we may write (13.15) as

$$f = g \mathcal{D}_c \cdot A_k \times E_k.$$  

The function $\mathcal{D}_c$ has no closed form, but can be expanded in a power series in $g$. The first approximation is just the Green’s function for the Laplacian, and

$$\mathcal{D}_{cd}^{ab}(x, y; A) = \frac{\delta^{ab}}{4\pi|x - y|} + g \int d^3z \frac{1}{4\pi|x - z|} e^{acb} A_k^c \nabla_k \frac{1}{4\pi|y - z|} + \ldots \tag{13.16}$$

in analogy to the method for finding the Green’s function for $H_0 + H'$ where $H'$ is small and the Green’s function for $H_0$ is known.

We obtain an equation for $A_\alpha$ by taking the divergence of eq. (13.5) and using (13.10) and (13.11),

$$(\nabla^2 + g A_j \times \nabla_j) A_\alpha = \nabla^2 f \tag{13.17}$$

which can be solved using $\mathcal{D}_c$, since the operator in brackets is the same as in eq. (13.13):

$$A^a_\alpha(x, t) = \int d^3y \mathcal{D}_{cd}^{ab}(x, y; A) \nabla^2 f^b(y, t)$$

or

$$A_\alpha = \mathcal{D}_c \nabla^2 f. \tag{13.17a}$$

Now we construct the Hamiltonian density $\mathcal{H}$:
\[ \mathcal{A} = \mathbf{E}_i \cdot \frac{\partial \mathbf{A}_i}{\partial t} - \mathcal{L} \quad (13.18) \]

From (13.5), (13.11), (13.12) and (13.17a) we find that

\[ \frac{\partial}{\partial t} A_i = E_i - \nabla_i f + (\nabla_i + gA_i \times) \mathcal{D}_c \cdot \nabla^2 f \]

\[ = E_i - [\nabla_i - (\nabla_i + gA_i \times) \mathcal{D}_c] \nabla^2 f \quad (13.18a) \]

Because of (13.14), the operator in brackets operating on \( f \) is explicitly transverse. From (13.18a) and (13.13) or (13.15),

\[ \int d^3x E_i \cdot \mathcal{A}_i = \int d^3x [E_i^2 + g(E_i \times A_i) \cdot \mathcal{D}_c \cdot \nabla^2 f] \]

\[ = \int d^3x [E_i^2 - f \cdot \nabla^2 f] = \int d^3x [E_i^2 + (\nabla_i f)^2] \]

and

\[ \mathcal{L} = \frac{1}{4} F_{\mu \nu} \cdot F^{\mu \nu} - \frac{1}{2} F_{\mu \nu} \cdot (\partial^\mu A^\nu - \partial^\nu A^\mu + gA^\mu \times A^\nu) \]

\[ = \frac{1}{2} (F_{\mu \nu})^2 - \frac{1}{2} (B_i)^2 = \frac{1}{2} (E_k - \nabla_k f)^2 - \frac{1}{2} (B_k)^2 \quad (13.19) \]

where

\[ B_i = \frac{1}{2} \epsilon^{ijk} F_{jk} \]

So the Hamiltonian is

\[ H = \frac{1}{2} \int d^3x [E_i^2 + B_i^2 + (\nabla_i f)^2] \quad (13.20) \]

The last term is like the familiar instantaneous Coulomb interaction which occurs in electrodynamics when quantized in this gauge.

Now we can write the Coulomb gauge generating functional \( W_c[J] \) in terms of the independent co-ordinates and momenta, \( A_i \) and \( E_i \), where \( T \) stands for "Transverse", according to eq. (12.3):

\[ W_C[J] = \int [dE_i^T] [dA_i^T] \exp \{ i \int d^4x [E_k \cdot \dot{A}_k - \frac{1}{2} E_k^2 - \frac{1}{2} B_k^2 - \frac{1}{2} (\nabla_k f)^2 - A_k \cdot J_k] \} \quad (13.21) \]

where \( f \) is a function of \( E_i^T \) and \( A_i^T \) as expressed in (13.15). [We write the source term with a negative sign, so that the covariant version below will have \( +A^\mu \cdot J_\mu \).]

The transverse field \( E_i^T \) is difficult to compute with. Therefore we introduce a dummy variable \( E^L \) by

\[ \int [dE_i^T] = \int [dE_i^T] [dE^L] \prod_x \delta(E^L) \quad (13.22) \]

and define three independent components \( E_i \) by

\[ E_i = \left( \delta_{ij} - \nabla_j \frac{1}{\nabla^2} \nabla_i \right) E_i^T + \nabla_i \frac{1}{\nabla^2} E^L \quad (13.23) \]
in an obvious notation. From (13.23),
\[ \mathbf{E}^L = \nabla_i E_i \]
and therefore
\[ \int [dE^T_i] = \int \prod_x \delta(\nabla_i E_i) \]
where \( \mathcal{G} \) is the Jacobian of the transformation from the three \( E_i \) to \( E^T_i, E^L \), and
\[ [dE_i] \equiv \prod_x \prod_a \prod_i dE_i^a(x). \]

To give \( \mathcal{G} \) a meaning, we should go back to the definition of \([dE_i]\) as a limit of an approximation with a finite number of lattice points. In the limit, \( \mathcal{G} \to \infty \), but in a way independent of the fields, so it is just a multiplicative factor in \( W_c[J] \) which doesn’t matter. The same construction works for \( A_i^T \). Therefore,
\[ W_c[J] = \int [dE_i] [dA_i] \prod_x \delta(\nabla_i E_i) \delta(\nabla_i A_i) \exp\left[ i \int d^4x \left( E_i \cdot \nabla_i A_i - \frac{1}{2} B_i^2 - \frac{1}{2} (\nabla_i f)^2 - f \cdot J_i \right) \right]. \]  

At this point, we could examine (13.24) and obtain the Feynman rules in the Coulomb gauge. But they wouldn’t be covariant, and the Lorentz covariance of the \( S \)-matrix will not be obvious throughout the calculation. It isn’t useful to do calculations in the Coulomb gauge; the Coulomb gauge is the one in which the form \( W[J] \) is most easily obtained from first principle.

The \( S \)-matrix, of course, is covariant and gauge invariant, so it must be possible to find a more covariant-looking form of \( W_c[J] \) than (13.24). In (13.24), \( f \) is a function of \( E \) and \( A \) given by eq. (13.15). We introduce \( f \) as a dummy variable by multiplying eq. (3.24) by the constant
\[ \int [df] \delta(f - g \cdot \partial_c \cdot A_i \times E_i) \]
where by \( \partial_c \cdot \) we mean the operation in (13.15). Since (13.15) is equivalent to (13.13), we write (13.25) as
\[ \int [df] \det M_c \delta((\nabla^2 + g A_i \times \nabla_i)f - g A_i \times E_i) \]
where \( \det M_c \) is the Jacobian of the transformation from \( f \) to \( (\nabla^2 + g A_i \times \nabla_i)f \). \( M_c \) is a matrix in \( x - y \) space as well as isospin space:
\[ M_{ca}^b(x, y) = (\nabla^2 \delta_{ab} + g \epsilon^{abc} A_i^f(y) \nabla_i) \delta^4(x - y) \]
\[ = \nabla^2[\delta_{ab} \delta_3(x - y) + g \epsilon^{abc} G(x, y) A_i^f(y) \nabla_i] \delta(x_o - y_o) \]
where \( \nabla^2 G(x, y) = \delta_3(x - y) \). Now, eq. (13.24) becomes
\[ W_c[J] = \int [dA_i] [dE_i] [df] \det M_c \prod_x \delta(\nabla_i A_i) \prod_x \delta(\nabla_i E_i) \delta(\nabla^2 + g A_i \times \nabla_i)f \]
\[ - g A_i \times E_i \exp \left[ i \int \left( E_i \cdot A_k - \frac{1}{2} (f_k^2 + B_k^2 + (\nabla_k f)^2 - J_i \cdot A_i \right) d^4x \right]. \]
Next we change variables from $E_i$ to $F_{oi}$, defined by

$$F_{oi} = E_i - \nabla_i f.$$  \hfill (13.29)

Then, in (13.28), we write

$$[dE_i][df] \prod_x \delta(\nabla_i E_i) \delta[(\nabla^2 + g A_i \times \nabla_i)f - g A_i \times E_i]$$

$$= [dF_{oi}][df] \prod_x \delta(\nabla_i F_{oi} + \nabla^2 f) \delta[\nabla^2 f - g A_i \times F_{oi}]$$

$$= [dF_{oi}][df] \prod_x \delta(\nabla_i F_{oi} + g A_i \times F_{oi}) \delta(\nabla^2 f - g A_i \times F_{oi}).$$  \hfill (13.30)

Now we consider the $[df]$ integration, using the last $\delta$-function in (13.30). The Jacobian is just $\det \Delta^2$, an infinite constant which we drop (or absorb into the definition of $M_C$). Thus

$$W_C[J] = \int [dA_i][dF_{oi}] \delta(\nabla_i F_{oi} + g A_i \times F_{oi})$$

$$\times \exp \{i \int d^4x [F_{oi} \partial_o A_i - \frac{1}{2} F_{oi}^2 - \frac{i}{2} (\nabla_i A_j - \nabla_j A_i + g A_i \times A_j)^2 - J_i \cdot A_j] \}.$$  \hfill (13.31)

To obtain the exponent in (13.31), we have written in the exponent in (13.28)

$$[E_k + (\nabla_k f)]^2 = (E_k - \nabla_k f)^2 = F_{oi}^2$$

omitting the cross-term which vanishes upon integration over $x$.

Next we write the $\delta$-function as an integral, using $A_o$ as the dummy variable:

$$\prod_x \delta(\nabla_i F_{oi} + g A_i \times F_{oi}) = \prod_x \int \frac{dA_o}{2\pi} \exp \{i A_o \cdot (\nabla_i F_{oi} - g A_i \times F_{oi})\}$$

$$\sim \int [dA_o] \exp \{i \int d^4x F_{oi} \cdot (g A_o \times A_i - \nabla_i A_o)\}.$$  \hfill (13.32)

Finally, we write the term $\frac{1}{4} (\nabla_i A_j - \nabla_j A_i + g A_i \times A_j)^2$ in the exponent in (13.31) as

$$\int [dF_{ij}] \exp \{i \frac{1}{4} F_{ij} \cdot (\nabla_i A_j - \nabla_j A_i + g A_i \times A_j)\}$$  \hfill (13.33)

which is a standard Gaussian integral. Putting (13.33) and (13.32) into (13.31), and restricting $J_o$ to be zero, we obtain

$$W_C[J] = \int [dA_u][dF_{\mu\nu}] \det M_C \prod_x \delta(\nabla_i A_i)$$

$$\times \exp \left[ i \int d^4x \left\{ -\frac{1}{4} F_{oi} \cdot F_{oi} + \frac{1}{4} F_{ij} \cdot F_{ij} - \frac{1}{2} F_{ij} \cdot (\nabla_i A_j - \nabla_j A_i + g A_i \times A_j) + F_{oi} (\partial_o A_i - \nabla_i A_o + g A_o \times A_i) \right\} \right]$$

$$= \int [dA_u][dF_{\mu\nu}] \det M_C \prod_x \delta(\nabla_i A_i) \exp \{ i \int d^4x [\mathcal{L} + J^\mu \cdot A_\mu] \}. \hfill (13.34)$$

Were it not for the factor $\det M_C$, (13.34) implies that one could get the Feynman rules directly from $\mathcal{L}$. The extra factor is analogous to the correction obtained in section 11 for a velocity-dependent potential.
How do we interpret \( \det M_C \)? From (13.27),

\[
\det M_C = \det \nabla^2 \cdot \det [I + L]
\]

where

\[
L = g e^{abc} G(x, y) A^c_i(y) \cdot \nabla_i \delta(x_o - y_o), \quad I = \delta^{ab} \delta^4(x - y).
\]

Now \( \det \nabla^2 \) is an infinite constant, and

\[
\det(I + L) = \exp \text{Tr} \log(I + L)
\]

\[
= \exp \sum \frac{(-1)^{n-1}}{n} \int d^4x_1 \ldots d^4x_n \text{Tr} L(x_1, x_2) L(x_2, x_3) \ldots L(x_n, x_1).
\]

The trace inside the integral is over isospin indices only.

We shall encounter Jacobians like \( \det M_C \) in the next few sections. Eq. (13.37) is a general formula for evaluating them. In our case

\[
\det(I + L) = \exp \left[ \text{Tr} \left( \text{Tr} \left( -i \int d^4A^\mu \right) \delta^2(x) + J^\mu(x) \cdot A^\mu(x) \right) \right]
\]

where \( (T^a)_{bc} = e^{abc} \) and \( \text{Tr} \) means trace over isospin indices.

Since (13.38) is a power series in the exponent, it is an effective correction in each order to the Feynman rules obtained from \( \mathcal{L} \) alone.

### Bibliography

The presentation in this lecture is similar to and inspired by


See also


### 14. Intuitive approach to the quantization of gauge fields

Equation (13.34) can be further simplified. We can perform the functional integration over \( F^a_{\mu \nu} \) and obtain

\[
W[J] = \int [dA^\mu] \det M_C \left[ \left[ \delta(\nabla_i A_i(x)) \right] \exp \left\{ i \int d^4x \left[ \mathcal{L}(x) + J^\mu(x) \cdot A^\mu(x) \right] \right\}
\]

where \( \mathcal{L}(x) \) is the second-order Lagrangian:

\[
\mathcal{L}(x) = \frac{1}{4} \left( \partial_\mu A^\mu - \partial_\nu A^\nu + g A^\mu \times A^\nu \right)^2.
\]

Except for the factor \( \det M_C \prod_x \delta(\nabla_i A_i(x)) \), eq. (14.1) is in the standard form for simple field theories.
\[ W[J] \sim \int [d\phi] \exp \{ i \int d^4x [\mathcal{L}(x) + J(x)\phi(x)] \} \]  

(14.2)

The following intuitive argument due to Faddeev and Popov shows very clearly the raison d'être for this extra factor.

The reason eq. (14.2) is not applicable to the gauge theory is that the quadratic part of the Lagrangian

\[ L_0 = -\frac{1}{4} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})^2 d^4x = \int d^4x d^4y \frac{1}{2} A_{\mu}(x) \cdot K^{\mu\nu}(x, y) \cdot A_{\nu}(y), \]

is singular, in the sense that the operator \( K^{\mu\nu} \) which defines the quadratic form is singular and cannot be inverted. In fact, the operator \( K^{\mu\nu} \) is essentially a projection operator for the transverse components of \( A_{\mu} \). This means, in particular, that the Euclidean version of the functional integral of eq. (12.2) [see the discussion following eq. (12.10)] has no Gaussian damping factor with respect to the variation of the longitudinal component of \( A_{\mu} \), and eq. (14.2) is meaningless at this elementary level even in the Euclidean formulation. More generally, the action is invariant under the gauge transformation \( A_{\mu} \rightarrow A_{\mu}^g \) where \( A_{\mu}^g \) is the result of applying the element \( g \) of the gauge group \( G \) to the field \( A_{\mu} \):

\[ A_{\mu}^g \cdot L = U(g) \left[ A_{\mu} \cdot L + \frac{1}{ig} U^{-1}(g) \partial_{\mu} U(g) \right] U^{-1}(g). \]  

(14.3)

To put it differently, the action is constant on the orbits of the gauge group, which are formed by all \( A_{\mu}^g \) for fixed \( A_{\mu} \) and \( g \) ranging all over \( G \). Thus, the path integral for the vacuum-to-vacuum amplitude \( W[J] \) diverges even in the Euclidean formulation, since for those variations of \( A_{\mu} \) which are along the orbits, the action does not provide necessary damping. Faddeev and Popov pointed out that the amplitude \( W[J = 0] \) is therefore proportional to the "volume" of orbits \( \Pi A \cdot dg(x) \), and this factor should be extracted before defining \( W[J] \). In other words, for the gauge fields, the path integral is to be performed not over all variations of the gauge fields, but rather over distinct orbits of \( A_{\mu} \) under the action of the gauge group.

To implement the above idea, we shall choose a "hypersurface" in the manifold of all fields which intersects each orbit only once. This means that if

\[ f_a(A_{\mu}) = 0, \quad a = 1, 2, \ldots N \]

(14.4)

is the equation of the hypersurface, \( N \) being the dimension of the group, the equation

\[ f_a(A_{\mu}^g) = 0 \]

must have a unique solution \( g \) for given \( A_{\mu} \). We are going to integrate over this hypersurface, instead of integrating over the manifold of all fields. The conditions \( f_a(A_{\mu}) = 0 \) define a gauge; the Coulomb gauge \( f_a(A_{\mu}) = \nabla_i A_{\mu}^i \) is an example.

Before proceeding further, let us pause here to review briefly a few simple facts about group representations. Let \( g, g' \in G \). Then \( gg' \in G \), and

\[ U(g)U(g') = U(gg'). \]

The invariant Hurwitz measure over the group \( G \) is an integration measure on the group space which is invariant in the sense that

\[ dg' = d(gg'). \]  

(14.5)
If we parametrize $U(g)$ in the neighborhood of the identity as

$$U(g) = 1 + iu \cdot L + O(u^2),$$

then in the neighborhood of the identity we may always choose

$$dg = \prod_a du_a, \quad g \approx 1.$$  \hspace{1cm} (14.6)

Let us define the quantity $\Delta_f[A_\mu]$ by

$$\Delta_f[A_\mu] \int \prod_x dg(x) \prod_{x,a} \delta[f_a(A^g_\mu(x))] = 1.$$  \hspace{1cm} (14.7)

The "naive" expression for the vacuum-to-vacuum amplitude is

$$\int [dA_\mu] \exp \{i \int d^4x L(x)\}.$$  \hspace{1cm} (14.8)

We may insert the left-hand side of eq. (14.7) into the integrand of eq. (14.8) without changing anything:

$$\int \prod_x dg(x) [dA_\mu] \Delta_f[A_\mu] \prod_{x,a} \delta[f_a(A^g_\mu(x))] \exp \{i \int d^4x L[A_\mu(x)]\}.$$  \hspace{1cm} (14.9)

Now, in the integrand of eq. (14.9) we can perform a gauge transformation on $A_\mu(x)$:

$$A_\mu(x) \rightarrow [A_\mu(x)]g^{-1}.$$  \hspace{1cm}

Under the gauge transformation (14.3) the action and the metric are invariant, and one can verify easily from eqs. (14.5) and (14.7) that $\Delta_f[A_\mu]$ is gauge invariant:

$$\Delta_f^{-1}[A^g_\mu] = \int \prod_x dg'(x) \prod_{x,a} \delta[f_a(A^{gg'}_\mu(x))]$$

$$= \int \prod_x d(g(x)g'(x)) \prod_{x,a} \delta[f_a(A^{gg'}_\mu(x))]$$

$$= \int \prod_x dg''(x) \prod_{x,a} \delta[f_a(A^{gg''}_\mu(x))] = \Delta_f^{-1}[A_\mu],$$

or

$$\Delta_f[A^g_\mu] = \Delta_f[A_\mu].$$  \hspace{1cm} (14.10)

So, eq. (14.9) is equal to

$$\int \prod_x dg(x) [dA_\mu] \Delta_f[A_\mu] \prod_{x,a} \delta[f_a(A_\mu(x))] \exp \{i \int d^4x L[A_\mu(x)]\}$$

and we find that the integrand of the group integration is independent of $g(x)$. This is the observation of Fadeev and Popov, who saw that $\prod_x dg(x)$ is simply an infinite factor independent of fields. Therefore, it can be divided out, and $W[J]$ may be defined as

$$W_f[J] = \int [dA_\mu] \Delta_f[A_\mu] \prod_{x,a} \delta[f_a(A_\mu(x))] \exp \{i \int d^4x [L(x) + J^\mu(x) \cdot A_\mu(x)]\}.$$  \hspace{1cm} (14.11)

It is to the credit of Faddeev and Popov that they also gave the canonical derivation of eq. (13.34) as discussed in the preceding section, as well as this elegant argument. Before demonstrating the connection between eqs. (14.1) and (14.11) above, we shall compute $\Delta_f[A_\mu]$. 

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Since the factor $\Delta_f[A_\mu]$ is multiplied by $\Pi_a \delta[f_a(A_a(x))]$ in eq. (14.11), it suffices to compute the former only for $A_\mu$ which satisfies eq. (14.4). Now define $M_f$ by

$$f_a(A_\mu^x(x)) = f_a(A_\mu(x)) + \int d^4y \sum_b [M_f(x, y)]_{ab} u_b(y) + O(u^2). \tag{14.12}$$

Then from eq. (14.7) we find that

$$\Delta_f^{-1}[A_\mu] = \int x a \{ d_u(x) \delta[f_a(A_\mu^x(x))] \} = \int x a \{ d_u(x) \delta(M_f u) \}$$

for $A_\mu$ satisfying $f_a(A_\mu) = 0$, so that

$$\Delta_f[A_\mu] = \det M_f = \exp \{ \text{Tr} \ln M_f \}. \tag{14.13}$$

The hypersurface equation $f_a = 0$ is just the gauge condition, and for the Coulomb gauge adopted in the preceding section, we have

$$f_a(A_\mu) = \nabla_i A_i^a = 0$$

and

$$f_a(A_\mu^x) = \nabla_i A_i^a + \frac{1}{g} (\nabla^2 \delta^{ab} - g e^{abc} A_i^c \nabla_i) u_b(x) + O(u^2)$$

so

$$[M_f(x, y)]_{ab} \sim \frac{1}{g} \nabla^2 \left( \delta^{ab} - g e^{abc} \frac{1}{g} \nabla_i A_i^c \nabla_i \right) \delta^4(x - y) \sim [M_c(x, y)]_{ab}, \tag{14.14}$$

which shows that eq. (14.1) is indeed a special case of eq. (14.11) above for $f_a = \nabla_i A_i^a$.

The form of eq. (14.11) suggests using a wide range of gauges other than the Coulomb gauge. For the moment, we shall not ask what relations the Green's functions generated in such a gauge bear to those defined in the Coulomb gauge, but merely note the explicit form of $\Delta_f$ when the manifestly covariant Landaau gauge condition

$$\partial^\mu A_\mu(x) = 0$$

is chosen. Equation (14.12) takes the form

$$\partial^\mu A_\mu^x(x) = \partial^\mu A_\mu(x) + \frac{1}{g} [\partial^2 u + g \partial^\mu(A_\mu \times u)] + O(u^2)$$

so that $M_f$ is given by

$$[M_f(x, y)]_{ab} = \frac{1}{g} (\partial^2 \delta_{ab} - g e_{abc} A_i^c \partial_i) \delta^4(x - y) \tag{14.15}$$

when $A_\mu$ is restricted to $\partial^\mu A_\mu = 0$. Therefore removing the trivial factor $(1/g)\partial^2$, we have

$$\Delta_L \equiv \det M_L \sim \exp \{ \text{Tr} \ln(1 + L) \} \tag{14.16}$$
where

$$(x, a \mid L \mid y, b) = g \varepsilon_{abc} \int D_F(x - z) A^c_\mu(z) \frac{\partial}{\partial z_\mu} \delta^4(z - y) d^4 z.$$ \hspace{1cm} \text{(14.11)}$$

More explicitly we can write

$${\Delta_L} = \exp \left\{ - \sum \frac{(-g)^n}{n} \int d^4 x_1 \ldots d^4 x_n \text{Tr} [\partial^\Lambda D_F(x_1 - x_2) \cdot A_\mu(x_2) \partial_\mu D_F(x_2 - x_3) \ldots D_F(x_n - x_1) \cdot \lambda_\Lambda(x_1)] \right\}. \hspace{1cm} \text{(14.17)}$$

Here we have used the conventional notation

$$(-\partial^2 + ie)D_F(x - y) = \delta^4(x - y).$$

The $ie, \epsilon > 0$, is chosen according to the Euclidicity postulate.

The necessity of having the extra factor $\Delta_f \Pi \delta [f(A_\mu(x))]$ was first noted by Feynman. We can write eq. (14.11) as

$$W_f[J] = \int [dA_\mu] \prod_x \delta[f(A_\mu(x))] \exp \{ i S_{\text{eff}} + \int d^4 x J^\mu(x) \cdot A_\mu(x) \}$$ \hspace{1cm} \text{(14.18)}$$

where

$$S_{\text{eff}} = \int d^4 x L(x) - i \text{Tr} \ln M_f.$$ \hspace{1cm} \text{(14.19)}$$

In the case of the Landau gauge, it has been observed that the additional term in the effective action $-i \text{Tr} \ln M_L$ can be viewed as arising from loops generated by a fictitious isotriplet of complex scalar fields $c$ obeying Fermi statistics, whose presence and interactions can be described by the action

$$S_c = -\int d^4 x [\partial^\mu c^\dagger(x) \cdot \partial_\mu c(x) + g \partial^\mu c^\dagger(x) \cdot A_\mu(x) \times c(x)]$$

$$\sim \int d^4 x d^4 y \sum_{a, b} c^\dagger_a(x) [M_L(x, y)]_{ab} c_b(y).$$

That is, eq. (14.18) may be written as

$$W_L[J] = \int [dA_\mu] \prod_x \delta[f(A_\mu(x))] \int [dc^\dagger] [dc] \exp \left\{ i [S + S_c + \int d^4 x J^\mu(x) \cdot A_\mu(x)] \right\}.$$ \hspace{1cm} \text{(14.19)}$$

In fact it is not difficult to show that the $c$- and $c^\dagger$-integrations could be carried out trivially if they were commuting $c$-numbers, yielding

$$\int [dc^\dagger] [dc] \exp(i S_c) \sim (\det M_L)^{-1} = \exp \{ -\text{Tr} \ln M_L \},$$

and

$$\exp \{ -\text{Tr} \ln M_L \} \sim \exp \{ -\text{Tr} \ln(1 + L) \}$$

$$= \exp \left\{ -\text{Tr} L + \frac{1}{2} \text{Tr} L^2 + \ldots (-)^n \frac{1}{n} \text{Tr} L^n + \ldots \right\}.$$
where the terms in the exponent may be viewed as arising from loops of the complex boson fields \( c \). If the \( c \) are fermion fields, then the terms \( \text{Tr} L^n \) have to be multiplied by an extra \(-1\) sign, so that we have

\[
\int [dc^+] [dc] \exp(iS_c) \sim \exp \left\{ + \text{Tr} L - \frac{1}{2} \text{Tr} L^2 \ldots + \frac{(-1)^{n+1}}{n} \text{Tr} L^n \ldots \right\}
\]

\[
= \exp \{ \text{Tr} \ln(1 + L) \} \sim \det M_L.
\]

The Feynman rules for \( W_L[J] \) of eq. (14.19) can be worked out in much the same way as we did for a scalar field theory in section 12. The gauge boson propagator is determined from

\[
W_L[J] = \int [dA] \prod_x \delta[\partial^\mu A^\mu(x)] \exp \left[ i \int d^4x \left( -\frac{1}{4} (\partial_\mu A^\nu - \partial_\nu A^\mu)^2 + J^\mu(x) \cdot A^\mu(x) \right) \right], \quad (14.20)
\]

A convenient way of computing eq. (14.20) is to write

\[
\prod_x \delta[\partial^\mu A^\mu(x)] \sim \lim_{\alpha \to 0} \exp \left\{ \frac{i}{2\alpha} \int d^4x [\partial^\mu A^\mu(x)]^2 \right\}.
\]

[We have discarded an infinite constant \( \Pi_x \sqrt{2\pi\alpha} \).] Then we have

\[
W_L[J] = \lim_{\alpha \to 0} \int [dA] \exp \left[ i \int d^4x \left( -\frac{1}{4} \partial^\mu \partial_\nu A^\mu(x) \cdot \left[ \partial^\nu A^\mu \left( 1 - \frac{1}{\alpha} \right) \right] + \int d^4x J^\mu(x) \cdot A^\mu(x) \right) \right]
\]

\[
= \lim_{\alpha \to 0} \exp \left( \frac{-i}{2} \right) \int d^4x d^4y J^\mu(y) \cdot D_F^{\mu\nu}(x - y; \alpha) J_\nu(y) \quad (14.21)
\]

where the vector boson propagator \( D_F^{\mu\nu} \) in this gauge is

\[
D_F^{\mu\nu}(x - y; \alpha) = \frac{d^4k}{(2\pi)^4} \exp \{ ik \cdot (x - y) \} \frac{1}{k^2 + i\epsilon} \left[ g^{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right] \frac{1}{(1 - \alpha)} \quad \alpha \to 0
\]

\[
D_F^{\mu\nu}(x - y) = -\int d^4k (2\pi)^4 \exp \{ ik \cdot (x - y) \} \frac{1}{k^2 + i\epsilon} \left( g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (14.22)
\]

and is four-dimensionally transverse. The rest of the Feynman rules can be derived as in the scalar case. They are recorded in the following fig. 14.1. In addition, the following rules must kept in mind: the ghost-ghost-vector vertex is "dotted", the dot indicating which ghost line is differentiated; a ghost line cannot be dotted at both ends; a ghost loop carries an extra minus sign.

Bibliography

We have given the references to the work of Faddeev and Popov in the proceeding section; in addition, we cite

1. N.P. Konopleva and V.N. Popov, Kalibrovochnye Polya (Atomizdat, Moscow, 1972), in Russian.

Quantization of the gauge fields has been discussed also in

15. Equivalence of the Landau and Coulomb gauges

Formally, the S-matrix computed in the Landau gauge is the same as that computed in the Coulomb gauge. An element of the unrenormalized S-matrix is obtained from the corresponding Green's functions by removing single particle propagators corresponding to external lines, taking the Fourier transform of the resulting "amputated" Green's function and placing external momenta on the mass shell. The demonstration we shall present is rigorous except that the S-matrix of a gauge theory is plagued by infrared divergences and may not even be defined. In fact this may be the reason why massless Yang-Mills particles are not seen in Nature. The point of presenting this demonstration is purely pedagogical: the spirit and the technique we espouse here will become useful when we discuss spontaneously broken versions of gauge theories.

We shall first establish the connection between $W_C[J]$ and $W_L[J]$. Recall that [eq. (14.1)]

$$W_C[J] = \int \{dA_\mu\} \Delta_C[A_\mu] \prod_x \delta(\nabla_i A_i(x)) \exp\left[iS[A_\mu] + i \int d^4x J^\mu \cdot A_\mu\right]$$

(15.1)

where $\Delta_C = \det M_C$ and that

$$\Delta_L[A_\mu] \int \prod_x dg(x) \prod_x \delta(\partial^\mu A_\mu(x)) = 1.$$  

(15.2)

Inserting the left-hand side of eq. (15.2) in the integrand of the functional integration in eq. (15.1), we write

---

### Fig. 14.1. Feynman rules in the Yang-Mills theory. Solid lines are vector mesons. Dashed lines are scalar ghosts.
\[ W_C[J] = \int \prod_x \delta(\partial^\mu A^\mu(x)) \exp \{ iS[A^\mu] + i \int d^4x J^\mu \cdot A^\mu \} \]

We now make a gauge transformation of the integration variables \( A^\mu(x) \): \( A^\mu(x) \rightarrow [A^\mu(x)]g^{-1} \). Recalling the gauge invariance of the action \( S \), \( \Delta_\mu \) and the metric \( \left[ dA^\mu(x) \right] \), we find that

\[ W_C[J] = \int \prod_x \delta(\partial^\mu A^\mu(x)) \exp \{ iS[A^\mu] + i \int d^4x J^\mu \cdot A^\mu \} \]

\[ \times \Delta_C[A^\mu] \int \prod_x \delta(\partial^\mu A^\mu(x)) \exp \left\{ i \int d^4x J^\mu \cdot \dot{A}^\mu \right\} \]

\[ = \int \prod_x \delta(\partial^\mu A^\mu(x)) \exp \left\{ iS[A^\mu] + i \int d^4x J^\mu \cdot A^\mu \right\} \]

(15.3)

where \( A^\mu_0 \) is the gauge transform of \( A^\mu(x) \), which satisfies \( \partial^\mu A^\mu = 0 \), such that

\[ L \cdot \nabla_i A^\mu_i = \nabla_i \left[ \frac{U(g_o)}{L} \left[ L \cdot A_i + \frac{1}{ig} U^{-1}(g_o) \nabla_i U(g_o) \right] U^{-1}(g_o) \right] = 0. \]

(15.4)

In deriving eq. (15.3), we have used the fact that

\[ \Delta_C[A^\mu] \int \prod_x \delta(\nabla_i A^\mu_i^{-1}) = \Delta_C[A^\mu] \int \prod_x \delta(\partial^\mu A^\mu(x)) \prod_x \delta \left( \nabla_i A^\mu_i^{-1} - \frac{1}{g} M_C[A^\mu_0 \right) u \]

\[ \sim \Delta_C[A^\mu] \Delta_C^{-1}[A^\mu_0] = 1. \]

Now, we must find out \( A^\mu_0 \) by solving eq. (15.4). It is possible to construct \( A^\mu_0 \) in a power series in \( A^\mu \). We leave it as an exercise to construct first few terms in this expansion. For our purpose it suffices to note that

\[ A^\mu_i = \left( \delta_{ij} - \nabla_i \frac{1}{\sqrt{g}} \nabla_j \right) A_j + O(A^\mu_0 \right).

The source \( J_\mu \) in the Coulomb gauge shall be restricted to

\[ J_\mu = 0, \quad \nabla_i J_i = 0. \]

Therefore, we may write

\[ \int d^4x J^\mu \cdot A^\mu_0 = \int d^4x J^\mu \cdot F^\mu(x; \lambda) \]

where

\[ F^\mu(x; \lambda) = A^\mu(x) + O(A^\mu_0 \right) \]

(15.5)

We can finally write down an equation for \( W_C \) in terms of \( W_L \). It is

\[ W_C[J] = \left[ \exp \left( i \int d^4x J^\mu(x) \cdot F^\mu(x; \lambda) \right) \right] W_L[j] \left|_{j=0} \right. \]

(15.6)
It is helpful to visualize eq. (15.3) or eq. (15.6) in terms of Feynman diagrams. These equations say that Green's functions in the Coulomb gauge are the same as those in the Landau gauge, when the source is suitably restricted [eq. (13.26)] except that one must take into account extra vertices between a source and fields, represented by the term

$$\int d^4x J^\mu \cdot (F_\mu - A_\mu)$$

when one tries to construct Coulomb gauge Green's functions by the Feynman rules of the Landau gauge. This connection becomes much simpler, if we go to the mass shell. In this case, we ought to compare only the terms having a pole in each of the external momenta, $p_i$, when $p_i^2 \to 0$. Of all the diagrams generated by the extra couplings of (15.7), only those in which the whole effect of the extra vertices can be reduced to a type of self energy insertion to the corresponding external line survive in this limit. The other corrections introduced by (15.7) will not contribute to poles of the Green's functions at $p_i^2 = 0$, and therefore not to the $S$-matrix. Therefore in the limit $p_i^2 \to 0$, the Coulomb gauge and the Landau gauge (unrenormalized) $S$-matrix elements will differ by a factor $\sigma^n$ where $n$ is the number of external lines and $\sigma$ is a factor independent of $n$. Comparing the two-point Green's functions in the two gauges $C$ and $L$:

$$\lim_{p^2 \to 0} D'_{\mu\nu}(p; C) = \frac{Z_C}{p^2 + i\epsilon} (g_{\mu\nu} + \ldots), \quad \lim_{p^2 \to 0} D'_{\mu\nu}(p; C) = \frac{Z_L}{p^2 + i\epsilon} (g_{\mu\nu} + \ldots)$$

we find

$$\sigma^2 = \frac{Z_C}{Z_L}.$$  

In general, unrenormalized $S$-matrix elements in the two gauges $C$ and $L$ are related to each order by

$$S_C = \sigma^n S_L = (Z_C/Z_L)^{n/2} S_L$$

so that the renormalized $S$-matrix element

$$S_{\text{ren}} = Z_C^{-n/2} S_C = Z_L^{-n/2} S_L$$

is independent of the gauge chosen to compute it.

In sum, what we have shown here is that $W_C[J]$ is equal to the expression (15.3) which would be $W_L[J]$ except that the coefficient of $J^\mu$ is $A_\mu^\text{so}$ instead of $A_\mu$. For the $S$-matrix, the only consequence of this difference is that the renormalization constants attached to each external line depend on the gauge.

Thus we have shown that the $S$-matrix can be calculated from $W_L[J]$, not just by the intuitive argument of section 14, but more formally, by obtaining $W_C[J]$ from first principles, and then demonstrating the equivalence of $S_C$ and $S_L$.

As pointed out earlier, the only flaw in the above argument is that the singularity at $p_i^2 = 0$ is not in general a simple pole.

**Bibliography**

This section is an explication of the discussion on the same subject in


A similar discussion was given for quantum electrodynamics in the operator field theory language by

16. Generating functionals for Green's functions and proper vertices

In this section we develop the formalism of generating functionals of connected Green's functions and of proper vertices. This topic is slightly out of the main line of development of this review. However, many recent papers on spontaneously broken symmetry make use of this elegant formalism for a very good reason: this formalism allows the discussion of the conditions for spontaneous breakdown of symmetry which goes beyond the one based on the classical Lagrangian and which is valid to all orders in perturbation theory.

Let us go back to the discussion of section 12 on scalar fields. We define the generating functional $Z[J]$ of connected Green's functions by

$$
\exp[i Z[J]] = \int [d\phi] \exp \{ i \int d^4x [\mathcal{L}(\phi(x)) + J(x) \cdot \phi(x)] \} 
$$

(16.1)

where $\phi$ and $J$ are multicomponent fields and sources, respectively.

The first derivative of $Z[J]$ with respect to $J_i$ is

$$
\frac{\delta Z[J]}{\delta J_i(x)} = \frac{1}{W[J]} \int [d\phi] \phi_i(x) \exp \{ i \int d^4x [\mathcal{L}(x) + J(x) \cdot \phi(x)] \} .
$$

(16.2)

We give it a special name, $\Phi_i(x)$:

$$
\delta Z[J]/\delta J_i(x) = \Phi_i(x).
$$

(16.3)

$\Phi_i(x)$ is the vacuum expectation value of $\phi_i(x)$ in the presence of $J(x)$; i.e., it is the classical field.

The value of eq. (16.2) when the external source is turned off ($J(x) = 0$) is the vacuum expectation value of the field $\phi$:

$$
\delta Z[J]/\delta J_i(x) \big|_{J=0} = \Phi_i.
$$

(16.4)

Note that $\Phi$ is independent of space-time, since in the limit $J = 0$, the left-hand side of (16.4) is translationally invariant.

It turns out that higher derivatives of $Z[J]$ at $J = 0$ are Green's functions of the field $\phi = \phi - \Phi$ whose vacuum expectation value vanishes. For example

$$
\frac{\delta^2 Z[J]}{\delta J_i(x) \delta J_j(y)} \big|_{J=0} = i \frac{1}{W[0]} \int [d\phi] [\phi(x) - \Phi][\phi(y) - \Phi] J_i \exp(i \int d^4x \mathcal{L})
$$

$$
= i \frac{1}{W[0]} \int [d\phi] \phi_i(x) \phi_j(y) \exp(i \int d^4x \mathcal{L}(x))
$$

(16.5)

as can be verified by differentiating eq. (16.2) with respect to $J_i$ and letting $J \to 0$. More generally we have

$$
\frac{\delta^n Z[J]}{\delta J_{i_1}(x_1) \ldots \delta J_{i_n}(x_n)} \bigg|_{J=0} = (i)^{n-1} \langle T(\phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n)) \rangle^c
$$

(16.6)

(where the superscript $c$ denotes the connected part of the Green's function) as can be shown by induction.
We shall now define the Legendre transform $\Gamma[\Phi]$ of $Z[J]$. It is defined as

$$\Gamma[\Phi] = Z[J] - \int d^4x J(x) \cdot \Phi(x), \quad \delta Z[J]/\delta J = \Phi.$$

(16.7)

The meaning of eq. (16.7) is this: $\Gamma$ is a functional of $\Phi(x)$ as defined by the right-hand side of the first equality. In it, $J$ is to be expressed in terms of $\Phi$ by inverting eq. (16.3), which defined $\Phi$ as a function of $J$. The Legendre transform (16.7) is a functional version of the well-known transformation familiar in classical mechanics and thermodynamics. By differentiating eq. (16.7) with respect to $\Phi_i$, we find that

$$\delta \Gamma[\Phi]/\delta \Phi_i(x) = \sum_j \int d^4y \{ \delta Z[J]/\delta J_j(y) \} \{ \delta J_j(y)/\delta \Phi_i(x) \} - J(x) \sum_j \int d^4y \Phi_j(y) \{ \delta J_j(y)/\delta \Phi_i(x) \},$$

or

$$\delta \Gamma[\Phi]/\delta \Phi_i(x) = -J_i(x).$$

(16.8)

Equation (16.8) is dual to eq. (16.3): by this we mean that the relation (16.3) which expresses $\Phi$ in terms of $J$ is the inverse of eq. (16.8) which expresses $J$ in terms of $\Phi$. This, in particular, means that eq. (16.4) can be written as

$$\delta \Gamma[\Phi]/\delta \Phi_i(x) \mid_{\Phi = v} = 0,$$

(16.9)

i.e., when $J = 0$, $\Phi$ takes the value $v$, and vice versa. Equation (16.9) is very important. It expresses the vacuum expectation value $v$ of the field $\phi$ as the solution to a variational problem: $v$ is the value of $\Phi$ which extremizes $\Gamma[\Phi]$.

What is the physical significance of $\Gamma$? To streamline our discussion, let us agree on the following convention: We will denote by subscripts $i, j, \ldots$ any labels $J$ or $\Phi$ carry, including the space-time variable $x$. We will adopt the convention that summations and integrations are always to be carried out over repeated indices. Differentiating eq. (16.3) with respect to $\Phi$, we obtain

$$\frac{\delta^2 Z[J]}{\delta J_j \delta J_i} = \delta_{ik}.$$

(16.10)

From eq. (16.9) we learn that

$$\delta J_i/\delta \Phi_j = -\delta^2 \Gamma[\Phi]/\delta \Phi_j \delta \Phi_i.$$

(16.11)

Define

$$\{X^{-1}[J]\}_{ij} = -\delta^2 Z[J]/\delta J_i \delta J_j$$

(16.12)

and

$$\{X[\Phi]\}_{ij} = \delta^2 \Gamma[\Phi]/\delta \Phi_i \delta \Phi_j.$$

(16.13)

Equations (16.10) and (16.11) mean that

$$(X^{-1})_{ij}(X)_{jk} = \delta_{ik}.$$

(16.14)

Since
\{X^{-1}[J = 0]\}_{ij} = -\delta^2 Z[J]/\delta J_i \delta J_j \bigg|_{J=0} + [\Delta'_F]_{ij}

is the full propagator for the barred field, and \(J = 0\) implies \(\Phi = v\), it follows that

\{X[\Phi = v]\}_{ij} = \delta^3 \Gamma[\Phi]/\delta \Phi_i \delta \Phi_j \bigg|_{\Phi = v}

is the inverse of the full propagator.

Next differentiate eq. (16.10) with respect to \(J_i\). We obtain

\[-\frac{\delta^3 Z[J]}{\delta J_i \delta J_j \delta J_k} X_{jk}(X^{-1})_{ij} \frac{\delta^3 \Gamma}{\delta \Phi_i \delta \Phi_k \delta \Phi_m} (X^{-1})_{lm} = 0\]

or

\[
\frac{1}{i^2} \frac{\delta^3 Z[J]}{\delta J_i \delta J_j \delta J_k} = (i X^{-1})_{ij}(i X^{-1})_{jk} \left[ \frac{i \delta^3 \Gamma[\Phi]}{\delta \Phi_i \delta \Phi_j \delta \Phi_k} \right]. \tag{16.15}
\]

Now take the limit \(J = 0, \Phi = v\). In this limit \(X^{-1}[J = 0]\) is the full propagator, so that

\[
\delta^3 \Gamma[\Phi]/\delta \Phi_i \delta \Phi_j \delta \Phi_k \bigg|_{\Phi = v} = \Gamma^{(3)}_{ij} \tag{16.16}
\]

is the three-point proper vertex. A proper vertex (or one-particle irreducible vertex) is a Green's function which cannot be made disconnected by cutting a single internal propagator, and from which (by convention) full propagators corresponding to external lines are removed. The three-point function has no such disconnected graphs except corrections to the propagators, which are explicitly removed in (16.15).

In general, the \(n\)th derivative of \(\Gamma\) at \(\Phi = v\) is the \(n\)-point proper vertex:

\[
\delta^n \Gamma/\delta \Phi_i \delta \Phi_j \ldots = \Gamma^{(n)}_{ij} \ldots .
\]

The proof of this statement proceeds inductively. Assume that \(\delta^n Z[J]/\delta J_i \delta J_j \ldots\) can be expressed as a sum of tree diagrams, each diagram consisting of proper vertices corresponding to \(\delta^n \Gamma[\Phi]/\delta \Phi_i \delta \Phi_j \ldots\), internal lines corresponding to \(\Delta'_F\) connecting pairs of proper vertices, and external lines. In particular,

\[
\frac{1}{i^{n-1}} \frac{\delta^n Z[J]}{\delta J_i \delta J_j \ldots} = (i X^{-1})_{ij}(i X^{-1})_{jk} \ldots \left[ \frac{i \delta^n \Gamma[\Phi]}{\delta \Phi_i \delta \Phi_j \ldots} \right] + \text{one-particle reducible terms}. \tag{16.17}
\]

Now, differentiate eq. (16.17) with respect to \(J_k\). Recall that

\[
\frac{\delta}{\delta J_k} = \frac{\delta \Phi_r}{\delta J_k} \frac{\delta}{\delta \Phi_r} = \frac{\delta^2 Z}{\delta J_k \delta J_r} \frac{\delta}{\delta \Phi_r} = -(X^{-1})_{kr} \delta/\delta \Phi_r. \tag{16.18}
\]

The differential operator \(\delta/\delta \Phi_r\), when applied to the right-hand side of eq. (16.17) can act either on some \(X^{-1}\), or on some \(\delta^m \Gamma/\delta \Phi_i \delta \Phi_j \ldots\). In the former case, we have

\[
\frac{1}{i} \frac{\delta}{\delta J_i} (i X^{-1})_{kl} = (i X^{-1})_{km}(i X^{-1})_{ln}(i X^{-1})_{ij} \frac{1}{i} \frac{\delta^3 \Gamma}{\delta \Phi_m \delta \Phi_n \delta \Phi_j}.
\]

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which amounts to adding a new external line to a newly created three point vertex, and in the latter

\[
\frac{1}{i} \frac{\delta}{\delta J_i} \frac{\delta^{m+1}\Gamma}{\delta \Phi_k \delta \Phi_l \ldots} = (i \, X^{-1})_{ij} \frac{\delta^{m+1}\Gamma}{\delta \Phi_j \delta \Phi_k \delta \Phi_l \ldots}
\]

which amounts to adding a new external line to what used to be an \( m \)-point proper vertex. In any case, when the differential operator of eq. (16.15) is applied to the right-hand side of eq. (16.14), we generate all tree diagrams for the \( (n + 1) \)-point Green’s function, and

\[
\frac{1}{i^n} \frac{\delta^{n+1}Z[J]}{\delta J_i \delta J_l \ldots} = (i \, X^{-1})_{ij} (i \, X^{-1})_{km} \ldots \frac{i}{\delta \Phi_j \delta \Phi_m \ldots} \delta^{n+1}\Gamma[\Phi] + \text{one-particle reducible terms.} \tag{16.19}
\]

Therefore, in the limit \( J = 0, \Phi = v \),

\[
\delta^{n+1}\Gamma[\Phi] / \delta \Phi_i \delta \Phi_m \ldots \bigg|_{\Phi = v} = \Gamma_{lm\ldots}^{(n+1)}
\]

is the \( (n + 1) \)-point proper vertex. Now our proof is complete, since the induction hypothesis is true for \( n = 3 \), as shown in eq. (16.15).

The generating functional of proper vertices \( \Gamma[\Phi] \) has the representation:

\[
\Gamma[\Phi] = \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_{i_1,i_2,i_3,...,i_n}^{(n)} (\Phi - v)_{i_1} (\Phi - v)_{i_2} ... (\Phi - v)_{i_n} \tag{16.20}
\]

with

\[
\Gamma_i^{(2)} = [\Delta_F^{-1}]_{ij} \tag{16.21}
\]

Let us revert to the standard notation:

\[
\Gamma_{i_1,i_2,...,i_n}^{(n)} = \Gamma_{i_1,i_2,...,i_n}^{(n)} (x_1, x_2, ..., x_n).
\]

Because of the translational invariance \( \Gamma^{(n)} \) depends only on \( n - 1 \) differences \( x_i - x_j \), so that its Fourier transform \( \tilde{\Gamma}^{(n)} \) is defined as

\[
\tilde{\Gamma}^{(n)}_{i_1,...,i_n} (p_1, ... p_n) = (2\pi)^n \delta^n(p_1 + ... + p_n) = \left( \prod_{i=1}^{n} \int d^4 x \exp(ip_i x_i) \right) \Gamma^{(n)}_{i_1,...,i_n} (x_1, ..., x_n). \tag{16.22}
\]

This means that four-momentum must be conserved at vertices.

In discussing the implications of the condition (16.9), it is convenient to consider the case in which \( \Phi \) is a constant \( \phi \) independent of space-time. Define the super-potential \( \mathcal{W} \) by

\[
\Gamma[\Phi = 0] = -(2\pi)^4 \delta^4(0) \mathcal{W}(\phi),
\]

\[
\mathcal{W}(\phi) = -\sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_{i_1,i_2,...,i_n}^{(n)} (0, 0, ..., 0) (\phi - v)_{i_1} (\phi - v)_{i_2} ... (\phi - v)_{i_n}, \tag{16.23}
\]
so that

\[
\frac{\delta^N \mathcal{V} (\phi)}{\delta \phi_{i_1} \delta \phi_{i_2} \ldots \delta \phi_{i_n}} \bigg|_{\phi = \nu} = -\tilde{\Gamma}^{(N)}_{i_1, i_2, \ldots, i_n} (0, 0, \ldots, 0)
\]

is the negative of the \(N\)-point proper vertex evaluated at the point where all external momenta vanish. The condition (16.9) translates into

\[
\frac{\delta \mathcal{V} (\phi)}{\delta \phi_i} \bigg|_{\phi = \nu} = 0.
\]  

Furthermore,

\[
\frac{\delta^2 \mathcal{V} (\phi)}{\delta \phi_i \delta \phi_j} = -[\Delta^{-1}_F (0)]_{ij}
\]

is positive semi-definite, since \(\Delta^{-1}_F\) behaves like \((p^2 - m^2)\) near \(p^2 \approx m^2\) and it cannot have any other zero for \(p^2 < m^2\). Thus the vacuum expectation value \(\phi = \nu\) is the value of \(\phi\) which minimizes \(\mathcal{V} (\phi)\). The discussion in section 2 suggests that \(\phi = \nu\) must be the absolute minimum of \(\mathcal{V}\), but we do not prove it here.

When \(\mathcal{L}\) is invariant under

\[
\phi_i \rightarrow \phi_i - i \theta^a L^a \phi_j
\]

it follows from the structure of eq. (16.1) that \(Z[J]\) is invariant under

\[
J_i \rightarrow J_i - i \theta^a L^a \phi_j,
\]

and so on, and finally the superpotential \(\mathcal{V} (\phi)\) is an invariant function of \(\phi\) under the above transformation. The analysis of section 2 on the potential \(\mathcal{V}\) can now be applied verbatim to the superpotential \(\mathcal{V}\), with \(-[\Delta' (0)]_{ij}\) of eq. (16.22) taking the place of \(M^2_{ij}\) of eq. (2.19). We find therefore that the occurrence and the number of the Goldstone bosons discussed there are true to all orders of perturbation theory.

We can construct \(Z[J], \Gamma [\Phi]\) and \(\Psi [\phi]\) in perturbation theory. For simplicity we shall consider the case of a single-component field. An effective way of expanding these quantities in a series is to write eq. (16.1) with a fictitious parameter \(a\):

\[
\exp \{iZ[J]\} = \int \left[ d\phi \right] \exp \left[ i \int d^4 x \left( \frac{1}{a} \mathcal{L}(x) + J(x) \cdot \phi(x) \right) \right]
\]

\[
\approx \exp \left[ i \int d^4 x \frac{1}{a} \mathcal{L} \left[ \frac{1}{i} \delta \mathcal{J}(x) \right] \right] \exp \left[ \frac{i}{2} \int d^4 x d^4 y a J(x) \Delta_F (x - y) J(y) \right],
\]  

and expand \(Z[J]\) in powers of \(a\) and let \(a = 1\) afterwards. Since each propagator is multiplied by \(a\) and each vertex by \(a^{-1}\) when we use eq. (16.23) as the definition of \(Z\), it follows that a Feynman diagram with \(E\) external lines, \(I\) internal lines and \(V\) vertices is multiplied by the factor, \(a^{E+I-V}\). There is a topological relation that holds for any Feynman diagram. It is
\[ L = I - V + 1 \]

where \( L \) is the number of loops (i.e., the number of independent four-momentum integrations) in the diagram. Therefore the expansion in this fictitious parameter \( a \) corresponds to expanding a Green's function in the number of loops in the Feynman diagrams. The reason this expansion is preferable over the expansion in powers of some coupling constant is that in the former any symmetry of the Lagrangian is preserved in each order of perturbation theory since, effectively, \( a \) multiplies the whole Lagrangian. In contrast, if we were to split up the Yang-Mills Lagrangian into a free and perturbing parts to develop a perturbation expansion, for example, each part would not be separately gauge invariant and the consequences of gauge invariance of the Lagrangian might not manifest themselves in each order of perturbation series. (Recall that non-Abelian gauge transformations depend on the coupling constant.)

In the following we shall discuss explicit constructions of \( Z, \Gamma \) and \( \mathcal{O} \) in the first two orders of loop expansion for a simple model:

\[ \mathcal{L} = \frac{1}{2} (\partial^\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4 \quad (16.27) \]

The method can be generalized easily to other models. Our discussion will not show that our construction is in fact the expansion in the number of loops, but the interested student can convince himself of this fact by first referring to Nambu's paper which shows that the loop expansion is also an expansion in the Planck constant \( h \), and then noting that our method is an asymptotic evaluation of these quantities in \( h \).

Imagine that eq. (16.1) is written in the Euclidean space as explained in section 12. Since the exponent in the right-hand side is bounded from above in this case, we are tempted to evaluate the functional integral by the method of steepest descent. We shall keep the Minkowsky notation for simplicity, but the ultimate justification of this method lies in the Euclidicity postulate.

We shall expand the exponent on the right-hand side of eq. (16.1):

\[ S[\phi] + \int d^4 x J(x) \phi(x) = \int d^4 x \{ \mathcal{L}(x) + J(x) \phi(x) \}, \]

about a point \( \phi(x) = \phi_0(x) \):

\[ S[\phi] + \int d^4 x J(x) \phi(x) = S[\phi_0] + \int d^4 x J(x) \phi_0(x) + \int d^4 x \left\{ \frac{\delta S[\phi_0]}{\delta \phi_0(x)} + J(x) \right\} [\phi(x) - \phi_0(x)] + \frac{1}{2!} \int d^4 x d^4 y [\phi(x) - \phi_0(x)] [\phi(y) - \phi_0(y)] \frac{\delta^2 S[\phi_0]}{\delta \phi_0(x) \delta \phi_0(y)} + \ldots \quad (16.28) \]

and choose \( \phi_0 \) so that the term linear in \( \phi - \phi_0 \) is missing from the expansion of eq. (16.28). This will be achieved provided

\[ \frac{\delta S[\phi_0]}{\delta \phi_0(x)} = -J(x) \quad (16.29) \]

which means that \( \phi_0 \) is the solution of the classical (non-quantized) field equation in the presence of the external source \( J(x) \). For the Lagrangian (16.27), eq. (16.29) is

\[ (\partial^2 + \mu^2) \phi_0(x) + \lambda \phi_0^3(x) = J(x) \quad (16.30) \]

In any case, \( \phi_0 \) is obtained from eq. (16.29) as a functional of the external source \( J \).
When eq. (16.28) is substituted in eq. (16.1), we obtain
\[
\exp(i Z[J]) = \exp \{ i S[\phi_0] + i \int d^4x J(x) \phi_0(x) \} \\
\times \int [d\phi] \exp \left\{ i \int d^4x d^4y \left( \frac{1}{2!} \frac{\delta^2 S[\phi_0]}{\delta \phi(x) \delta \phi(y)} (\phi(x) - \phi_0(x))(\phi(y) - \phi_0(y)) + \ldots \right) \right\} \quad (16.31)
\]

The lowest order approximation (which is one order lower than the steepest descent approximation) is obtained if we ignore the functional integral over \(\phi(x)\) altogether and set
\[
Z[J] \approx S[\phi_0] + \int d^4x J(x) \phi_0(x) \equiv Z^0[J] \quad (16.32)
\]
which is a functional of \(J\) only, because \(\phi_0\) is a functional of \(J\). We can evaluate \(Z^0\) explicitly by first solving for \(\phi_0\) in eq. (16.30) and then substituting that \(\phi_0\) in eq. (16.32). Equation (16.32) can be solved in powers of \(\lambda\):
\[
\phi_0(x) = - \int d^4y \Delta_F(x - y; \mu^2) J(y) - \lambda \left[ \int d^4y \Delta_F(x - y; \mu^2) J(y) \right]^3 + \ldots \quad (16.33)
\]
where the use of \(\Delta_F\) is dictated by the Euclidianity postulate. When eq. (16.33) is substituted in eq. (16.32), one finds that \(Z^0[J]\) is the generating functional of Green’s functions in the tree- (i.e., no loop) approximation:
\[
Z^0[J] = - \frac{1}{2} \int d^4x d^4y J(x) \Delta_F(x - y; \mu^2) J(y) + \lambda \left[ \int d^4y \right] \left[ \int d^4x \right] \Delta_F(x - y; \mu^2) + \ldots \quad (16.34)
\]
We can see more readily that \(Z^0\) is the tree approximation to \(Z\) if we compute \(\Gamma[\Phi]\) in this approximation. Since
\[
\frac{\delta Z}{\delta J(x)} \approx \frac{\delta Z^0}{\delta J(x)} = \int d^4y \left( \frac{\delta^2 S[\phi_0]}{\delta \phi_0(y) \delta \phi_0(x)} + J(y) \frac{\delta^2 \phi_0(y)}{\delta \phi_0(x)} \right) + \phi_0(x),
\]
we have, to this order,
\[
\Phi(x) = \phi_0(x). \quad (16.35)
\]
Therefore, \(\Gamma[\Phi]\) can be computed to this order:
\[
\Gamma^0[\Phi] \approx Z^0[J] - \int d^4x J(x) \Phi(x) \\
= \{ S[\Phi] + \int d^4x J(x) \Phi(x) \} - \int d^4x J(x) \Phi(x) = S[\Phi]. \quad (16.36)
\]
So, to this order, proper vertices are generated by the Lagrangian itself and Green’s functions are built up of these unmodified vertices by the rules of tree graphs. The superpotential \(\mathcal{V}[\phi]\) eq. (16.36) is, to this order
\[
\mathcal{V}(\phi) = - S(\phi) = V(\phi)
\]
where \(\phi\) is independent of space-time and \(V\) is the negative of the part of the Lagrangian which is independent of derivative of fields. That is, \(V(\phi)\) is the potential of the field \(\phi\). This justifies the name “super-potential” for \(\mathcal{V}\).
We can proceed further by applying the steepest descent method to the functional integral in eq. (16.31). This consists of neglecting terms higher than quadratic in \((\phi - \phi_0)\) in the exponent of the integrand and performing the functional Gaussian integration. In this way we obtain

\[
\int \! [d\phi] \exp \left\{ \frac{i}{2} \oint d^4 y \frac{1}{2!} \frac{\delta^2 S[\phi_0]}{\delta \phi_0(x) \delta \phi_0(y)} [\phi(x) - \phi_0(x)][\phi(y) - \phi_0(y)] \right\}
\]

\[
\approx \frac{1}{\sqrt{\det \delta^2 S[\phi_0]/\delta \phi_0(x) \delta \phi_0(y)}} = \exp(-\frac{1}{2}) \text{Tr} \ln \left( \frac{\delta^2 S[\phi_0]}{\delta \phi_0(x) \delta \phi_0(y)} \right),
\]

so that

\[
Z[J] \approx Z^0[J] + \frac{1}{2} i \text{Tr} \ln \{\delta^2 S[\phi_0]/\delta \phi_0(x) \delta \phi_0(y)\} \equiv Z^1[J]. \tag{16.37}
\]

For the Lagrangian (16.27), for example,

\[
\delta^2 S/\delta \phi(x) \delta \phi(y) = (-\partial^2 - \mu^2 - 3\lambda \phi^2(x)) \delta^4(x - y),
\]

so that

\[
\frac{i}{2} \text{Tr} \ln \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \approx \frac{i}{2} \text{Tr} \ln \left(1 - 3\lambda \frac{1}{-\partial^2 - \mu^2 + i\epsilon} \phi^2 \right)
\]

\[
= \frac{i}{2} \sum_{n=1}^{\infty} \frac{(-3i\lambda)^n}{n} \oint d^4 x_1 ... d^4 x_n \Delta_F(x_1) \phi^n(x_2) \Delta_F(x_2) \phi^n(x_3) ... \Delta_F(x_n) \phi^n(x_1). \tag{16.38}
\]

Let us now construct \(\Gamma[\Phi]\) to this order:

\[
\Gamma^1[\Phi] \equiv Z^1[J] - \int d^4 x J(x) \Phi(x), \tag{16.39}
\]

where

\[
\Phi(x) = \delta Z^1[J]/\delta J(x) \equiv \phi_0(x) + \epsilon(x) \tag{16.40}
\]

and \(\epsilon(x)\) is given by

\[
\epsilon(x) = \frac{i}{\delta J(x)} \sum_{n=1}^{\infty} \frac{(-3i\lambda)^n}{n} \oint d^4 x_1 ... d^4 x_n \Delta_F(x_1) \phi^n(x_2) \Delta_F(x_2) \phi^n(x_3) ... \Delta_F(x_n) \phi^n(x_1).
\]

Fortunately, it is not necessary to know the form of \(\epsilon(x)\) to construct \(\Gamma[\Phi]\) to first order in \(\epsilon(x)\), as we shall demonstrate presently. First, note that

\[
Z^0[J] = S[\phi_0] + \int d^4 x J(x) \phi_0(x)
\]

\[
= S[\Phi] + \int d^4 x J(x) \Phi(x) - \int d^4 y \left( \frac{\delta S[\phi_0]}{\delta \phi_0(y)} + J \right) \epsilon(y) + O(\epsilon^2)
\]

\[
= S[\Phi] + \int d^4 x J(x) \Phi(x) + O(\epsilon^2) \tag{16.41}
\]
by virtue of eq. (16.29). Therefore to order $\epsilon$, we have from eqs. (16.37), (16.39) and (16.41)

$$\Gamma^I[\Phi] = S[\Phi] + \frac{i}{2} \textrm{Tr} \ln \frac{\delta^2 S[\Phi]}{\delta \Phi(\xi) \delta \Phi(\eta)}. \quad (16.42)$$

The second term is the one-loop correction to the generating functional of proper vertices.

The super-potential $\mathcal{W}$ can be evaluated explicitly from eqs. (16.38) and (16.42). Recalling the definition of $\mathcal{W}$ of eq. (16.23), we find

$$\mathcal{W}(\phi) = \frac{\mu_0^2}{2} \phi^2 + \frac{\lambda_0}{4} \phi^4 + \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \sum N = 1 \frac{1}{N} \left( \frac{-3\lambda \phi^2}{k^2 - \mu^2 + i\epsilon} \right)^N. \quad (16.43)$$

The terms for $N = 1$ and 2 are divergent. However these terms are proportional to $\phi^2$ and $\phi^4$ and the divergences in these terms can be amalgamated with $\mu_0^2$ and $\lambda_0$.

We may write

$$\mathcal{W}(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + J(\phi^2)$$

where

$$J(\phi^2) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \sum N = 1 \frac{1}{N} \left( \frac{-3\lambda \phi^2}{k^2 - \mu^2 + i\epsilon} \right)^N \quad (16.44)$$

and $\mu^2$ and $\lambda$ are defined as the value of the two- and four-point vertices at the point where all external momenta vanish.

Bibliography

The idea of using generating functionals for Green's functions and proper vertices was originated by


The following paper contains the first explicit construction of the generating functional for proper vertices by the Legendre transformation method


This paper also contains the derivation of the Goldstone theorem by this technique. Recent reviews of this method may be found in


The observation that the expansion in the number of loops is equivalent to the expansion in $\hbar$ is due to


The evaluation of the one-loop corrections by the steepest descent approximation is discussed in


For an extensive use of this method in a recent literature, see, for example,

17. Renormalization in the $\sigma$-model

The formalism developed in the preceding section is useful in discussing renormalization of spontaneously broken symmetry models and, in particular, the $\sigma$-model. In the generic sense, the $\sigma$-model is a model in which a symmetry is broken by a term of dimension one, i.e., by a term proportional to a boson field.

A simple example of this kind of models is

$$\mathcal{L} = \frac{1}{2} [ (\partial_\mu \pi)^2 + (\partial_\mu \sigma)^2 ] - \frac{1}{2} \mu_\sigma^2 (\sigma^2 + \pi^2) - \frac{1}{4} \lambda_\sigma (\sigma^2 + \pi^2)^2 + c\sigma = \mathcal{L}_{\text{sym}} + c\sigma$$

(17.1)

which is a two-dimensional generalization of the model discussed in the preceding section. Except for the last term $c\sigma$, the Lagrangian (17.1) is the one studied in section 2, and it was noted there that this Lagrangian is invariant under a U(1) transformation of the fields $\sigma$ and $\pi$. The salient features of this model are that the "almost" conserved current

$$A_\mu = \pi \partial_\mu \sigma - \sigma \partial_\mu \pi$$

(17.2)

has a divergence proportional to the $\pi$-field

$$\partial^\mu A_\mu = c\pi$$

(17.3)

and that the $\sigma$-field acquires a nonvanishing vacuum expectation value thanks to the last term in eq. (17.1). Equation (17.3) is a version of the PCAC condition, and for this reason the model is of some physical interest.

It pays to study first the classical solution of the Lagrangian (17.1). The potential is given by

$$V(\sigma, \pi) = \frac{1}{2} \lambda (\sigma^2 + \pi^2)^2 - \frac{1}{4} \lambda_\sigma (\sigma^2 + \pi^2)^2 - c\sigma$$

(17.4)

(we drop the subscript 0 on $\lambda$ and $\mu_\sigma$ for the moment). The minimum of the potential occurs at $\pi = 0$ and $\sigma = u$ where

$$u(\mu_\sigma^2 + \lambda u^2) = c,$$

(17.5)

$u$ being the vacuum expectation value of the $\sigma$-field in this approximation. If we displace the field $\sigma$ by the amount $u$ and define $s$ by $s = \sigma - u$ eq. (17.1) takes the form

$$\mathcal{L} = \frac{1}{2} [ (\partial_\mu s)^2 + (\partial_\mu \pi)^2 ] - \frac{1}{2} \mu_\sigma^2 s^2 - \frac{1}{2} \mu_\pi^2 \pi^2 - \frac{1}{4} \lambda(s^2 + \pi^2)^2 - \lambda u(s^2 + \pi^2) s$$

(17.6)

so that in this approximation the $s$-field represents a particle of mass $\mu_\sigma^2$:

$$\mu_\sigma^2 = \mu^2 + 3\lambda u^2$$

(17.7)

and the $\pi$-field a particle of mass $\mu_\pi^2$:

$$\mu_\pi^2 = \mu^2 + \lambda u^2.$$  

(17.8)

In this approximation, when $c = 0$, i.e., when the Lagrangian is invariant under the U(1) transformation, either $u = 0$ or $\mu_\pi^2 = \mu^2 + \lambda u^2 = 0$ according to eq. (17.5). If $u^2 = 0$, then $\mu^2 > 0$ in order that $\mu_\sigma^2 = \mu_\pi^2 = \mu^2 > 0$. This is the "usual" way the symmetry of the Lagrangian manifests itself: the particles corresponding to the fields $\sigma$ and $\pi$ are degenerate. On the other hand, if $\mu_\pi^2 = 0$, we
must have \( \mu^2 < 0 \) since \( \lambda \mu^2 > 0 \). The second case is the Goldstone mode of the symmetry with the field \( \pi \) playing the role of the Goldstone boson. In that case, \( \mu^2 = -\lambda \mu^2 \), and \( \mu_0^2 = -2\mu^2 > 0 \). For more thorough discussion of the \( \sigma \)-model, see the monograph "Chiral Dynamics" by one of us.

We return to the discussion of the full solution, including radiative corrections. An important fact about the \( \sigma \)-model is that the Green's function of this model are generated by the generating functional of Green's functions of the symmetric theory. The latter is given by

\[
\exp \{ i Z[J] \} = \int [d\sigma] [d\pi] \exp \{ i \int d^4x [L_{\text{sym}}(x) + J_\sigma(x)\sigma(x) + J_\pi(x)\pi(x)] \},
\]

\( J = (J_\sigma, J_\pi) \). \( (17.9) \)

Now, expand \( Z[J] \) in \( J \) about \( J_\sigma = c \) and \( J_\pi = 0 \). We have

\[
\frac{1}{i^{n+m-1}} \frac{\delta^{n+m+1} Z}{\delta J_\sigma(x_1) \cdots \delta J_\sigma(x_n) \delta J_\pi(y_1) \cdots \delta J_\pi(y_n)} \bigg|_{J_\sigma = c, J_\pi = 0} = \frac{1}{W[c, 0]} \int [d\sigma] [d\pi] s(x_1) \cdots s(x_n) \pi(y_1) \cdots \pi(y_m) \exp \{ i \int d^4x L(x) \} - \text{disconnected pieces},
\]

where \( s = \sigma - u \), \( u \) being the vacuum expectation value of \( \sigma \) so that

\[
\int [d\sigma] [d\pi] s(x) \exp \{ i \int d^4y L(y) \} = 0, \quad s(x) \equiv \sigma(x) - u.
\]

and

\[
W[c, 0] = \int [d\sigma] [d\pi] \exp \{ i \int d^4x [L_{\text{sym}}(x) + c\sigma(x)] \} = \int [d\sigma] [d\pi] \exp \{ i \int d^4x L(x) \}, \quad (17.12)
\]

is the vacuum-to-vacuum amplitude of the \( \sigma \)-model. To recapitulate; if we expand \( [J] \) about \( J = 0 \), the expansion coefficients are the Green's functions of the symmetric model (i.e., the theory given by the Lagrangian \( L_{\text{sym}} \)); if we expand \( Z[J] \) about \( J = (c, 0) \), they are the Green's functions of the \( \sigma \)-model.

The point is simply that the symmetry-breaking term \( c\sigma \) has the form of an external source term \( J\sigma \) for constant \( J = c \). This important theorem has an analog in terms of \( \Gamma \). Since

\[
\frac{\delta \Gamma[\Phi]}{\delta \Phi_i(x)} = -J_i(x), \quad \Gamma[\Phi] = Z[J] - \int d^4x J(x) \cdot \Phi(x)
\]

\( (17.13) \)

where

\[
\frac{\delta Z[J]}{\delta J_i(x)} = \Phi_i(x)
\]

we have from eq. (17.13)

\[
\frac{\delta \Gamma[\Phi]}{\delta \Phi_i(x)} \bigg|_{\Phi = u} = -c_i
\]

(17.15)

which is the analog of eq. (16.9). Eq. (17.5) is the lowest order version of (17.15). Furthermore, we can repeat the analysis leading to eq. (16.17), but this time taking the limit \( J = c \) and \( \Phi = u \), to find that

\[
\frac{\delta^n \Gamma[\Phi]}{\delta \Phi_{i_1} \delta \Phi_{i_2} \cdots \delta \Phi_{i_n}} \bigg|_{\Phi = u} \equiv \Gamma^{(n)}_{i_1, i_2, \ldots, i_n}(u)
\]

\( (17.16) \)
is the proper n-point vertex of the σ-model. (In eq. (17.16) we have reverted to the convention of representing the internal symmetry index and the space-time variable x collectively by an index i.) To recapitulate, the generating functional of proper vertices of the symmetric theory generates the Green's functions of the σ-model when it is expanded about \( \Phi = u(c) \), where \( u(c) \) is given by eq. (17.15). As was shown in the preceding section, \( \Gamma[\Phi] \sim S[\Phi] \) to lowest order, so that eq. (17.5) follows from eq. (17.15).

Let us now consider the limit \( c \to 0 \) of eq. (17.15). Equation (17.15) is really an equation which determines the vacuum expectation value \( u \) in terms of \( c \). To study the ramifications of eq. (17.15) it suffices to consider the superpotential defined in eq. (16.20):

\[
\Gamma[\Phi = \phi] = -(2\pi)^4 \delta^4(0) \mathcal{V}(\phi)
\]

where \( \phi \) is independent of space-time. Eq. (16.15) is equivalent to

\[
\delta \mathcal{V}(\phi)/\delta \phi \bigg|_{\phi = u} = c_i \ldots
\]

The limit of \( u(c) \) as \( c \to 0 \) may or may not vanish, depending on the parameters of the symmetric Lagrangian. If it does not, i.e., \( u(0) = v \neq 0 \), the symmetry of the Lagrangian is spontaneously broken.

Let us consider, however, the case in which the parameters of the symmetric Lagrangian are such that \( u(0) = 0 \), that is, the case in which the symmetry is manifested in the usual way. From eq. (17.16) it follows that

\[
\Gamma[\Phi] = \sum_{n=2} \frac{1}{n!} (\Phi - u)_{i_1} (\Phi - u)_{i_2} \ldots (\Phi - u)_{i_n} \Gamma^{(n)}_{i_1, i_2 \ldots i_n} (u). \tag{17.18}
\]

Further, the analog of the relation

\[
\left( \frac{d}{dx} \right)^n f(x) \bigg|_{x=a} = \sum_{m=0} a^m \left( \frac{d}{dx} \right)^{n+m} f(x) \bigg|_{x=0}
\]

gives

\[
\Gamma^{(n)}_{i_1, i_2 \ldots i_n} (u) = \sum_{m=0} \frac{1}{m!} \frac{1}{u_{j_1} u_{j_2} \ldots u_{j_m}} \Gamma^{(n+m)}_{i_1, i_2 \ldots i_n, j_1, j_2 \ldots j_m} (u = 0)
\]

or, in momentum space,

\[
\tilde{\Gamma}^{(n)}_{i_1, i_2 \ldots i_n} (p_1, p_2, \ldots, p_n; u) = \sum_{m=0} \frac{1}{m!} u^m \tilde{\Gamma}^{(n+m)}_{i_1, i_2 \ldots i_n, 1, 1, \ldots, 1} (p_1, p_2, \ldots p_m, 0, 0, \ldots, 0). \tag{17.19}
\]

In eq. (17.19), the indices \( i \)'s and \( j \)'s stand for \( \sigma \) or \( \pi \) and \( \tilde{\Gamma}_{ij \ldots} (p, q, \ldots) \equiv \tilde{\Gamma}_{ij \ldots} (p, q, \ldots; u = 0) \) is the momentum space proper vertex of the symmetric theory \( (c = 0, u = v = 0) \).

Equation (17.19) is important in that it affords us a handle for removing the divergences from the σ-model if we know how to renormalize the symmetric model, since eq. (17.19) expresses the proper vertex of the σ-model in terms of proper vertices of the symmetric theory. We shall give a brief review of the renormalization theory in the next section, but suffice it to say for the moment that if we write the Lagrangian of the symmetric theory as
\[ \mathcal{L} = \frac{1}{2} [(\partial \sigma)^2 + (\partial \pi)^2 - \mu^2(\sigma^2 + \pi^2)] - \frac{1}{4} \lambda (\sigma^2 + \pi^2)^2 \]

\[ + \frac{1}{2} (Z_3 - 1) [(\partial \sigma)^2 + (\partial \pi)^2 - \mu^2(\sigma^2 + \pi^2)] - \frac{1}{2} \delta \mu^2(\sigma^2 + \pi^2) - \frac{1}{4} \delta \lambda (\sigma^2 + \pi^2)^2 \]  \tag{17.20}

where \( \mu^2 \) and \( \lambda \) are finite constants, and choose \( Z_3, \delta \mu^2, \delta \lambda \) in an appropriate way, then all infinities of the theory can be removed. Thus starting from the Lagrangian it is possible to construct a \emph{finite} generating functional \( \Gamma[\Phi] \) for \( u = 0 \). Once we have a renormalized (i.e., finite) expression for \( \Gamma[\Phi] \), we can expand it about \( \Phi = u \), where \( u \) is determined from eq. (17.15), to recover the proper vertices of the \( \sigma \)-model characterized by the parameters \( \lambda, \mu^2 \) and \( c \).

Finally, we turn to the Ward-Takahashi identities of the model. Since \( \mathcal{V}(\Phi) \) is the generating function of zero-momentum proper vertices of the symmetric theory when we expand it about \( \Phi = 0 \), it follows that \( \mathcal{V} \) is a function of the invariant of \( \Phi \), i.e., of \( \Phi^2 = \phi_o^2 + \phi_\pi^2 \). Thus eq. (17.17) takes the form

\[ 2\phi_o \delta \mathcal{V}(\Phi)/\delta(\Phi^2) \bigg|_{\phi_o = u, \phi_\pi = 0} = c. \] \tag{17.21}

Since the inverse \( \pi \)-propagator at zero momentum is given by [see (16.25)]

\[ -\Delta_\pi^{-1}(0) = \delta^2 \mathcal{V}(\Phi)/\delta \phi_\pi \delta \phi_o \bigg|_{\phi_\pi = 0, \phi_o = u} = 2\delta \mathcal{V}(\Phi)/\delta(\Phi^2) \bigg|_{\Phi = u} \] \tag{17.22}

it follows that

\[ -u \Delta_\pi^{-1}(0) = c \] \tag{17.23}

from which the value of \( u \) can be determined conveniently, if we know \( \Delta_\pi^{-1}(0) \) in terms of \( \lambda, \mu^2 \) and \( u \).

The above prescription for constructing renormalized proper vertices of the \( \sigma \)-model works if \( \mu^2 > 0 \), since in that case there is a comparison symmetric theory that makes sense. However, once \( \Gamma[\Phi] \) is constructed in terms of \( \lambda, \mu^2 \) and \( c \) there is nothing that stops us from expressing \( \Gamma[\Phi] \) in terms of \( \lambda, u \) and \( m_\pi^2 \), where the last is defined as

\[ m_\pi^2 = -\Delta_\pi^{-1}(0) = 2\delta \mathcal{V}(\Phi)/\delta(\Phi^2) \bigg|_{\Phi = u} \]

and taking the limit \( m_\pi^2 \to 0 \). Then eq. (17.23) reads

\[ um_\pi^2 = c. \] \tag{17.24}

Equation (17.24) is the renormalized Goldstone theorem: if \( c = 0 \) either \( u = 0 \), or \( m_\pi^2 = 0 \). The latter corresponds to the Goldstone mode. In this case the basic parameters of the theory can be taken to be \( \lambda \) and \( u = v \), instead of \( \lambda \) and \( -\mu^2 \).

The moral of the above discussion is that the renormalizability of the \( \sigma \)-model in the Goldstone mode depends only on the renormalizability of the symmetric theory. The process of renormalization does not induce additional symmetry breaking, in the sense that the symmetric counter-terms exhibited in (17.20) suffice to remove infinities from the theory whether or not the symmetry is broken externally (\( c \neq 0 \)) or internally (\( v \neq 0 \)).

Later we will discuss a way of renormalizing the \( \sigma \)-model without making explicit reference to the symmetric theory. This method makes use of the Ward-Takahashi identities. Let us derive them. The generating functional \( Z[J] \) in eq. (17.9) is invariant under the \( U(1) \) transformation of
the external sources:

\[
\left( \begin{array}{c} J_o \\ J_{\pi} \end{array} \right)' = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} J_o \\ J_{\pi} \end{array} \right)
\]

as can be seen by making the change of integration variables

\[
\left( \begin{array}{c} \alpha' \\ \pi' \end{array} \right) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \alpha \\ \pi \end{array} \right)
\]

which leaves the scalar product \( J_o \alpha + J_{\pi} \pi \) invariant. Therefore,

\[
dZ/d\theta = 0,
\]

or

\[
\int d^4x \left( \frac{\delta Z[J]}{\delta J_o(x)} J_{\pi}(x) - \frac{\delta Z[J]}{\delta J_{\pi}(x)} J_o(x) \right) = 0.
\]

Substituting eqs. (17.13) and (17.14) into eq. (17.27), we find that

\[
\int d^4x \left( \Phi_o(x) \frac{\delta \Gamma[\Phi]}{\delta \Phi_o(x)} - \Phi_{\pi}(x) \frac{\delta \Gamma[\Phi]}{\delta \Phi_{\pi}(x)} \right) = 0
\]

which shows that \( \Gamma \) is an invariant functional of \( \Phi \) under the U(1) transformation:

\[
\left( \begin{array}{c} \Phi_o' \\ \Phi_{\pi}' \end{array} \right) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \Phi_o \\ \Phi_{\pi} \end{array} \right).
\]

Note that the invariance of \( \Gamma \) under the transformation (17.29) is true whether \( \mu^2 > 0 \) or \( \mu^2 < 0 \).

The renormalized \( \Gamma \) constructed according to the prescription above, thus satisfies eq. (17.28) as we continue \( m_o^2 \) to zero.

Equation (17.28) is the Ward-Takahashi identity for the generating functional of proper vertices. An infinite number of Ward-Takahashi identities is obtained if we differentiate eq. (17.28) with respect to \( \Phi_{\pi} \) and \( \Phi_o \) repeatedly, and set \( \Phi_{\pi} = 0, \Phi_o = u \). If we differentiate eq. (17.28) with respect to \( \Phi_{\pi} \) and set \( \Phi_{\pi} = 0, \Phi_o = u \), we obtain the "eigenvalue" equation for \( u \), eq. (17.23). If we differentiate it with respect to \( \Phi_{\pi} \) and \( \Phi_o \) and take the limit, we obtain

\[
\Delta_o^{-1}(p^2) - \Delta_{\pi}^{-1}(p^2) = u \Gamma_{o\pi\pi}(p; 0, -p).
\]

An important lesson to be learned here is that the Ward-Takahashi identity for the generating functional for proper vertices is the same, whether or not the symmetry is spontaneously broken. It is satisfied by the generating functional constructed first in the symmetric theory and then continued to the Goldstone mode by varying an appropriate parameter of the theory.

Bibliography

This section is based on

18. BPHZ renormalization

In this section we will give a brief survey of renormalization theory developed and perfected in recent years by Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ). Nothing will be proved, but we will try to give definitions and theorems in a precise manner.

First, we will give some definitions. The interaction Lagrangian is a sum of terms \( \mathcal{L}_i \) which is a product of \( b_i \) boson fields and \( f_i \) fermion fields with \( d_i \) derivatives. The vertex of the \( i \)th type arising from \( \mathcal{L}_i \) has the index \( \delta_i \) defined as

\[
\delta_i = b_i + \frac{3}{2} f_i + d_i - 4 = \dim \mathcal{L}_i - 4. \tag{18.1}
\]

Let \( \Gamma \) be a one-particle irreducible (IPI) diagram (i.e., a diagram that cannot be made disconnected by cutting only one line). Let \( E_B \) and \( E_F \) be the numbers of external boson and fermion lines, \( I_B \) and \( I_F \) the numbers of internal boson and fermion lines, \( n_i \) the number of vertices of the \( i \)th type. Then

\[
E_B + 2I_B = \sum_i n_ib_i \tag{18.2}
\]

\[
E_F + 2I_F = \sum_i n_if_i. \tag{18.3}
\]

The *superficial degree of divergence* of \( \Gamma \) is the degree of divergence one would naively guess by counting the powers of momenta in the numerator and denominator of the Feynman integral. It is

\[
D(\Gamma) = \sum n_i d_i + 2I_B + 3I_F - 4V + 4 \tag{18.4}
\]

the last two terms arising from the fact that at each vertex there is a four dimensional delta function which allows one to express one four-momentum in terms of other momenta, except that one delta function expresses the conservation of external momenta. Making use of eqs. (18.1), (18.2), and (18.3) we can write eq. (18.4) as

\[
D = \sum n_i \delta_i - E_B - \frac{3}{2} E_F + 4, \tag{18.5}
\]

or,

\[
D + E_B + \frac{3}{2} E_F - 4 = \sum n_i \delta_i. \tag{18.6}
\]

The purpose of renormalization theory is to give a definition of the *finite part* of the Feynman integral corresponding to \( \Gamma \):

\[
F_\Gamma = \lim_{\epsilon \to 0} \int d\mathbf{k}_1 \ldots d\mathbf{k}_L I_\Gamma. \tag{18.7}
\]
where $I_\Gamma$ is a product of propagators $\Delta_F$ and vertices $P$:

$$I_\Gamma = \prod_{a,b,\sigma} \Delta_{F,a}^{a\sigma} \prod_a P_a. \quad (18.8)$$

The finite part of $F_\Gamma$ will be denoted by $J_\Gamma$ and written

$$J_\Gamma = \lim_{\epsilon \to 0^+} \int dk_1...dk_L R_\Gamma. \quad (18.9)$$

We shall describe Bogoliubov's prescription of constructing $R_\Gamma$ from $I_\Gamma$.

Let us first consider a simple case, in which $\Gamma$ is primitively divergent. The diagram $\Gamma$ is primitively divergent if it is proper (i.e., $|\Pi|$), superficially divergent (i.e., $D(F) \gg 0$) and becomes convergent if any line is broken up. In this case, we may use the original prescription of Dyson. We write

$$J_\Gamma = \int dk_1...dk_L (1 - t^\Gamma) I_\Gamma,$$

i.e.,

$$R_\Gamma = (1 - t^\Gamma) I_\Gamma.$$  

The operation $t^\Gamma$ must be defined to cancel the infinity in $J_\Gamma$. $I_\Gamma$ is a function of $E_F + E_B - 1 = E - 1$ external momenta $p_1, ... p_{E-1}$:

$$I_\Gamma = f(p_1, ... p_{E-1}).$$

The operation $(1 - t^\Gamma)$ on $f$ is defined by subtracting from $f$ the first $D(\Gamma) + 1$ terms in a Taylor expansion about $p_i = 0$:

$$t^\Gamma f(p_1, ... p_{E-1}) = f(0, ..., 0) + ... + \frac{1}{d!} \sum_{i_1, ..., i_d=1}^{E-1} (p_{i_1})_\lambda (p_{i_2})_\mu ... (p_{i_d})_\nu \frac{\partial^d f}{(\partial p_{i_1})_\lambda (\partial p_{i_2})_\mu ... (\partial p_{i_d})_\nu} \quad (18.10)$$

where $d = D(\Gamma)$. The operation $(1 - t^\Gamma)$ amounts to making subtractions in the integrand $I_\Gamma$, the number of subtractions being determined by the superficial degree of divergence of the integral.

Some more definitions: A renormalization part is a proper diagram which is superficially divergent ($D \gg 0$). Two diagrams (subdiagrams) are disjoint, $\gamma_1 \cap \gamma_2 \neq \emptyset$ if they have no lines or vertices in common. Let $\{\gamma_1, ..., \gamma_c\}$ be a set of mutually disjoint connected subdiagrams of $\Gamma$. Then

$$F \equiv \Gamma/\{\gamma_1, ..., \gamma_c\}$$

is defined by contracting each $\gamma$ to a point and assigning the value 1 to the corresponding vertex.

We are now in a position to describe Bogoliubov's $R$ operation:

1. If $\Gamma$ is not a renormalization part (i.e., $D(\Gamma) \leq -1$),

$$R_\Gamma = \bar{R}_\Gamma; \quad (18.11)$$

2. If $\Gamma$ is a renormalization part ($D(\gamma) \gg 0$),

$$R_\Gamma = (1 - t^\Gamma) \bar{R}_\Gamma. \quad (18.12)$$

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where $\tilde{R}_\Gamma$ is defined as

$$\tilde{R}_\Gamma = I_\Gamma + \sum_{\{\gamma_1, \ldots, \gamma_c\}} I_{\Gamma/\{\gamma_1, \ldots, \gamma_c\}} \prod_{\tau=1}^c O_\gamma$$

(18.13)

and $O_\gamma = -t^\gamma \tilde{R}_\gamma$, where the sum is over all possible different sets of $\{\gamma_i\}$. This definition of $\tilde{R}_\Gamma$ in terms of $\tilde{R}_\gamma$ appears to be recursive; in perturbation theory there is no problem; the $\tilde{R}_\gamma$ appearing in the definition of $\tilde{R}_\Gamma$ is necessarily of lower order.

It is possible to “solve” eq. (18.13). We refer the interested reader to Zimmermann’s lectures and merely present the result. Again we need some more definitions before we can do this. Two diagrams $\gamma_1$ and $\gamma_2$ said to overlap, $\gamma_1 \cap \gamma_2$, if none of the following holds:

$$\gamma_1 \cap \gamma_2 = \emptyset, \quad \gamma_1 \supset \gamma_2, \quad \gamma_2 \supset \gamma_1.$$

A $\Gamma$-forest $U$ is a hierarchy of subdiagrams satisfying (a)—(c) below: (a) elements of $U$ are renormalization parts; (b) any two elements of $U$, $\gamma'$ and $\gamma''$ are nonoverlapping; (c) $U$ may be empty. A $\Gamma$-forest $U$ is full or normal respectively depending on whether $U$ contains $\Gamma$ itself or not. The theorem due to Zimmermann is

$$R_\Gamma = \sum_{\text{all } U} \prod_{\lambda \in U} (-t^\lambda)I_\Gamma$$

(18.14)

where $\Sigma$ extends over all possible (full, normal and empty) $\Gamma$-forests, and in the product $\Pi(-t^\lambda)$ the factors are ordered such that $t^\lambda$ stands to the left of $t^\sigma$ if $\lambda \supset \sigma$. If $\lambda \cap \sigma = \emptyset$, the order is irrelevant. A simple example is in order. Consider the diagram in fig. 18.1. The forests are $\emptyset$ (empty); $\gamma_1$ (full); $\gamma_2$ (normal); $\gamma_1, \gamma_2$ (full). Equation (18.14) can be written in this case as

$$R_\Gamma = (1 - t^{\gamma_1} - t^{\gamma_2} + t^{\gamma_1 \gamma_2})I_\Gamma = (1 - t^{\gamma_1})(1 - t^{\gamma_2})I_\Gamma.$$

Note that in the BPH program, the $R$-operation is performed with respect to subdiagrams which consist of vertices and all propagators in $\Gamma$ which connect these vertices. By the BPH definition, the subdiagram $\gamma_2$ above does not contain renormalization parts other than itself and in this sense the present treatment differs from Salam’s discussion.

In formulating the BPH theorem it is necessary first to regularize the propagators in eq. (18.9) by some device such as by

$$\Delta_F(p) \to \Delta_F(p; r, \epsilon) = -i \int_0^\infty \! d\alpha \exp\{i\alpha(p^2 - m^2 + i\epsilon)\}$$

Fig. 18.1. Example of the BPHZ definition of subdiagrams in a particular contribution to the four-point function in a $\lambda\phi^4$ coupling theory.
and define $I_\Gamma (r, e)$ as in eq. (18.9) in terms of $\Delta_\Gamma (r, e)$, and then construct $R_\Gamma (r, e)$ by the $R$-operation. The BPH theorem states that $R_\Gamma$ exists as $r \to 0$ and $e \to 0^+$, as a boundary value of an analytic function in the external momenta. Another theorem, the proof of which can be found in the book by Bogoliubov and Shirkov, section 26, and which is combinatoric in nature, states that the subtractions implied by the ($1 - \tilde{t}$) prescription in the $R$-operation can be formally implemented by adding counterterms in the Lagrangian.

A theory which has a finite number of renormalization parts is called renormalizable. A theory in which all $\delta_j$ are less than, or equal to zero is renormalizable. In this case the index of a subtraction term in the $R$-operation is bounded by $D + E_B + \frac{1}{2} E_F - 4$ which is at most equal to zero by eq. (18.5). In such a theory, only a finite number of renormalization counterterms to the Lagrangian suffice to implement the $R$-operation. In the $\sigma$ model we considered in the proceeding section, all two-, three- and four-point proper vertices are superficially divergent. The two-point vertices (self-energy parts) are quadratically divergent so the $R$ operation makes two subtractions in $p^2$ from the Feynman integrals. The other vertices are only logarithmically divergent.

The BHPZ renormalization can be combined with the Ward-Takahashi identities discussed in the proceeding section to produce a systematic scheme for renormalizing the $\sigma$-model without explicit reference to the symmetric theory. This was first worked out by Symanzik. Construction of a renormalized perturbation series according to the BPHZ prescription requires prescribing values of renormalization parts at subtraction points. Suppose these values are determined up to the $(n - 1)$ loop approximation, in such a way as to satisfy the Ward-Takahashi identities, and we are to construct proper vertices up to the $n$-loop approximation. Suppose further that we have a regularization scheme so that the Ward-Takahashi identities hold for regularized proper vertices. For example, we have, from eq. (17.30),

$$\Delta^{-1}_\sigma (p^2; r) - \Delta^{-1}_\pi (p^2; r) = u \tilde{\Gamma}_{\sigma \pi \sigma} (p; 0, -p; r)$$  
\hspace{1cm} (18.15)

where $r$ is a cutoff parameter which should be set equal to zero at the end. We apply the $R$ operation to relevant vertices and write

$$\Delta^{-1}_\sigma (p^2; r) = p^2 Z - m^2 + (1 - \tilde{t}) \{ \Delta^{-1}_\sigma (p^2; r) \},$$
$$\Delta^{-1}_\pi (p^2; r) = p^2 - m^2 + (1 - \tilde{t}) \{ \Delta^{-1}_\pi (p^2; r) \},$$
$$\tilde{\Gamma}_{\sigma \pi \sigma} (p; g, k; r) = -2 \lambda u + (1 - \tilde{t} \beta) \{ \tilde{\Gamma}_{\sigma \pi \sigma} (p; g, k; r) \},$$  
\hspace{1cm} (18.16)

where $D(\Gamma_1) = 2, D(\Gamma_2) = 2, D(\Gamma_3) = 0$, and the symbol $\{ \}$ signifies the quantity constructed by the $R$ operation as in (18.13), wherein the vertices $P_q \in \text{eq. (18.8)}$ take the values of the corresponding renormalization parts at subtraction points as determined up to the $(n - 1)$ loop approximation. In (18.16), the degrees of the subtraction polynomials are determined by the superficial degrees of divergence of the proper vertices in question. We have chosen the coefficient of $p^2$ in $\Delta^{-1}_\pi$ equal to one by convention, i.e., by renormalizing the $\pi$ and $\sigma$ fields appropriately. Likewise we have chosen the value of $\tilde{\Gamma}_{\sigma \pi \sigma} (0; 0, 0; r)$ to be $-2 \lambda u$ by convention. Now substituting the expressions in (18.16) into (18.15) and identifying terms proportional to $(p^2)^0$ and $(p^2)$, we obtain

$$Z = 1 + u \left( \frac{d}{dp^2} \tilde{\Gamma}_{\sigma \pi \sigma} (p; 0, -p; r) \right)_{p^2 = 0},$$  
\hspace{1cm} (18.17)
\[ m_o^2 = m_{\pi}^2 + 2\lambda u^2. \]

The BPH theorem then asserts that the quantities appearing in eq. (18.16) together with \( Z \) defined in (18.17) are cutoff independent, i.e., well-defined in the limit \( r \to 0 \); furthermore this procedure determines the values \( Z, m_{\pi}^2 \) of the renormalization part \( \Delta_0^{-1} \) up to the \( n \) loop approximation. In fact, by a systematic exploitation of the Ward-Takahashi identities, it is possible, as Symanzik first showed, to determine the values of three- and four-point renormalization parts at subtraction points completely in terms of \( m^2, -2\lambda u \) and \( \Gamma_{\pi \pi \pi \pi}(0, 0, 0, 0) \equiv -6\lambda \).

The inductive procedure described above becomes complete when we realize that in the tree (zero loop) approximation the values of renormalization parts at subtraction points are those read off the Lagrangian (they, of course, satisfy the Ward-Takahashi identities). Thus the values of renormalization parts at subtraction points have the expansion

\[
Z = 1 + z_1 \lambda + z_2 \lambda^2 + \ldots, \\
\tilde{\Gamma}_{\sigma \sigma \sigma}(0, 0, 0) = -6\lambda u [1 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \ldots], \\
\tilde{\Gamma}_{\sigma \sigma \sigma}(0, 0, 0, 0) = -6\lambda [1 + \beta_1 \lambda + \beta_2 \lambda^2 + \ldots], \\
\tilde{\Gamma}_{\sigma \sigma \pi \pi}(0, 0, 0, 0) = -2\lambda [1 + \gamma_1 \lambda + \gamma_2 \lambda^2 + \ldots].
\]

The symmetry breaking parameter \( c \) is given by eq. (17.23), or (17.24).

This discussion makes sense only if there is a regularization scheme which preserves the Ward-Takahashi identities, and this leads us to the subject of the next lecture.

The Symanzik procedure outlined above is equivalent to the renormalization procedure discussed in section 17. This statement is clearly true in the tree approximation. Let us recall that the Lagrangian is first written in terms of bare quantities as

\[
\mathcal{L} = \frac{1}{2}[(\partial_{\mu}\sigma_0)^2 + (\partial_{\mu}\pi_0)^2] - \frac{1}{2}\mu_0^2(\sigma_0^2 + \pi_0^2) - \frac{1}{4}\lambda_0(\sigma_0^2 + \pi_0^2)^2 + c_0\sigma_0. \tag{18.18}
\]

After making the renormalization transformations

\[
\sigma_0 = Z_3^{1/2}(u + s), \quad \pi_0 = Z_3^{1/2}\pi, \quad c_0 = Z_3^{-1/2}c,
\]

\[
\lambda_0 = (\lambda + \delta\lambda)Z_3^{-2}, \quad \mu_0^2 = Z_3[m_{\pi}^2 + \delta m_{\pi}^2 - u(\lambda + \delta\lambda)] \tag{18.19}
\]

we can write the Lagrangian as

\[
\mathcal{L} = \frac{1}{2}(\partial_{\mu}s)^2 - \frac{1}{2}(m_{\pi}^2 + 2\lambda u^2)s^2 + \frac{1}{2}(\partial_{\mu}\pi)^2 - \frac{1}{2}m_{\pi}^2\pi^2 - \lambda us(s^2 + \pi^2) - \frac{1}{4}\lambda(s^2 + \pi^2)^2 + \mathcal{L}_c, \tag{18.20}
\]

where \( \mathcal{L}_c \) is the sum of the renormalization counterterms:

\[
\mathcal{L}_c = \frac{1}{2}(Z_3 - 1)[(\partial_{\mu}s)^2 + (\partial_{\mu}\pi)^2] - \frac{1}{2}\delta m_{\pi}^2\pi^2 - \frac{1}{2}(\delta m_{\pi}^2 + 2u^2\delta\lambda)s^2 - u\delta\lambda s(s^2 + \pi^2) - \frac{1}{4}\delta\lambda(s^2 + \pi^2)^2 + [c - u(m_{\pi}^2 + \delta m_{\pi}^2)]s. \tag{18.21}
\]

Now, suppose that the Symanzik procedure is equivalent to the subtractions of infinities by the above counterterms up to the \((n - 1)\) loop approximation. We then have, in the \( n \) loop approximation,
\[ \Delta_{\pi}^{-1}(p^{2}; r) = [1 + A(r) + (Z_{3} - 1)]p^{2} - (m_{\pi}^{2} + B(r) + \delta m_{\pi}^{2}) + (1 - t^{\Gamma_{1}})[\Delta_{\pi}^{-1}(p^{2}; r)], \]
\[ \Delta_{\gamma}^{-1}(p^{2}; r) = [1 + C(r) + (Z_{3} - 1)]p^{2} - (m_{\pi}^{2} + 2\mu u + D(r) + \delta m_{\pi}^{2} + 2u^{2}\delta \lambda) + (1 - t^{\Gamma_{2}})[\Delta_{\gamma}^{-1}(p^{2}; r)], \]
\[ \widetilde{\Gamma}_{\sigma \pi \pi} = -2\mu u[1 + E(r)] - 2u\delta \lambda + (1 - t^{\Gamma_{3}})[\widetilde{\Gamma}_{\sigma \pi \pi}] \]

where \( A(r), \ldots E(r) \) are infinite (i.e., \( r \)-dependent) quantities. We choose \( Z_{3}, \delta m_{\pi}^{2} \) and \( \delta \lambda \) such that
\[ Z_{3} = 1 - A(r), \quad \delta m_{\pi}^{2} = -B(r), \quad \delta \lambda = \lambda E(r). \]

Then the Ward-Takahashi identity (18.16) tells us that
\[ C(r) - A(r) = u \left( \frac{\partial}{\partial p^{2}} [\widetilde{\Gamma}_{\sigma \pi \pi}(p; 0, -p; r)] \right)_{p^{2} = 0} \]
which is convergent as \( r \to 0 \), and
\[ D(r) + \delta m_{\pi}^{2} + 2u^{2}\delta \lambda = 0. \]

The combination of (18.22), (18.24) and (18.25) is clearly equivalent to eqs. (18.16) and (18.17).

Bibliography

For renormalization theory see:
2. A. Salam, Phys. Rev. 82 (1951) 217; 84 (1951) 426.
   Chapter IV, and references cited in p. 330 thereof.

For the renormalization of the \( \sigma \)-model discussed here, refer to Symanzik's papers cited in the preceding section.

19. The regularization scheme of 't Hooft and Veltman

Recently, 't Hooft and Veltman proposed a scheme for regularizing Feynman integrals which preserves various symmetries of the underlying Lagrangian. This method is applicable to the \( \sigma \)-model, electrodynamics, and non-Abelian gauge theories, and depends on the idea of analytic continuation of Feynman integrals in the number of space-time dimensions. The critical observations here are that the global or local symmetries of these theories are independent of space-time dimensions, and that Feynman integrals are convergent for sufficiently small, or complex \( N \), where \( N \) is the "complex dimension" of space-time.

Let us first review the nature of ultraviolet divergence of a Feynman diagram. For this purpose, it is convenient to parametrize the propagators as
\[ \Delta_F(p^2) = \frac{1}{i} \int_0^\infty d\alpha \exp \{ i\alpha(p^2 - m^2 + i\epsilon) \}. \] (19.1)

Making use of this representation, we can write a typical Feynman integral as

\[ F_{\Gamma} \sim \left( \prod_{i=1}^l \int_0^\infty d\alpha_i \right) \left( \prod_{j=1}^m \int d^4k_j \right) (k_{i_1})_{\lambda}(k_{i_2})_{\mu}... (k_{i_n})_{\nu} \times \exp \{ i \sum_i \alpha_i(q_i^2 - m_i^2 + i\epsilon) \} \] (19.2)

where \( I \) is the number of internal propagators in \( \Gamma \), \( L \) the number of loops, and \( l_1, ..., l_n \) may take any values from 1 to \( L \). The momentum \( q_j \) carried by the \( j \)th propagator is a linear function of loop momenta \( k_j \) and external momenta \( p_m \). The exponent on the right-hand side of eq. (19.2) can therefore be written as

\[ \sum_{i=1}^I \alpha_i(q_i^2 - m_i^2 + i\epsilon) = \frac{1}{2} \sum_{i,j} k_i A_{ij}(\alpha) k_j + \sum_{i,m} k_i B_{im}(\alpha) p_m - \sum_i \alpha_i(m_i^2 - i\epsilon) \]

\[ \equiv \frac{1}{2} k^T \cdot A \cdot k + k \cdot B \cdot p - \sum_i \alpha_i(m_i^2 - i\epsilon) \]

where \( k \) is a column matrix with entries which are four-vectors. The matrices \( A \) and \( B \) are homogeneous functions of first degree in \( \alpha \)'s, and \( A \) is symmetric. Upon translating the integration variables

\[ k \rightarrow k' = k + A^{-1}Bp \]

and diagonalizing the matrix \( A \) by an orthogonal transformation on \( k' \), we can perform the loop integrations over \( k_j \) in eq. (19.2). The result is a sum over terms each of which has the form

\[ F_{\Gamma} \sim T_{\lambda \mu... \nu} \left( \prod_{i=1}^l \int_0^\infty d\alpha_i \right) \frac{1}{\Pi_i[A_i(\alpha)]^{s_i}} \exp \left[ -i \left\{ \frac{1}{2} p \cdot C(\alpha) \cdot p + \sum_i \alpha_i(m_i^2 - i\epsilon) \right\} \right] \] (19.3)

where \( T_{\lambda \mu... \nu} \) is a tensor typically a product of \( g_{\rho \sigma} \)'s, \( A_i(\alpha) \) is the \( i \)th eigenvalue of the matrix \( A \), and \( s_i \) is a positive number which is determined by the tensorial structure of \( F_{\Gamma} \). Note that \( A_i(\alpha) \) is homogeneous of first degree in \( \alpha \)'s. The matrix \( C \) is

\[ C = B^T A^{-1} B, \]

and is also a homogeneous function of first degree in \( \alpha \)'s. In this parametrization, the ultraviolet divergences of the integral appear as the singularities of the integrand on the right-hand side of eq. (19.3) arising from the vanishing of some factors \( \Pi_i[A_i(\alpha)]^{s_i} \) as some or all \( \alpha \)'s approach to zero in certain orders, for example,

\[ \alpha_{r_1} < \alpha_{r_2} < ... < \alpha_{r_I} \]

where \( (r_1, r_2, ..., r_I) \) is a permutation of \( (1, 2, ..., I) \). See, for instance, a more detailed and careful discussion of Hepp.
The 't Hooft-Veltman regularization consists in defining the integral $F_r$ in $n$ dimensions, $n > 4$ (one-time and $(N-1)$-space dimensions) while keeping external momenta and polarization vectors in the first four dimensions (i.e., in the physical space), performing the $n-4$ dimensional integrals in the space orthogonal to the physical space, and then continuing the result in $n$. (For single-loop graphs one may perform all $n$ integrations together.) For sufficiently small $n$, or complex $n$, the subsequent four-dimensional integrations are convergent.

To see how it works, consider the integral

$$F_r(n) \sim \left( \prod_i \int \frac{d\alpha_i}{\pi} \right) \left( \prod_j \int d^n k_j \right) \prod (k_a \cdot k_b) \prod (k_c \cdot p_m) \prod (k_a \cdot e_1) \exp \left[ i \sum \alpha_i (q_i^2 - m_i^2 + i\epsilon) \right]$$

(19.4)

where, now, the $k_j$ are $n$-dimensional vectors. As before we can express the $q_i$ as linear functions of the $k_j$ and the external momenta $p_j$, where the $p_j$ have only first four component nonvanishing. From now, we shall denote an $n$-dimensional vector by $(\hat{k}, K)$, where $\hat{k}$ is the projection of $k$ onto the physical space-time and $K = k - \hat{k}$. Thus, $p = (\hat{p}, 0)$. Equation (19.4) may be written as a sum of terms of the form

$$F_r(n) \sim \left( \prod_i \int \frac{d\alpha_i}{\pi} \right) \left( \prod_j \int d^4 k_j \right) \left( \prod_j \int d^{n-4} K_j \right) \prod (k_a \cdot k_b)$$

$$\times \left( \prod_{\hat{c}, \hat{m}} \hat{k}_c \cdot \hat{p}_m \right) \left( \prod_{\hat{d}, \hat{l}} \hat{k}_d \cdot \hat{e}_l \right) \exp \left[ i \left( \hat{k}_e \cdot \hat{A} \hat{k} + \hat{k} \cdot B p - \hat{k} \cdot \hat{K} \cdot A K - i \sum \alpha_i (m_i^2 + \epsilon) \right) \right]$$

(19.5)

The integrals over $K_j$ can be performed immediately, using the formulas

$$\int d^{n-4} K K_{\alpha_1} K_{\alpha_2} \ldots K_{\alpha_{2r}} \exp(-iAK^2)$$

$$= \frac{\pi^{n/2}}{2r!} \sum_{\sigma} \delta_{\sigma(\alpha_1), \sigma(\alpha_2)} \delta_{\sigma(\alpha_3), \sigma(\alpha_4)} \ldots \delta_{\sigma(\alpha_{2r-1}), \sigma(\alpha_{2r})} (iA)^{-n/2 + 2 - r}$$

where the summation is over the elements $\sigma$ of the symmetric group on $2r$ objects ($\alpha_1, \alpha_2, \ldots \alpha_{2r}$), and

$$\delta_{\alpha\beta} \delta_{\beta\alpha} = n - 4.$$

Thus $F_r$ of eq. (19.5) will have the form

$$F_r(n) \sim \left( \prod_i \int \frac{d\alpha_i}{\pi} \right) \left( \prod_i \int d^4 k_i \right) \left( \prod_j \int d^{n-4} K_j \right) \prod (k_a \cdot k_b) \prod (k_a \cdot e_1) \exp \left[ i \sum \alpha_i (q_i^2 - m_i^2 + i\epsilon) \right]$$

$$\times \prod \left( \hat{k}_d \cdot \hat{e}_l \right) \exp \left[ i \sum \alpha_i (q_i^2 - m_i^2 + i\epsilon) \right]$$

where $f(n)$ is a polynomial in $n$ and $r_i$ is a nonnegative integer depending on the structure of $\Pi K_a \cdot K_b$ in eq. (19.5). For sufficiently small $n < 4$, the singularities of the integrand as some or all $\alpha$'s go to zero disappear.

The reasons this regularization preserves the Ward-Takahashi identities of the kind discussed in
the preceding sections are, firstly, that the vector manipulations such as

\[ k^\mu (2p + k)_\mu = [(p + k)^2 - m^2] - (p^2 - m^2) \]

or partial fractioning of a product of two propagators, which are necessary to verify these identities "by hand", are valid in any dimensions, and, secondly, that the shifts of integration variables, dangerous when integrals are divergent, are justified for small enough, or complex \( n \), since the integral in question is convergent.

The divergence in the original integral is manifested in the poles of \( F_R(n) \) at \( n = 4 \). These poles are removed by the \( R \)-operation, so that \( J_R(n) \) as defined by the \( R \)-operation is finite and well-defined as \( n \to 4 \). Actually, to our knowledge the proof of this has not appeared in the literature, except for the original discussion of 't Hooft and Veltman. Hepp's proof, for example, does not really apply here, since the analytical discussion of Hepp is not tailored for this kind of regularization. However, the argument of 't Hooft and Veltman is sufficiently convincing and we have no reason to believe why a suitable modification of Hepp's proof, for example, of the BPHZ theorem should not go through with the dimensional regularization.

The above discussion is fine for theories with bosons only. When there are fermions in the theory, a complication may arise. This has to do with the occurrence of the so-called Adler-Bell-Jackiw anomalies, which we discussed briefly in section 5. The subject of anomalies in Ward-Takahashi identities has been discussed thoroughly in two excellent lectures by Adler, and by Jackiw, and we shall not go into any further details here. In short, the Adler-Bell-Jackiw anomalies may occur when the verification of certain Ward-Takahashi identities depends on the algebra of Dirac gamma matrices with \( \gamma_s \), such as \( \gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu = 0 \). Typically, this happens when a proper vertex involving an odd number of axial vector currents cannot be regularized in a way that preserves all the Ward-Takahashi identities on such a vertex, and as a consequence some of the Ward-Takahashi identities have to be broken. The occurrence of these anomalies is not a matter of not being clever enough to devise a proper regularization scheme: for certain models such a scheme is impossible to devise. The dimensional regularization does not help in such a case, due to the fact that \( \gamma_5 \) and the completely antisymmetric tensor density \( \epsilon_{\lambda\mu\nu} \) are unique to four dimensions and do not allow a logically consistent generalization to \( n \) dimensions. When there are anomalies in a spontaneously broken gauge theory, the unitarity of the \( S \)-matrix is in jeopardy since, as we shall see in the forthcoming sections, the unitarity of the \( S \)-matrix, i.e., cancellation of spurious singularities introduced by a particular choice of gauge is inferred from the Ward-Takahashi identities. Gross and Jackiw have shown that, in an Abelian gauge theory, the occurrence of anomalies runs afoul of the dual requirements of unitarity and renormalizability of the theory.

Thus, a satisfactory theory should be free of anomalies. Fortunately, it is possible to construct models which are anomaly-free, by a judicious choice of fermion fields to be included in the model. There are two "lemmas" which make the above assertion possible. One is that the anomalies are not "renormalized", which in particular means that the absence of anomalies in lowest order insures their absence to all orders. This was shown by Adler and Bardeen in the context of an \( SU(3) \) version of the \( \sigma \)-model, and by Bardeen in a more general context which encompasses non-Abelian gauge theories. The second is the observation that all anomalies are related; in particular, if the simplest anomaly involving the vertex of three currents is absent in a model, so are all other anomalies. This can be inferred from an explicit construction of all anomalies by Bardeen, or from a more general and elegant argument of Wess and Zumino.
To make sure that a non-Abelian gauge theory is anomaly-free, therefore, it suffices to check that one-fermion-loop contribution to the three-gauge-boson-vertex is free of anomaly. Let

\[ \bar{\psi} \gamma^a \Gamma_a \psi \]

be the coupling term of the gauge boson \( A^{a} \) to the fermions. Here \( \psi \) is a column matrix of all fermion fields in the theory and \( \Gamma_a \) is a matrix whose elements may depend on \( \gamma \). Now the one-fermion-loop contribution to the cubic coupling of the gauge bosons is

\[
\Gamma_{\lambda_{\mu
u}e}^{abc}(p, q, r) \sim \frac{1}{(2\pi)^{4}} \text{Tr} \left\{ \gamma^a \Gamma_a \frac{1}{\gamma \cdot (k+p) - M} \gamma^b \Gamma_b \frac{1}{\gamma \cdot (k-r) - M} \gamma^c \Gamma_c \frac{1}{\gamma \cdot k - M} + \left( \frac{b}{q} \right) \right\}
\]

(19.6)

where \( M \) is the mass matrix of the fermions, and \( p + q + r = 0 \). As can be deduced from the discussion of Gross and Jackiw, for example, the vertex of eq. (19.6) is anomaly-free if the part of this vertex proportional to \( \epsilon_{\mu_{\nu_{\rho}}l}p^\rho \) or \( \epsilon_{\nu_{\mu_{\rho}}l}q^\rho \) is convergent. This calls for

\[
\text{Tr} \gamma_s \Gamma_a \{ \Gamma_b, \Gamma_c \} = 0.
\]

(19.7)

Equation (19.7) is a sufficient condition for the absence of anomalies in a gauge theory.

Georgi and Glashow have discussed various ramifications of this condition. Physically, eq. (19.7) implies that the anomaly caused by one kind of fermions is cancelled by that caused by another. In some models, this cancellation may be arranged among leptons and among hadrons, separately; in some other models this cancellation takes place between leptons and hadrons. In any case anomaly-free theories tend to contain more leptons and hadrons (quarks) than the phenomenology warrants at this time.

The rather restrictive constraints which the consideration of the absence of anomalies imposes on model building may in fact be a blessing in disguise. The possibility of a certain correspondence between leptonic and hadronic building blocks or of new-quantum numbers and new dimensions in hadron spectroscopy is intriguing and perhaps exciting.

Let us conclude with a simple example of dimensional regularization: the vacuum polarization in scalar electrodynamics. The Lagrangian is

\[
\mathcal{L} = (\partial \mu \phi^* - i e A^\mu \Phi^*) (\partial_{\mu} \phi + i e A_{\mu} \phi) - \frac{1}{4} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})^2 - V(\phi)
\]

and the relevant vertices are shown in fig. 19.1. There are two diagrams which contribute to the vacuum polarization, shown in fig. 19.2. The sum of these contributions is

\[
I = e^2 \frac{d^n k}{(2\pi)^n} \int \frac{1}{(k+p)^2 - \mu^2} \frac{1}{k^2 - \mu^2} \left[ (2k+p)_\mu (2k+p)_\nu - 2((k+p)^2 - \mu^2)g_{\mu\nu} \right].
\]

(19.8)

We use the exponential parametrization of the propagators to obtain

\[
I = \frac{e^2}{(i)^2} \int_0^\infty d\alpha \int_0^{2\pi} d\theta \int \frac{d^n k}{(2\pi)^n} \exp \left\{ i[(\alpha(k+p)^2 + \beta k^2 - (\alpha + \beta)(\mu^2 - i\epsilon))] \right\} [(2k+p)_\mu (2k+p)_\nu - 2((k+p)^2 - \mu^2)]
\]

(19.9)

The exponent is proportional to
\[(\alpha + \beta)k^2 + 2k \cdot p\alpha + \alpha p^2 - (\alpha + \beta)(\mu^2 - i\epsilon) = (\alpha + \beta)\left(k + \frac{\alpha}{\alpha + \beta} p\right)^2 + \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon),\]

so we may write

\[I = -e^2 \int \frac{d\alpha}{0} \int \frac{d\beta}{0} \int \frac{d^nk}{(2\pi)^n} \exp \left(i(\alpha + \beta)k^2 + i\left[\frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon)\right]\right)\]

\[\times \left\{4k_\mu k_\nu + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 p_\mu p_\nu - g_{\mu\nu} \left[2(k^2 - \mu^2) + \frac{\alpha^2 + \beta^2}{(\alpha + \beta)^2} p^2\right]\right\}\]

\[= e^2(p_{\mu}p_{\nu} - p^2g_{\mu\nu}) \int \frac{d\alpha}{0} \int \frac{d\beta}{0} \int \frac{d^nk}{(2\pi)^n} \exp \left(i(\alpha + \beta)k^2 + i\left[\frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon)\right]\right)\]

\[\times \left\{4k_\mu k_\nu - 2g_{\mu\nu}(k^2 - \mu^2) - g_{\mu\nu} \frac{\alpha\beta}{(\alpha + \beta)^2} p^2\right\},\]

(19.10)

The first term is explicitly gauge invariant and only logarithmically divergent, so that a subtraction will make it convergent. It is the second term that requires a careful handling. We need the formulas

\[\int \frac{d^nk}{(2\pi)^n} \exp(i\lambda k^2) = \frac{\exp(i\pi n/4)}{(2\sqrt{\pi\lambda})^n}\]

\[\int \frac{d^nk}{(2\pi)^n} k^2 \exp(i\lambda k^2) = \frac{1}{i\lambda} \left(-\frac{n}{2}\right) \frac{\exp(i\pi n/4)}{(2\sqrt{\pi\lambda})^n}\]

\[\int \frac{d^nk}{(2\pi)^n} k_\mu k_\nu \exp(i\lambda k^2) = \frac{1}{n} g_{\mu\nu} \frac{d^nk}{(2\pi)^n} k^2 \exp(i\lambda k^2) = g_{\mu\nu} \frac{1}{i\lambda} \left(-\frac{1}{2}\right) \frac{\exp(i\pi n/4)}{(2\sqrt{\pi\lambda})^n}.\]

(19.11)
so that the second term, $I_2$, is

$$I_2 = -e^2 g_{\mu\nu} \left( \frac{\exp(i\pi n/4)}{(2\sqrt{\pi})^n} \right) \int_0^\infty d\alpha \int_0^\infty d\beta \frac{1}{(\alpha+\beta)^{n/2}} \exp\left\{ i \left[ \frac{\alpha\beta}{\alpha+\beta} p^2 - (\alpha+\beta)(\mu^2-\epsilon) \right] \right\} \right]$$

$$(19.12)$$

$$\times \frac{2}{\alpha+\beta} \left[ i \left( 1 - \frac{n}{2} \right) - \left[ \frac{\alpha\beta}{\alpha+\beta} p^2 - (\alpha+\beta)\mu^2 \right] \right]$$

$$= -2ie^2 g_{\mu\nu} \left( \frac{\exp(i\pi n/4)}{(2\sqrt{\pi})^n} \right) \int_0^\infty d\beta \delta(1-\alpha-\beta) \int_0^\infty d\lambda \frac{\lambda^{n/2-1}}{\lambda^{n/2-1}} \exp\{i\lambda(\alpha\beta p^2-\mu^2+i\epsilon)\} \left[ \frac{1-n/2}{\lambda} + i(\alpha\beta p^2-\mu^2) \right].$$

For sufficiently small $n$, $n < 2$, the $\lambda$-integration is convergent, and

$$\int_0^\infty \frac{d\lambda}{\lambda^{n/2-1}} \exp\{i\lambda(A+i\epsilon)\} \left( \frac{1-n/2}{\lambda} + iA \right) = \int_0^\infty d\lambda \frac{d}{d\lambda} \{ \lambda^{1-n/2} \exp[i\lambda(A+i\epsilon)] \} = 0. \quad (19.13)$$

So the dimensional regularization gives the gauge invariant result,

$$I_2 = 0.$$

Bibliography

The dimensional regularization, in the form discussed here, is due to

See also

A closely related regularization method – analytic regularization – is discussed in

Excellent reviews on the Adler-Bell-Jackiw anomalies are:

For a complete list of anomalies vertices, involving only currents (not pions), see

The following papers discuss the problem of anomalies in gauge theories

The last reference gives a concise algorithm for dimensional regularization valid for scalar loops. Our prescription agrees with it for this case.

20. Feynman rules and renormalization of spontaneously broken gauge theories: Landau gauge

The reader who has followed the developments so far should have no difficulty in comprehending the recent literature on various renormalizable formulations of spontaneously broken gauge
symmetries. In the following sections, we shall try to convey the general ideas underlying the discussions of Lee and Zinn-Justin on this subject, without getting involved too much in mathematical manipulations.

For concreteness let us consider an O(3) gauge theory in which the triplet gauge bosons are interacting with a triplet of real scalar fields $\phi_0$. As we explained in section 14, the generating functional $W_L$ of Green's functions in the Landau gauge is written as [the subscript "0" refers to unrenormalized quantities]

$$W_L[J_\mu, J] = \int [dA_{0\mu}] [dc_0] [dc_0^c] \exp \left[ i \left( S + S_c + \int d^4x \left[ -\frac{1}{2c_0} (\partial_\mu A_{0\mu}(x))^2 + J_0(x) \cdot A^0_0(x) + J_0(x) \cdot \phi_0(x) \right] \right) \right]$$  \hspace{1cm} (20.1)

where, we recall, $c_0$ is a triplet of fictitious complex scalar fields of the wrong statistics and the limit $c_0 \to 0$ is understood. The action, $S$, and $S_c$, are given by

$$S = \int d^4x \left\{ -\frac{1}{4} (\partial_\mu A_{0\mu} - \partial_\nu A_{0\nu} + g_0 A_{0\mu} \times A_{0\nu})^2 + \frac{1}{2} (\partial_\mu \phi_0 + g_0 A_{0\mu} \times \phi_0)^2 - \frac{1}{2} \mu_0^2 \phi_0^2 - \frac{1}{2} \lambda_0 (\phi_0^2)^2 \right\},$$ \hspace{1cm} (20.2)

$$S_c = \int d^4x \left\{ -\partial_\mu c_0^+(x) \cdot \partial_\mu c_0(x) - g_0 \partial_\mu c_0^+(x) \cdot A_{0\mu}(x) \times c_0(x) \right\}.$$ \hspace{1cm} (20.3)

If $\mu^2 > 0$, the theory is the usual one of massless gauge bosons interacting with a multiplet of scalar fields of mass $\mu$. Let us consider the renormalization of the theory in this case. All the three-point and four-point vertices are logarithmically divergent (i.e., the superficial degrees of divergence $D = 0$), and all the self-energy parts, for $A_{0\mu}, \phi_0$ and $c_0$, have $D = 2$, i.e., are quadratically divergent, according to the power counting procedure discussed in section 18.

The Ward-Takahashi identity for $W_L$ of eq. (20.1) is obtained by considering the effects on $W_L$ of the transformation

$$A_{0\mu}(x) \to A_{0\mu}(x) - \omega(x) \times A_{0\mu}(x) + \frac{1}{g_0} \partial_\mu \omega(x)$$

$$\phi_0(x) \to \phi_0(x) - \omega(x) \times \phi_0(x)$$ \hspace{1cm} (20.4)

which leaves $S$ invariant, after eliminating the $c_0^-$ and $c_0^c$ fields. Since we are going to derive the Ward-Takahashi identity for $W$ for a more general class of gauge conditions in a later section, we shall forego writing it down here. Now, when momentum-space Green's functions are dimensionally regularized, they satisfy the Ward-Takahashi automatically. When we generate renormalization counterterms in the manner described below (or scale fields and parameters in the way specified below), the renormalized Green's functions are finite as $n \to 4$, satisfy the renormalized form of the Ward-Takahashi identities [see eq. (2.8) of Lee and Zinn-Justin II]. It is necessary to ensure that the renormalized Green's functions satisfy the renormalized Ward-Takahashi identities, because the latter will be used to show that the renormalized $S$-matrix is free of spurious singularities.

A simple way of generating all the necessary renormalization counterterms in the effective action is to perform the following scale transformations on the quantities appearing in eq. (20.1):

$$A_{0\mu} = Z_1^{1/2} A_\mu, \quad J_{0\mu} = Z_3^{1/2} J_\mu, \quad \phi_0 = Z_2^{1/2} \phi,$$
\[ J_0 = Z_2^{1/2} J, \]
\[ c_0 = \tilde{Z}_3^{1/2} c, \quad g_0 = g Z_1 / Z_3^{1/2} = g \tilde{Z}_1 / \tilde{Z}_3^{1/2}, \]
\[ (\mu_0)^2 = \mu^2 + \delta \mu^2 / Z_2, \quad \lambda_0 = \lambda Z_4 / Z_3^{1/2} \]
and
\[ \alpha_0 = Z_3 \alpha, \]
where the superscript "0" signifies the quantities appearing in eq. (20.1). In terms of the new (renormalized) quantities, the generating functional \( W' \) has the same form as eq. (20.1), except that \( S \) and \( S'_c \) acquire additional pieces \( \Delta S \) and \( \Delta S'_c \), where
\[
\Delta S = \int \! d^4 x \left\{ -\frac{1}{4} (Z_3 - 1)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2} g(Z_1 - 1) A_\mu \times A_\nu \cdot (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \right.
\]
\[ - \frac{1}{4} g^2 (Z_3 / Z_2 - 1)(A_\mu \times A_\nu)^2 + \frac{1}{2} g[1 / 2] (\partial_\mu \phi)^2 - \mu^2 \phi^2 - g[Z_1 (Z_2 / Z_3 - 1)] A_\mu \cdot (\phi \times \partial_\mu \phi) \]
\[ + 1/2 g^2 [(Z_1 / Z_3)(Z_2 / Z_3 - 1)] (A_\mu \times \phi)^2 - \frac{1}{2} \delta \mu^2 \phi^2 - \frac{1}{4} \lambda Z_4 (Z_4 - 1)(\phi^2)^2 \right\} \quad (20.6) \]
and
\[
\Delta S'_c = \int \! d^4 x \left\{ - (Z_3 - 1) \delta \mu^2 c^\dagger \cdot \partial_\mu c - g(Z_1 - 1) \delta \mu^2 c^\dagger \cdot A_\mu \times c \right\}. \quad (20.7) \]

When properly regulated, the self energy-parts of \( A_\mu \) and \( c \) are only logarithmically divergent, and if we choose \( Z_1, Z_3, Z_2 \) and \( \delta \mu^2 \) to make the propagators for \( A_\mu, c \) and \( \phi \) finite; \( Z_1, \tilde{Z}_1 \) and \( Z_4 \) to make the \( A_\mu \phi^2, A_\mu c^\dagger c \) and \( \phi^4 \) proper vertices finite, then the counterterms exhibited in eqs. (20.6) and (20.7) render finite all renormalization parts of the theory. In particular, the renormalization constants \( Z_1, Z_3, \tilde{Z}_1 \) and \( \tilde{Z}_3 \) can be chosen so that
\[ Z_1 / Z_3 = \tilde{Z}_1 / \tilde{Z}_3. \quad (20.8) \]

This is first shown by A. Slavnov and J.C. Taylor. Also, if we choose the renormalization counter-terms in the above manner, then the counterterms for the \( A_\mu \phi^2 \) and \( A_\mu c^\dagger c \)-vertices shown in eqs. (20.6) and (20.7) remove divergences from these vertices.

The proof for this is considerably complicated by the fact that we should not perform subtractions from renormalization parts at the points where all external momenta vanish, since at these points infrared divergences of the renormalization parts are uncontrollable. For this reason, the BPHZ R-operation has to be performed at some points where all external momenta \( p_m \) are Euclidean, \( p_m^2 < 0 \). In any case, the gauge invariant renormalizability of Green's functions in the Landau gauge, i.e., the possibility of renormalizing Green's functions in terms of the scaling as in eq. (20.5) as indicated above, were shown in paper I and paper II, section 2 of Lee and Zinn-Justin*.

Let us now consider the case \( \mu^2 < 0 \). For this case, let us mimic the developments of the \( \sigma \)-model we presented in section 17. From the generating functions \( Z[J_\mu, J] \) of the connected Green's functions

*It should be borne in mind that the gauge transformation for the renormalized gauge fields is
\[
A_\mu \rightarrow A_\mu - \omega \times A_\mu + \frac{1}{i} \frac{Z_2}{Z_3} \partial_\mu \omega = A_\mu - \omega \times A_\mu + \frac{1}{i} \frac{Z_3}{Z_4} \partial_\mu \omega. \]
\[ W(J_\mu, J) = \exp \{ iZ[J_\mu, J] \}, \]  
we define the generating functional of proper vertices

\[ \Gamma [\mathcal{A}_\mu, \Phi] = Z[J_\mu, J] - \int d^4x [\mathcal{A}_\mu(x) \cdot J^\mu(x) + J(x) \cdot \Phi(x)] \]

where

\[ \mathcal{A}_\mu(x) = \delta Z/\delta J^\mu(x) \]

and

\[ \Phi(x) = \delta Z/\delta J(x). \]

The Maxwell equations dual to eqs. (20.11) and (20.12) are

\[ -J_\mu(x) = \delta \Gamma/\delta \mathcal{A}_\mu(x) \]

\[ -J(x) = \delta \Gamma/\delta \Phi(x). \]

The expansion of \( \Gamma \) around \( \Phi = 0 \) and \( \mathcal{A}_\mu = 0 \) generates proper vertices of the symmetric theory, \( \mu^2 > 0 \), and conversely, the knowledge of the renormalized proper vertices amounts to knowing the renormalized form of \( \Gamma \). Now we consider the equation

\[ -\gamma = \delta \Gamma/\delta \Phi(x) \bigg|_{\Phi = u, \mathcal{A}_\mu = 0} \]  
which determines the vacuum expectation value of the scalar fields, when the system is subjected to a constant external source \( \gamma \):

\[ u = u(\gamma). \]

The direction of \( \gamma \) may be defined as the z-direction in the isospin space. The isospin invariance of \( \Gamma \) implies that \( u \) is along the z-direction. Just as for eq. (17.22) of section 17, we obtain from eq. (20.15)

\[ -u \Delta_X^{-1}(0) = \gamma \]

where \( \Delta_X \) is the momentum-space propagator of those components of the scalar field which are perpendicular to \( \gamma \). Denoting

\[ -\Delta_X^{-1}(0) = m_X^2, \]

we see that eq. (20.17) may be written as

\[ u m_X^2 = \gamma. \]

So the spontaneous breakdown of the gauge symmetry entails

\[ m_X^2 = 0 \]

and

\[ v \equiv u(\gamma = 0) \neq 0 \]

where \( v \) is the spontaneous vacuum expectation value of the scalar field. We can adjust \( \delta \mu^2 \) so
that* $m^2 \to 0$: the expansion coefficients of $\Gamma$ about $\Phi = v$ are the proper vertices of the spontaneously broken gauge theory. The generating functions $Z[J, J]$ satisfies the same Ward-Takahashi identity in the limit $m^2 \to 0$, for its response to the gauge transformation (20.4) is independent of the value of $\mu^2$.

In the spontaneously broken symmetry case, it is convenient to write the scalar field $\phi_0$ as

$$\phi_0 = (\psi_0 + \phi_0) + \chi_0$$

so that $\psi_0 \cdot \chi_0 = 0$, $\phi_0 \cdot \chi_0 = 0$. The action (20.2) can be written as

$$S[A_{0\mu}, \phi_0, \chi_0] = \int d^4x \left\{ -\frac{1}{4} (\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu} + g_0 A_{0\mu} \times A_{0\nu})^2 + \frac{1}{2} g_0^2 (\psi_0 \times \chi_0)^2 - \frac{1}{2} g_0 (\phi_0 \cdot \chi_0)^2 - \frac{1}{2} \Delta \mu^2 (\phi_0 \times \chi_0) + \frac{1}{2} \Delta \mu^2 (\phi_0 \times \chi_0) + \frac{1}{2} \Delta \mu^2 (\phi_0 \times \chi_0) + \frac{1}{2} \Delta \mu^2 (\phi_0 \times \chi_0) + \frac{1}{2} \Delta \mu^2 (\phi_0 \times \chi_0) + \frac{1}{2} \Delta \mu^2 (\phi_0 \times \chi_0) \right\}$$

where we have written

$$\Delta \mu^2 = \mu^2_0 + \lambda_0 v_0^2.$$ 

Since the vacuum expectation value of $\phi_0$ is $\psi_0$, $\phi_0$ must not have one. This leads to the condition that

$$v_0 [\Delta \mu^2 + S] = 0$$

(20.23)

where $v_0 \Delta \mu^2$ is the contribution of the last term on the right-hand side of eq. (20.22) to the vacuum expectation value of $\psi$, and $v_0 S$ is the higher order contribution to the process $\psi \to$ vacuum. Actually it can be shown that $(\Delta \mu^2 + S) = -[\Delta^{-1}(0)]$ unrenormalized so that eq. (20.23) is nothing but (20.18) in the limit $\gamma = 0$, and tells us that $\Delta \mu^2$ should be chosen to make $m^2 \to -\Delta^{-1}(0)$ vanish. We can perform the renormalization transformation of eq. (20.5) in $A_{0\mu}, \phi_0 = \psi_0 + \phi_0 + \chi_0, g_0$ and $\lambda_0$ in eq. (20.22) [note that $\psi_0, \phi_0$ and $\chi_0$ must all transform like $\phi_0$] to generate necessary renormalization counterterms. The discussion of the preceding paragraph then implies that choosing renormalization constants $Z_1, Z_2, Z_3, Z_4, Z_1$, and $Z_2$ to be the same as in a symmetric theory will eliminate divergences completely from the spontaneously broken symmetry version of the theory.

The Feynman rules for this theory are obtained if we write

$$S[A_{0\mu}, \phi_0] + S_c[A_{0\mu}, c_0, c_0^\dagger] = S_0[A_{\mu}, \psi, \chi, c, c^\dagger] + S_1[A_{\mu}, \psi, \chi, c, c^\dagger]$$

(20.24)

with

$$S_0 = \int d^4x \left\{ -\frac{1}{4} (\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})^2 + \frac{1}{2} g^2 (v \times A_{0\mu})^2 + \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} (\partial_\mu \phi_0)^2 + \frac{1}{2} (\partial_\mu \chi_0)^2 - g_0 \cdot (A_{\mu} \times \partial^\mu \chi_0) - g_0 \cdot (A_{\mu} \times \partial^\mu \phi_0) \right\}$$

(20.25)

where all quantities in the definition of $S_0$ refer to the renormalized ones, and $S_1$ is defined as the rest, including renormalization counterterms. The generating functional $W_L[J_{\mu}, J]$ may be written as

* A change in $\delta \mu^2$ affects $Z$'s only by finite multiplicative factors, see for example K. Symanzik, Comm. Math. Phys. 23 (1972) 46.
\[ W_L[J_\mu, J] = \exp \{iv \int d^4x J_1(x) \} \exp \{i S_\alpha[\delta/\delta J_\mu, \delta/\delta J_3, \delta/\delta J_{1,2}, \delta/\delta K^\dagger] \} \Big|_{K=K^\dagger=0} \]

where

\[
W_{L0} = \int [dA_\mu] [d\psi] [d\chi] [dc^\dagger] [dc] \exp \left[ i \left( S_0 + \int d^4x \left[ -\frac{1}{2\alpha} (\partial^\mu A_\mu(x))^2 + J_\mu(x) \cdot A_\mu(x) + J_3(x) \psi(x) + J(x) \cdot \chi(x) + K^\dagger(x) \cdot c(x) + c^\dagger(x) \cdot K(x) \right] \right) \right] ,
\]

\[ K \text{ and } K^\dagger \text{ being anticommuting c-numbers.} \]

The propagators of the theory are easily obtained from eq. (20.27), and perturbation theory is based on the formula (20.26) and on the idea of loop-wise expansion, as explained in section 16. The propagators are, as \( \alpha \to 0 \),

\[
A^{1,2} = -i(g_{\mu\nu} - k_\mu k_\nu/(k^2 + i\epsilon))/(k^2 - \mu^2 + i\epsilon), \quad \mu = g_{\mu\nu},
\]

\[
\chi^{1,2} = i/(k^2 + i\epsilon)
\]

\[
A_3^\mu = -i(g_{\mu\nu} - k_\mu k_\nu/k^2)/(k^2 + i\epsilon)
\]

\[
\psi = i/(k^2 - (2\mu^2) + i\epsilon)
\]

\[
c^{1,2,3} = i/(k^2 + i\epsilon).
\]

This model may be considered as the Georgi-Glashow model discussed in Part I, without fermions. \( A_3^\mu \) is the photon, \( \psi \) is the physical neutral Higgs boson, \( (A_\mu' = i A_\mu)/\sqrt{2} \) are the \( W^\pm \) boson fields. The \( W \) boson propagator in this gauge may be written as

\[
- i \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right) \frac{i}{k^2 - \mu^2 + i\epsilon} = - i \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{\mu^2} \right) \frac{1}{k^2 - \mu^2 + i\epsilon} - i \frac{k_\mu k_\nu}{\mu^2} \frac{1}{k^2 + i\epsilon}.
\]

The first term on the right-hand side is the canonical propagator for a massive vector boson with three degrees of polarization freedom. The second term corresponds to a massless scalar boson which couples to the source of the vector meson gradiently. The trouble is that this scalar particle is associated with a negative probability. So Green's functions of this theory are full of "ghosts".

What happens in an \( S \)-matrix element, which is obtained from the Green's function by removing external lines, setting external momenta on the mass-shell, and contracting tensor indices with appropriate physical polarization vectors, is that the poles at \( k^2 = 0 \) associated with the projection operator \( (g_{\mu\nu} - k_\mu k_\nu/k^2) \) in the vector boson propagators, and of the propagators for the unphysical Higgs scalars \( \chi^{1,2} \) and for the scalars \( c^{1,2,3} \) of the wrong statistics cancel, so that none of the massless scalar particles in the theory are physical. The physical particles are the photon \( (A_\mu^3) \) which is massless and has two polarizations, a neutral massive scalar meson \( (\psi) \) and a pair of massive charged vector bosons with three polarizations. This precisely is what is predicted by the Higgs-Kibble theorem discussed in Part I. The reader is invited to verify this fact for a simple process like \( W^+ + \psi \to W^+ + \psi \) in lowest order. In this case there are four diagrams which contri-
bute in lowest order (see fig. 20.1). When all external particles are physical, the pole in the $t$-channel at $t = 0$ is absent.

The proof of the cancellation of spurious poles at $k^2 = 0$ in the $S$-matrix proceeds from the Ward-Takahashi identities satisfied by renormalized Green’s functions. These relations are used to show that when the imaginary part of an $S$-matrix element is computed by the unitarity relation, the contributions from massless particles associated with the three kinds of the $k^2 = 0$ poles add up to zero. The proof is extremely tedious and was worked out explicitly and in detail for intermediate states containing one, two and three such unphysical quanta in paper II of Lee and Zinn-Justin.

The discussion to be presented in the forthcoming sections obviates the necessity of proving the cancellation of spurious singularities in this manner.

Bibliography

The quantization of spontaneously broken gauge theories in a renormalizable gauge was first suggested, and carried out by

This section is based largely on
2. B.W. Lee and J. Zinn-Justin, Phys. Rev. D5 (1972) 3121, 3137, which were referred to as I and II respectively in the text.

The Taylor-Slavnov identity was derived in

In this section, as well as in ref. [2], renormalization is discussed primarily in the context of an $O(3)$ gauge theory. A discussion applicable to any gauge theory will be presented in
5. B.W. Lee, to be published.

based in the Ward-Takahashi identities satisfied by irreducible vertices, derived in

21. The $\mathbf{R}_t$-gauges

In this section we will discuss a formulation of spontaneously broken gauge theories in a class of gauges in which the proof of the unitarity of the $S$-matrix is fairly simple. But first, let us describe spontaneous broken gauge theories in a general way, without making commitments as to the group involved and the representation of scalar fields.

Let $\phi_i$ ($i = 1, 2, ...K$) be a set of scalar fields transforming, in general, reducibly under $G$ of dimension $N$:

$$\phi \rightarrow (1 + i u^\alpha L_\alpha)\phi, \quad \alpha = 1, 2, ...N \quad (21.1)$$
where \( u^\alpha \) are infinitesimal parameters of the group, and \( L_\alpha \)'s are representations of the generators of \( G \). We include the coupling constants of the gauge theory in \( L \) so that the structure constants, \( C_{\alpha\beta\gamma} \), defined by
\[
[L_\alpha, L_\beta] = i C_{\alpha\beta\gamma} L_\gamma
\]  
(21.2)
deep on the coupling constants. We choose \( \phi \) to be real so that \( L \) can be made imaginary anti-symmetric, and \( C_{\alpha\beta\gamma} \) real and completely antisymmetric. With this convention, the gauge-invariant renormalizable Lagrangian is written as
\[
\mathcal{L} = \frac{1}{4} F_{\mu\nu}^\alpha F^{\alpha\mu\nu} + \frac{1}{2} (\partial^\mu \phi^T + i \phi^T A^\alpha_\mu)(\partial_\mu \phi - i L_\beta A^\beta_\mu \phi) - V(\phi)
\]  
(21.3)
where \( V(\phi) \) is an invariant quartic polynomial in \( \phi \), and
\[
F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + C_{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma.
\]  
(21.4)
Let \( v \) be the vacuum expectation value of \( \phi \) in the Landau gauge, and define \( \phi' \) by
\[
\phi = v + \phi'.
\]  
(21.5)
[As Appelquist et al. (see bibliography) have stressed, the vacuum expectation value of a scalar field \( v \) depends in general on the gauge. In final results, we can always trade \( v \) for the mass of a surviving Higgs scalar meson, for example, which is gauge invariant, as a fundamental parameter of the theory.] In terms of \( \phi' \), the Lagrangian (21.3) can be written as
\[
\mathcal{L} = \frac{1}{4} F_{\mu\nu}^\alpha F^{\alpha\mu\nu} + \frac{1}{2} (\partial^\mu \phi'^T + i \phi'^T A_\alpha^{\beta\mu})(\partial_\mu \phi' - i L_\beta A_\mu^\beta \phi') + i (v, L_\alpha \partial_\mu \phi') A_\mu^\alpha + (v, L_\alpha L_\beta \phi') A_\alpha^\mu A_\beta^\mu - V(\phi' + v).
\]  
(21.6)
The gauge invariant potential can be written as
\[
V(\phi) = V(v) + \phi_i \frac{\partial V(v)}{\partial v_i} + \frac{1}{2!} \phi'_i \phi'_j \frac{\partial^2 V(v)}{\partial v_i \partial v_j} + \frac{1}{3!} \phi'_i \phi'_j \phi'_k \frac{\partial^3 V(v)}{\partial v_i \partial v_j \partial v_k} + \frac{1}{4!} \phi'_i \phi'_j \phi'_k \phi'_\ell \frac{\partial^4 V(v)}{\partial v_i \partial v_j \partial v_k \partial v_\ell}.
\]  
(21.7)
Let us recall the discussion of section 2. The vector boson mass matrix
\[
(\mu^2)_{\alpha\beta} \equiv (v, L_\alpha L_\beta v), \quad \mu^2 = (\mu^2)^T
\]  
(21.8)
has rank \( N - M \) where \( M \) is the dimension of the little group of \( v \). We can decompose the representation space of \( \phi \) by the projection operators \( P \) and \( (1 - P) \):
\[
\delta_{ij} = P_{ij} + (1 - P)_{ij}, \quad P_{ij} = \sum_{\alpha, \beta} \langle L_\alpha v \rangle_i \left( \frac{1}{\mu^2} \right)_{\alpha\beta} \langle v^T L_\beta \rangle_j;
\]  
(21.9)
\( P \) is the projection operator onto the space of the Goldstone bosons which is \( N - M \) dimensional. Note that the sum over \( \alpha \) and \( \beta \) in (21.9) actually extends over the \( N - M \) generators which are not of the little group of \( v \). We may decompose the quadratic terms of \( \phi' \) of (21.7) into two parts:
\[
\frac{\partial^2 V(v)}{\partial v_i \partial v_j} = (M^2)_{ij} + P_{ik} \frac{\partial^2 V(v)}{\partial v_k \partial v_j},
\]  
\[
(M^2)_{ij} = (1 - P)_{ik} \frac{\partial^2 V(v)}{\partial v_k \partial v_j}.
\]  
(21.10)
We can always adjust the quadratic term of \( V(\phi), \frac{1}{2} \phi^T A \phi, [A, L_\alpha] = 0, \) so that \( M^2 \) is a positive semi-definite matrix.
To lowest order, $v$ is determined by the condition

$$
\frac{\partial V(v)}{\partial v_i} = 0, \quad v \neq 0,
$$

(21.11)

from which it follows that [see eq. (2.18)]

$$
P_{ik} \frac{\partial^2 V(v)}{\partial v_k \partial v_j} = 0.
$$

(21.12)

In higher orders in the Landau gauge, $v$ is given by

$$
P_{ik} \Delta^{-1}(0)_{kj} = 0
$$

(21.13)

which is the generalization of eq. (20.17) (with $\gamma = 0$). In (21.13) $\Delta(0)$ is the scalar meson propagator matrix at zero momentum, which is a function of $v$.

We shall write the Lagrangian $\mathcal{L}$ as

$$
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1
$$

where

$$
\mathcal{L}_0 = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^T (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{i}{2} \Lambda A_\mu \phi^\dagger \phi + \frac{i}{2} A_\mu \partial_\mu \phi^\dagger \phi + i A_\mu (v, L_\alpha \partial_\mu \phi^\dagger)
$$

(21.14)

and $\mathcal{L}_1$ is the rest [the sum of interaction terms and the counter terms $\phi^\dagger \phi \partial V(v)/\partial v_i + \frac{i}{2} \phi_\mu \phi_\nu \partial^2 V(v)/\partial v_k \partial v_j$]. In eq. (21.14), we have expressed $A_\mu$ as a column matrix.

The Lagrangian (21.14) is a perfectly straightforward free Lagrangian for $M$ massless vector bosons, $N - M$ massive vector bosons and a multiplet of scalar mesons, but for the last term, which couples the longitudinal components of the massive vector bosons to some of the scalar mesons.

The development in section 14 suggests that we consider a wider class of gauge conditions than the Landau gauge. Let us further generalize the considerations there and consider gauge conditions of the form

$$
F_\alpha (A_\mu, \phi) - a_\alpha = 0
$$

(21.15)

where $a_\alpha$ is in general an arbitrary function of space-time, so that

$$
F_\alpha (A_\mu^g, \phi^g) - a_\alpha = 0
$$

(21.16)

has a unique solution for $g$, given $A_\mu$ and $\phi$. Following the discussion of section 14, we define the determinant $\Delta_F$ by

$$
\Delta_F^\dagger [A_\mu, \phi] = \int \prod_x \prod_{x, \alpha} \delta [F_\alpha (A_\mu^g, \phi^g) - a_\alpha].
$$

(21.17)

The counterparts of eqs. (14.12) and (14.13) are

$$
\Delta_F [A_\mu, \phi] = \det M_F
$$

(21.18)

where

$$
[M_F(x, y)]_{\alpha \beta} = \frac{\partial^2 F_\alpha (A_\mu^g (x), \phi^g (x))}{\partial u_\beta (y)} \bigg|_{u=0}
$$

(21.19)
and for \( g \) in the neighborhood of the identity,
\[
F_{\alpha}(A^{\mu}_{\mu}(x), \phi^{\beta}(x)) = F_{\alpha}(A^{\mu}_{\mu}(x), \phi(x)) + \int d^{4}y \sum_{\beta} [M_{F}(x, y)]_{\alpha\beta} u^{\beta}(y) + O(u^2).
\]

The generating functional of Green's functions in this gauge is
\[
W_{F,\alpha}[J_{\mu}, J_{\phi}] = \int [dA^{\alpha}_{\mu}][d\phi_{i}] \Delta_{F}[A^{\mu}_{\mu}, \phi] \prod_{x,\alpha} \delta[F_{\alpha}(A^{\mu}_{\mu}, \phi)-a_{\alpha}] \exp\{i \int d^{4}x [\mathcal{L}(x)+J^{T}_{\mu}A^{\mu}+J^{T}\phi]\}.
\]  (21.20)

Repeating the argument in section 16, where we showed that the renormalized S-matrix is the same in the Coulomb and Landau gauges, we can show that the renormalized S-matrix is independent of the arbitrary function \( a_{\alpha}(x) \) in the gauge condition (21.15). One may therefore integrate over \( a_{\alpha}(x) \) (with an arbitrary weight factor) the right-hand side of eq. (21.20) without changing the renormalized S-matrix
\[
W_{F} \equiv \int \prod_{x,\alpha} da_{\alpha}(x) \exp\left\{-\frac{i}{2} \int a_{\alpha}^{2}(x) d^{4}x \right\} W_{F,\alpha}
\]  (21.21)
where we have inserted a Gaussian factor arbitrarily. We obtain finally
\[
W_{F} \sim \int [dA^{\alpha}_{\mu}][d\phi_{i}] \det M_{F} \exp\{i \int d^{4}x [\mathcal{L}(x)-\frac{i}{2} F^{T} F + J^{T}_{\mu}A^{\mu} + J^{T}\phi]\}.
\]  (21.22)

Equation (21.22) is the basis of a formulation based on the general gauge condition (21.15).

The idea is to choose \( F_{\alpha} \) in such a way that the part of \( \mathcal{L}(x)-\frac{1}{2} F^{T} F \) which is quadratic in \( A^{\mu}_{\mu} \) and \( \phi^{i} \) is nonsingular. We choose \( F_{\alpha} \) to be, with a real, nonnegative \( \xi \),
\[
F_{\alpha} = \sqrt{\xi} \left[ \delta^{\mu\nu} A^{\alpha}_{\mu} - \frac{i}{\xi}(\nu, L^a \phi^i) \right].
\]  (21.23)

Then the sum of \( \mathcal{L}_{0} \) in eq. (21.14) and \(-\frac{1}{2} F^{T} F\) is
\[
\mathcal{L} = -\frac{1}{2} F^{T} F = -\frac{1}{2} A^{\mu}[-g^{\mu\nu}\partial^{2} + \partial^{\mu}\delta^{\nu}(1-\xi) - g^{\mu\nu}\mu^{2}] A_{\nu} + \frac{1}{2} \phi^{T} \left[ -\partial^{2} - M^{2} - \frac{1}{\xi} \sum_{a} L_{a}(v^{T} L_{a}) \right] \phi
\]  (21.24)
where we have dropped terms which are four-divergences. The cross-term in the square of \( F_{\alpha} \) has cancelled the term which couples the longitudinal components of the massive vector bosons to some of the scalar bosons.

The propagators in this gauge are obtained from the formula
\[
W_{F_{0}} \sim \int [dA^{\alpha}_{\mu}][d\phi_{i}] \exp\{i \int d^{4}x [\mathcal{L}_{0}(x)-\frac{1}{2} F^{T} F + J^{T}_{\mu}A^{\mu} + J^{T}\phi]\}
\]  
\[\cdot \exp\left\{-\frac{i}{2} \int d^{4}x d^{4}y [J^{T}_{\mu}(x)\Delta_{F}^{\nu\mu}(x-y)J_{\nu} + iJ^{T}(x)\Delta_{F}(x-y)J(y)]\right\}.
\]  (21.25)

[We apologize for the proliferation of the symbol \( \Delta_{F} \): in eqs. (21.17)–(21.20), it is a Jacobian of field variables; here it stands for the Feynman propagator.] The propagators satisfy
\[
[(\partial^{2} + \mu^{2} - i\epsilon)g^{\mu\nu} - \partial^{\mu}\delta^{\nu}(1-\xi)] \Delta_{F_{\nu\lambda}}(x-y; \xi) = -g^{\mu\lambda}\delta^{4}(x-y),
\]
which incorporate the boundary condition given by the Euclidicity postulate. The momentum space propagators are given by

\[
\Delta_F^{\mu\nu}(k; \xi) = -\frac{1}{g^{\mu\nu}} \left(1 - \frac{1}{\xi} k^\mu k^\nu \right) \frac{1}{k^2 - \mu^2 + i\epsilon} \quad (21.27)
\]

and

\[
\Delta_F(k^2; \xi) = \frac{1}{k^2 - M^2 - (1/\xi) \sum_a L_a \nu \left( u^T L_a + i\epsilon \right)} = P \frac{1}{k^2 - \mu^2/\xi + i\epsilon} + \left(1 - P\right) \frac{i}{k^2 - M^2 + i\epsilon} \quad (21.28)
\]

where the projection operator P is defined in eq. (21.12). (The derivation of the second line of eq. (21.28) from the first is left as a challenge to the dedicated reader.)

Now let us consider \( \det M_v \). For infinitesimal \( u_\alpha \) we have

\[
\phi^g = \phi - i u^\alpha L_\alpha \phi, \quad (21.29)
\]

\[
[A_\mu^g]^a = A_\mu^a - i u^\beta C_{\alpha\beta\gamma} A_\mu^\gamma - \partial_\mu u^\alpha, \quad (21.30)
\]

so that

\[
F_\alpha (A_\mu^g, \phi^g) - F_\alpha (A_\mu, \phi) = \partial^2 u^\alpha - i C_{\alpha\beta\gamma} \partial^\mu (A_\mu^\gamma u^\beta) + \frac{i}{\xi} (v^T L^\alpha \phi) u^\beta + O(u^2). \quad (21.31)
\]

Making use of eqs. (21.18), (21.19) and discussions of section 14, we can write

\[
\det M_F = \det \left\{ \frac{\delta F_\alpha (A_\mu^g(x)\phi^g(x))}{\delta u_\beta(y)} \right\} \bigg|_{\xi = 0} = \int [d c_{\alpha}] [d c_{\alpha}^+] \exp \left[ i \int d^4 x \left( \partial^\mu \partial_\mu c_{\alpha}^{\dagger} c_{\alpha} - \frac{1}{\xi} \mu^2 \right) c_{\alpha} + i c_{\alpha\beta\gamma} \partial^\mu c_{\alpha}^{\dagger} A_\mu^\gamma c_{\beta} - \frac{1}{\xi} c_{\alpha}^{\dagger} (u^T L^\alpha \phi') c_{\beta} \right]
\]

where \( c \) and \( c_{\alpha}^{\dagger} \) are \( N \)-component complex fields of anticommuting \( c \)-numbers. The propagator for \( c \) is

\[
\Delta_c(k^2, \xi) = i/(k^2 - \mu^2/\xi + i\epsilon). \quad (21.33)
\]

The propagators (21.27), (21.28) and (21.33) become those of the Landau gauge as \( \xi \to \infty \). In fact in this limit, the Feynman rules for the \( \xi \)-gauge are identical to those of the Landau gauge. Incidentally, the vector propagator of the form of eq. (21.27) is precisely that devised by T.D. Lee and C.N. Yang over a decade ago in their attempt to construct a regularizable theory of weak interactions, which they called the \( \xi \)-limiting procedure. In this gauge, the spurious \( k^2 = 0 \) singularities in the vector and scalar propagators, and the propagators for the fictitious scalars of the wrong statistics in the Landau gauge, have been displaced to \( k^2 = \mu^2/\xi \). Furthermore, we shall show in the next lecture that the renormalized S-matrix is independent of the parameter \( \xi \). This can only mean that the poles at \( k^2 = \mu^2/\xi \) disappear in the S-matrix completely.

Note that the vector boson propagator of eq. (21.27) behaves as \( 1/k^2 \) for large \( k^2 \), and all the interactions of the theory are of the renormalizable type, so that the superficial degree of diver-
gence of any proper diagram in this gauge is at most two. One is at liberty to consider the limit \( \xi \to 0 \), after Feynman integrations are performed. To take this limit in the integrand is dangerous, since the integral may not then be well-defined (with the dimensional regularization, however, only-loop diagrams for \( S \)-matrix elements seem to be controllable even in this gauge). In any case, we may just take this limit in the propagators to see the particle content of the theory. In the limit \( \xi \to 0 \), there are \( M \) massless, and \( N - M \) massive vector bosons, \( K - N - M \) massive scalar bosons and no other spurious particles. Thus in this limit we are in the unitary gauge discussed in section 3. In this limit

\[
\det M_F \sim \int [dc_\alpha] [dc^\dagger_\alpha] \exp \left[ i \int d^4x c_\alpha^\dagger(x) \left( -\frac{1}{\xi} \mu_{\alpha\beta} - \frac{1}{\xi} (u^T L^\alpha L^\beta \phi'(x)) \right) c_\beta(x) \right] \\
\sim \exp \left[ \delta^4(0) \int d^4x \sum_\alpha \left[ \ln(1 + J(x)) \right]_{\alpha\beta} \right]
\]

where

\[
[J(x)]_{\alpha\beta} = (1/\mu^2)_{\alpha\gamma} (v^\gamma L^\beta \phi'(x))
\]

a result originally due to Weinberg.

### Bibliography


22. Proof that the renormalized \( S \)-matrix is independent of \( \xi \)

In this section we will derive the Ward-Takahashi identity for \( W_F \) and show that the renormalized \( S \)-matrix is independent of \( \xi \). In particular, it will then follow that the poles in propagators at \( \mu^2/\xi \) are spurious. We shall then comment on the practical way of performing renormalization in this scheme. In the following discussion all fields refer to unrenormalized ones, and all Feynman integrals are dimensionally regularized.

The Ward-Takahashi identity for \( W_F \) is so complicated that unless we use a compact notation it is almost impossible to print it. We shall let \( \phi_a \) be the set of all fields including the gauge fields, so that \( a \) runs over \( \alpha = 1, 2, \ldots, N \) and \( i = 1, 2, 3, \ldots, K \), in the notation of the previous section. As before, \( \alpha \) labels the generators of the group \( G \). We let the indices \( a \) and \( \alpha \) stand for the space-time
variables and tensor indices as well as the internal symmetry indices, and summation and integration over repeated indices will always be understood in this section. With this convention, the infinitesimal transformation law of the field \( \phi_a \) may be written as

\[
\phi'_a = \phi_a + [\Gamma_{ab}^a \phi_b + \Lambda_a^a] u_a + O(u^2)
\]

(22.1)

where \( \Gamma_{ab}^a \) is a reducible representation of the generator labeled by \( \alpha \), so that, for example,

\[
\Gamma_{bc}^a = i C_{bac} \delta^a_{(x_b - x_c)}
\]

if \( b \) and \( c \) refer to one of the \( \alpha \)'s, and

\[
\Lambda_{ab}^a = -\delta_{ab} \frac{\partial}{\partial x_b} \delta^a(x_b - x_a)
\]

so that

\[
\Lambda_{ab}^a u_a = \delta_{(a)\mu} \mu_\mu(x_b).
\]

The Ward-Takahashi identity for \( W_F[J] \) is obtained if we consider the change of integration variables \( \phi \) to \( \phi' \) where

\[
\phi_a \rightarrow \phi'_a = \phi_a + (\Gamma_{ab}^b \phi_b + \Lambda_{ab}^a) u_b
\]

(22.5)

where we shall restrict \( u_b \) such that

\[
(M_F)_{ab} u_b = \lambda_a
\]

(22.6)

\( \lambda_a \) being some constant independent of \( \phi \). Since \( M_F \) depends on \( \phi \), the allowed \( u ' s \) depend on \( \phi \):

\[
u_a = [M_F^{-1}(\phi)]_{ab} \lambda_b.
\]

(22.7)

The reason for this restriction is that the change in \( F_a \) is then simple. From eq. (22.4) we see that

\[
F_a(\phi') = F_a(\phi) + \lambda_a + O(\lambda^2).
\]

(22.8)
The functional metric \( [d\phi] \) and \( \Delta_F[\phi] \) defined by eq. (22.2) are not invariant under the transformation (22.5) when \( u \) is restricted by eq. (22.6). The reason is that the transformation is no longer linear in \( \phi \). A simple example is afforded by \( dxdy \) which is invariant under the transformation \( x \to x \cos \theta - y \sin \theta, y \to y \cos \theta + x \sin \theta \). The metric is not invariant if \( \theta \) depends on \( x \) and \( y \).

However there is an important lemma, derived by Fradkin and Tyutin, and Slavnov for the Landau gauge, and generalized to any gauge by Lee and Zinn-Justin in the Appendix of paper IV, which states that the product \( \Delta_F[\phi][d\phi] \) is invariant under such nonlinear gauge transformations:

\[
[d\phi] \Delta_F[\phi] = [d\phi'][\Delta_F[\phi']], \tag{22.9}
\]

for \( \phi' \) given by eqs. (22.5) and (22.6).

The proof is interesting but somewhat lengthy, so we refer the reader to Appendix of Lee - Zinn-Justin IV.

Since a change of integration variables does not change the value of the integral, the change in \( W_F \) with respect to \( \lambda \) must be zero when we change the variables by eqs. (22.5) and (22.6). Writing \( \phi \) for \( \phi' \), we have

\[
0 = \int [d\phi] \Delta_F[\phi] \exp[i\{S[\phi] - \frac{1}{2}F^2_\alpha(\phi) + J_\alpha\phi_\alpha\}] \frac{\delta}{\delta \lambda_\alpha} \{S[\phi] - \frac{1}{2}F^2_\alpha(\phi) + J_\alpha\phi_\alpha\}
\]

or

\[
\int [d\phi] \Delta_F[\phi] \exp[i\{S[\phi] - \frac{1}{2}F^2_\alpha(\phi) + J_\alpha\phi_\alpha\}] \{ -F_\alpha(\phi) + J_\alpha (\Gamma^\beta_{\alpha\beta}\phi_\beta + \Lambda^\beta_\alpha) [M^{-1}_F(\phi)]_{\beta\alpha}\} = 0. \tag{22.10}
\]

Equation (22.10) can be converted into a functional differential equation satisfied by \( W_F[J] \):

\[
\left\{ -F_\alpha \left( \frac{1}{i} \frac{\delta}{\delta J}\right) + J_\alpha \left( \Gamma^\beta_{\alpha\beta} \frac{1}{i} \frac{\delta}{\delta J}\right) + \Lambda^\beta_\alpha \left[ M^{-1}_F \left( \frac{1}{i} \frac{\delta}{\delta J}\right) \right]_{\beta\alpha}\right\} W_F[J] = 0. \tag{22.11}
\]

This is the Ward-Takahashi identity for \( W_F[J] \).

To make use of eq. (22.11), we must know what

\[
[M_F^{-1} \left( \frac{1}{i} \frac{\delta}{\delta J}\right)]_{\beta\alpha} W_F[J]
\]

is. Consider

\[
W_{\alpha\beta}[J] = \int \{ d\phi \} \{ dc_\alpha \} \{ dc^\dagger_\alpha \} c_\alpha c^\dagger_\alpha \exp[i\{S[\phi] - \frac{1}{2}F^2_\alpha(\phi) + c^\dagger_\alpha(M_F)_{\alpha\beta}c_\beta + J_\alpha\phi_\alpha\}] \tag{22.12}
\]

where \( c^\dagger_\alpha \) and \( c_\alpha \) are anticommuting fields, so that

\[
\Delta_F[\phi] = \text{det} M_F = \int \{ dc_\alpha \} \{ dc^\dagger_\alpha \} \exp[i c^\dagger_\alpha(M_F)_{\alpha\beta}c_\beta]. \tag{22.13}
\]

The functional \( W_{\alpha\beta}[J] \) is the Green's function for the fictitious scalar fields of the wrong statistics in the presence of external sources \( J \), and satisfies the equation

\[
[M_F \left( \frac{1}{i} \frac{\delta}{\delta J}\right)]_{\alpha\beta} W_{\beta\gamma}[J] = \int \{ d\phi \} \{ dc \} \{ dc^\dagger \} (M_F)_{\alpha\beta} c_\gamma c^\dagger_\gamma \exp[i\{S[\phi] - \frac{1}{2}F^2_\alpha(\phi) + c^\dagger_\alpha(M_F)_{\alpha\beta}c_\beta + J_\alpha\phi_\alpha\}]
\]

\[
= \int \{ d\phi \} \{ dc \} \{ dc^\dagger \} c^\dagger_\gamma \left( -\frac{\partial}{\partial c^\dagger_\gamma} \right) \exp[i\{S[\phi] - \frac{1}{2}F^2_\alpha(\phi) + c^\dagger_\alpha(M_F)_{\alpha\beta}c_\beta + J_\alpha\phi_\alpha\}],
\]

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\[
\left[ M_F \left( \frac{1}{i} \delta \right) \right]_{\alpha \beta} W_{\beta \gamma} [J] = \delta_{\alpha \gamma} W_F [J].
\] (22.14)

We see that
\[
\left[ M_F^{i} \left( \frac{1}{i} \delta \right) \right]_{\beta \alpha} W_F [J] = W_{\beta \alpha} [J].
\] (22.15)

Next, let us consider what happens to \( W_F [J] \) when we vary the gauge condition \( F \) by \( \Delta F \):
\[
W_{F+\Delta F} [J] - W_F [J] = \int [d\phi] \Delta_F \exp(i\{S[\phi] - \frac{1}{2} \Delta F^2(\phi) + J_a \phi_a \}) \left\{ -i F_a \Delta F_a + \frac{\delta \Delta F_a}{\delta \phi_a} (M_F^{-1}(\phi))_{\alpha \beta} \right\}
\] (22.16)

where we have used the fact that, from eq. (22.3),
\[
\Delta_{F+\Delta F} = \det \left( \frac{\delta F^a}{\delta \phi_a} + \frac{\delta \Delta F^a}{\delta \phi_a} (\Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha}) \right)
\] = \det \left( M_F + \frac{\delta \Delta F^a}{\delta \phi_a} (\Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha}) \right) \approx \Delta_F \frac{\delta \Delta F^a}{\delta \phi_a} (\Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha})(M_F^{-1})_{\beta \alpha}
\]
to first order in \( \Delta F_a \). We will operate \( i \Delta F_a (\delta / \delta J) \) on eq. (22.10). Noting that
\[
i \Delta F_a \left( \frac{1}{i} \delta \right) J_a \exp(iJ_a \phi_a) = \exp(iJ_a \phi_a) \{ J_a \Delta F_a(\phi) + \delta \Delta F_a(\phi) / \delta \phi_a \},
\]
we obtain
\[
0 = \int [d\phi] \Delta_F [\phi] \exp(i\{S[\phi] - \frac{1}{2} \Delta F^2(\phi) + J_a \phi_a \}) \left\{ -i F_a \Delta F_a + \frac{\delta \Delta F_a(\phi)}{\delta \phi_a} \right\} \left( \Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha} \right)(M_F^{-1})_{\beta \alpha}.
\] (22.17)

Combining eqs. (22.16) and (22.17), we finally obtain
\[
W_{F+\Delta F} [J] - W_F [J] = \int [d\phi] \Delta_F [\phi] \exp(i\{S[\phi] - \frac{1}{2} \Delta F^2(\phi) + J_a \phi_a \}) \{ J_a \Delta F_a(\phi) + \frac{\delta \Delta F_a(\phi)}{\delta \phi_a} \} (\Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha})(M_F^{-1})_{\beta \alpha}
\]
or, valid to first order in \( \Delta F \),
\[
W_{F+\Delta F} [J] = \int [d\phi] \Delta_F [\phi] \exp(i\{S[\phi] - \frac{1}{2} \Delta F^2(\phi) + J_a \phi_a \}) (\Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha})(M_F^{-1})_{\beta \alpha} \Delta F_a(\phi).
\] (22.18)

i.e., the effect of changing the gauge from \( F \) to \( F + \Delta F \) is merely to add extra vertices between the source and fields:
\[
J_a \phi_a \xrightarrow{F \rightarrow F + \Delta F} J_a \{ \phi_a + (\Gamma^{\varphi}_{\alpha \beta} \phi_\beta + \Lambda^a_{\alpha})(M_F^{-1}(\phi))_{\alpha \beta} \Delta F_a(\phi) \}.
\]
What happens to the S-matrix under such circumstances has been discussed thoroughly in section 15, in conjunction with the change in the S-matrix as one passes from the Coulomb gauge to the Landau gauge. We shall adopt the conclusion there to this case, that, if two \( W[J] \)'s differ only in the external source term as in (22.18), the result is only a redefinition of the renormalization of the unrenormalized S-matrix. The change in an S-matrix element \( S \) may be expressed as

\[
S_{F+\Delta F} = \prod_i (Z_{F+\Delta F}/Z_F)_i^{1/2} S_F
\]

where \( (Z_F)_i \) is the wave function renormalization for the \( i \)th external line and the product \( \Pi \) extends over all external lines. Thus, the quantity

\[
S = S_F/\prod_i (Z_F)_i^{1/2}
\]  

(22.19)

is independent of \( F \). Let us recall that we have been dealing with dimensionally regularized (with \( n = 4-\epsilon \)) quantities. In eq. (22.19) \( (Z_F)_i \) is to be calculated from the two-point function \( [G^{(2)}(\epsilon)]_i \) for the particle of type \( l \). In general \( S_F \) and \( Z_F \) depend both on \( F \) and \( \epsilon \), and are finite as long as \( \epsilon \neq 0, 2, \) etc.

When we specialize to the \( \xi \)-gauge, the quantity in (22.19) is

\[
S(\epsilon) = \lim_{p_i^2 \rightarrow m_i^2} \prod_{i=1}^{n} \frac{(p_i^2-m_i^2)}{[Z(\xi, \epsilon)]_i^{1/2}} G^{(n)}(\xi, \epsilon),
\]

(22.20)

where \( G^{(n)}(\xi, \epsilon) \) is the regularized \( n \)-point Green's function in the \( \xi \)-gauge. The discussion of the preceding paragraph implies that \( S(\epsilon) \) is independent of \( \xi \). Therefore it is devoid of spurious singularities in \( k^2 \) depending on \( \xi \) for all \( \epsilon \neq 0, 2 \), as can be seen by taking the limit \( \xi \rightarrow 0 \) in (22.20).

To obtain the physical S-matrix, we must take the limit \( \epsilon \rightarrow 0 \) after renormalization. One way of accomplishing this is renormalize the Green's function \( G^{(n)}(\xi, \epsilon) \) for arbitrary \( \xi \). In paper IV, Lee and Zinn-Justin discuss the renormalization scheme for arbitrary \( \xi \) based on the Ward-Takahashi identities for a particular model. The same has not been worked out for a general class of models.* However this is not necessary. Since \( S(\epsilon) \) is independent of \( \xi \) one may take the case \( \xi = \infty \) (the Landau gauge) and obtain

\[
S(\epsilon) = \lim_{p_i^2 \rightarrow m_i^2} \prod_{i=1}^{n} \frac{(p_i^2-m_i^2)}{[Z(\infty, \epsilon)]_i^{1/2}} G^{(n)}(\infty, \epsilon).
\]

This is precisely the regularized S-matrix element in the Landau gauge, and we now learn that this quantity is devoid of spurious singularities in \( k^2 \). We can now rescale the coupling constants and other parameters according to the discussion of section 20, i.e.,

\[
g^2 = g r \frac{Z_1(\epsilon)}{[Z_3(\epsilon)]^{3/2}} = g r \frac{Z_1(\epsilon)}{[Z_3(\epsilon)]^{1/2} Z_3(\epsilon)}, \text{ etc.}
\]

* Note added in proof: This has now been done; this will be reported in a forthcoming paper by B.W.L.
and take the limit $\epsilon \to 0$, while keeping $g_\tau$ and other renormalized parameters fixed. The result is a finite $S$-matrix element, which is devoid of spurious singularities.

Bibliography

This section is based on
Presumably, the discussion here is equivalent to a combinatoric discussion of

23. Anomalous magnetic moment of the muon in the Georgi-Glashow model

As a practical application of these ideas, we calculate the first weak correction to the anomalous magnetic moment of the muon in the Georgi-Glashow model, using the $R_\mu$ gauge. In general, the parity-conserving terms of the $\mu^-$ electromagnetic vertex have the form

$$V_\mu = \bar{u}(p + q/2) \left[ \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu} q^\nu}{2m} F_2(q^2) \right] u(p - q/2)$$

(23.1)

where $p \pm q/2$ is the final (initial) muon momentum, $q$ is the momentum transferred to the photon, and $m$ is the muon mass. Using the Gordon decomposition

$$2m \gamma_\mu = 2p_\mu + i\sigma_{\mu\nu} q^\nu$$

eq (23.1) can be rewritten

$$V_\mu = \bar{u}(p + q/2) \left[ \gamma_\mu \left( F_1(q^2) + F_2(q^2) \right) - \frac{p_\mu}{m} F_2(q^2) \right] u(p - q/2)$$

(23.2)

or

$$V_\mu = \bar{u}(p + q/2) \left[ \frac{p_\mu}{m} F_1(q^2) + \frac{i\sigma_{\mu\nu} q^\nu}{2m} \left( F_1(q^2) + F_2(q^2) \right) \right] u(p - q/2).$$

(23.3)

In any of these forms, $F_1(q^2)$ is the electric form factor, and $F_2(q^2)$ is the magnetic form factor. In particular, $F_1(0)$ is always renormalized to be 1, and $F_2(0)$ is the anomalous magnetic moment. In ordinary electrodynamics, $V_\mu$ is just a matrix element of the neutral, gauge-invariant, electro-

![Figure 23.1](image-url)

**Fig. 23.1.** Photon exchange contribution to the meson anomalous magnetic moment.
magnetic current, and therefore the form factors are independent of the gauge. In the more general case we have been considering, this current is not invariant under general transformations of the non-Abelian gauge group, and the form factors will be gauge-dependent. In particular, for the $R_f$ gauge, $F_1(q^2)$ and $F_2(q^2)$ will depend upon $\xi$.

However, the electric charge and the anomalous magnetic moment, $F_1(0)$ and $F_2(0)$ respectively, are physically measurable quantities; they are related to residues of the photon pole in the $S$-matrix for $\mu^+\mu^-$ elastic scattering, for example. Therefore, since the $S$-matrix is gauge independent, we expect them to be independent of $\xi$. This is the fact we wish to demonstrate by an explicit calculation.

In all models, there is a contribution to $F_2(0)$ from the photon-exchange graph, fig. 23.1, which was calculated long ago to be $\alpha/2\pi$. The remaining graphs are formally of order $\alpha$ also, but they are all proportional to $\alpha/\mu^2$ ($\mu$ is the W mass, as in the previous sections) and so are numerically of the order of the Fermi constant $G$, and may be thought of as weak corrections to $F_2(0)$. In the Weinberg-Salam model, all these corrections are equal to $G m^2$ times constants of the order one, so are very small indeed. In the Georgi-Glashow model, graphs with heavy-lepton exchange contribute terms of the order $G M_0 m$ ($M_0$ is the mass of the neutral heavy muon) and may be more interesting experimentally. We outline the calculation in a simple approximation.

Heavy muons $M^o$ and $M^+$, of mass $M_0$ and $M_+$, can be added to the Georgi-Glashow model discussed in section 9 in exact analogy to $E^o$ and $E^+$. In addition to the photon exchange graph (fig. 23.1), the graphs of figs. 23.2 and 23.3 all contribute to $F_2(0)$. In the U-gauge, graphs containing the charged scalars $s^\pm$ are absent, but the remaining ones are not all unambiguously convergent. In the $R_f$ gauge, we must include them, but this gauge has the advantage that the contribution of each graph to $F_2(0)$ is convergent. The propagators for the scalars and the W mesons are given in eqs. (21.27) and (21.28).

The neutral scalar exchange graph, fig. 23.2a, contributes a term proportional to $G m^2 (m^2/m_\psi^2)$, which we take to be negligible, even though the model does not strictly require $m_\psi$ to be very large. The neutrino exchange graphs, figs. 23.2b-e, are all of the order $G \mu^2$ while the $M^o$ exchange contributions of fig. 23.3 are proportional to $G M_0$ and are therefore the largest.
We calculate these $M^0$ exchange graphs only. Since $M_0$ is arbitrary, the term proportional to $M_0$ must in any case be independent of $\xi$. We make the approximation

$$m^2 \ll M_0^2 \ll \mu^2$$

and for convenience restrict $\xi$ so that

$$M_0^2 \ll \mu^2/\xi$$

as well. Our result for each graph separately will therefore not be valid in the Landau gauge $\xi \to \infty$, which may be treated as a separate case.

The vertices which occur in the graphs in fig. 23.3 can be obtained from the coupling terms we wrote down in section 9.

From the muon analog (19.12), the $\mu W^0$ vertex [fig. 23.4a] is

$$eM^0 \left[ \cos \gamma - \mu L^0 + \gamma S^0 + \mu R^0 \right].$$

The $sM^0 \mu$ vertex, fig. 23.4b, is obtained from the muon analog of eq. (19.22):

$$G_1[M^0_L s^0 \mu_R \cos \gamma + \mu_L s^0 M^0_R] + H.C. + G_2 \sin \gamma M^0_L s^0 \mu_R + H.C.$$  

where

$$G_1 = \frac{e}{\mu} [m - M_0 \cos \gamma], \quad G_2 = \frac{e}{\mu} M_0 \sin \gamma.$$

According to eq. (19.20), the $W^- A^- \mu$ vertex of fig. 23.4c is just

$$-e \mu g_{\alpha\mu}.$$  

The $W^- W^- A^- \mu$ vertex, fig. 23.4d, can be read off the trilinear term in $-F_{\mu\nu} \cdot F^{\mu\nu}$, as in fig. 14.1. The result is

$$e \Gamma_{\alpha\beta\mu} \equiv e[g_{\alpha\beta}(k_1 - k_2)_{\mu} + g_{\mu\alpha}(q - k_1)_{\beta} + g_{\beta\mu}(k_2 - q)_{\alpha}].$$  

A simplifying feature of the calculation is that we need only the term in $V_\mu$ linear in $q$ to obtain $F_2(0)$. Therefore, terms of higher order may be dropped, but the terms proportional to $q$ must be kept until $V_\mu$ is expressed in one of the forms (23.1–3). We use the following notation: $V^a_\mu$, $V^b_\mu$, $V^c_\mu$ and $V^d_\mu$ are the contributions to $V_\mu$ from the graphs in figs. 23. a–d, respectively, and $F^a_2$, $F^b_2$, $F^c_2$ and $F^d_2$ are defined similarly.

We turn to fig. 23.3a:
\[
V_\mu = \frac{-ie^2}{(2\pi)^4} \int d^4k \ u(p+q/2)N^{\alpha\gamma}u(p-q/2) \left[ -g_\alpha^\beta + \frac{(1-1/\xi)(k+q/2)\alpha(k+q/2)\beta}{(k+q/2)^2-\mu^2/\xi+ie} \right] \\
\times \left[ -g_\gamma^\delta + \frac{(1-1/\xi)(k-q/2)\delta(k-q/2)\gamma}{(k-q/2)^2-\mu^2/\xi+ie} \right] \Gamma_{\delta \beta \mu} [(p-k)^2-M_0^2+ie]^{-1} \\
\times [(k-q/2)^2-\mu^2+ie]^{-1}[(k+q/2)^2-\mu^2+ie]^{-1} \tag{23.10}
\]

where \( \Gamma_{\delta \beta \mu} \) is defined in (23.9), with \( k_1 = k-q/2, \ k_2 = -(k_1+q/2). \) From (23.6) and (23.9)

\[
\bar{u}(p+q/2)N^{\alpha\gamma}u(p-q/2) = \bar{u}(p+q/2) \left[ \cos \beta \gamma^\alpha \frac{(1-\gamma_s)}{2} + \gamma^\alpha \frac{(1+\gamma_s)}{2} \right] \\
\times [\gamma \cdot (p-k) + M_0] \left[ \cos \beta \gamma^\gamma \frac{(1-\gamma_s)}{2} + \gamma^\gamma \frac{(1+\gamma_s)}{2} \right] u(p-q/2) \\
= \bar{u}(p+q/2)[M_0 \cos \beta \gamma^\alpha \gamma + \frac{1}{2}(1+\cos^2 \beta)\gamma^\alpha \gamma \cdot (p-k)\gamma^\gamma + \text{terms in } \gamma_s] u(p-q/2). \tag{23.11}
\]

We ignore the parity-violating form factors proportional to \( \gamma_s. \) The first term in (23.11) is the one proportional to \( M_0, \) so we neglect the term in \( 1+\cos^2 \beta \) as well. Thus, we replace (23.11) by

\[
N^{\alpha\gamma} = M_0 \cos \beta \gamma^\alpha \gamma = M_0 \cos \beta [g^\alpha \gamma - ig^\alpha \gamma]. \tag{23.12}
\]

First, we compute the term in (23.10) proportional to \( g_\alpha^\beta g_\gamma^\delta. \) It is

\[
-\frac{i}{(2\pi)^4}e^2 \int d^4k \ u(p+q/2)N^{\alpha\gamma}u(p-q/2) \left[ 2k_\mu g_\alpha^\gamma + (\frac{3}{2}q-k)_\alpha g_\mu^\gamma \right] \\
-(k+q/2)\gamma g_\alpha^\mu \right] [(p-k)^2-M_0^2+ie]^{-1}[(k-q/2)^2-\mu^2+ie]^{-1}[(k+q/2)^2+\mu^2+ie]^{-1} \tag{23.13}
\]

We parametrize the denominator in the standard way:

\[
[(p-k)^2-M_0^2+ie]^{-1}[(k-q/2)^2-\mu^2+ie]^{-1}[(k+q/2)^2+\mu^2+ie]^{-1} \\
= 2 \int_0^1 dx \int_0^{1-x} dy \left[ k^2-2k \cdot (ap-\beta q) + a\alpha^2 + \frac{1}{4}(1-\alpha)q^2 - (1-\alpha)\mu^2 - \alpha M_0^2 \right]^{-3} \tag{23.14}
\]

where

\[
\alpha = 1 - x - y, \quad \beta = \frac{1}{2}(x - y) \tag{23.15}
\]

and change the integration variable from \( k \) to \( l: \)

\[
k = l + ap - \beta q. \tag{23.16}
\]

Then, using (23.12), the expression (23.13) becomes (we omit writing the spinors)

\[
-\frac{3e^2}{4\pi^4}M_0 \cos \beta \int d^4l \ dx \ dy \ \frac{[l_\mu + (\alpha - 1)p_\mu - \beta q_\mu + m\gamma_\mu]}{[l^2 - (1-\alpha)\mu^2 + ie]^3} \tag{23.17}
\]

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where, using (23.4), we have neglected terms in the denominator proportional to $m^2$, $q^2$, and $M_0^2$. The $l_\mu$ and $\beta q_\mu$ terms vanish after integration, leaving
\[
\frac{3 \alpha M_0 \cos \beta}{4\pi} \frac{m^2}{\mu^2} (p_\mu + m_\gamma \mu).
\]

From (23.2), we conclude that this term contributes
\[
\frac{3 \alpha m M_0 \cos \beta}{4\pi} \frac{m^2}{\mu^2}
\]

(23.18)

to $F^\alpha_2(0)$.

Next we compute the “crossed” terms in eq. (23.10), i.e., those proportional to $g_\alpha^\beta (1 - 1/\xi)$ and $g_\gamma^\delta (1 - 1/\xi)$.

We use the identities
\[
k_{1\delta} \Gamma_\mu^\delta \beta = k_{2\delta}^\beta k_{2\mu} - g_\mu^\delta k_{2\delta}^\beta
\]
\[
- k_{2\beta} \Gamma_\mu^\delta \beta = k_{1\delta} k_{1\mu} - g_\mu^\delta k_{1\delta}^\beta
\]
which hold up to terms quadratic in $q$. Using (23.19), the sum of the crossed terms in $V_\mu^\alpha$ can be written
\[
\frac{-ie^2}{(2\pi)^4} M_0 \cos \beta (1 - 1/\xi) \bar{u}(p+q/2)\gamma^\alpha \gamma^\mu \gamma(p-q/2)(f_{\alpha \gamma}^{(1)} + f_{\alpha \gamma}^{(2)})
\]
(23.20)

where
\[
f_{\alpha \gamma}^{(1)} = \int d^4k g_{\alpha \delta} k_{1\gamma} (k_{1\delta}^\beta k_{2\mu}^\beta - g_\mu^\delta k_{2\delta}^\beta)(k_1^2 - u^2/\xi + ie)(k_1^2 - \mu^2 + ie)^{-1}(k_1^2 - \mu^2 + ie)^{-1}[(k-p)^2 - M_0^2 + ie]^{-1}
\]
\[
= 6 \int d^4k dw dx dy dz \delta(w + x + y + z - 1)g_{\alpha \delta} k_{1\gamma} (k_{1\delta}^\beta k_{2\mu}^\beta - g_\mu^\delta k_{2\delta}^\beta)
\]
\[
[w(k_1^2 - \mu^2/\xi)^+ + x(k_1^2 - \mu^2)^+ + y(k_1^2 - \mu^2)^+] + z[(k-p)^2 - M_0^2 + ie]^{-1}
\]
and $f_{\alpha \gamma}^{(2)}$ is defined analogously.

We define $l$ by
\[
l = l + zp + \frac{1}{2} \lambda q
\]
(23.21)

where
\[
\lambda = x - y + w
\]
and make the same approximations in the denominator as before to obtain
\[
f_{\alpha \gamma}^{(1)} = 6 \int d^4l dx dy dz dw \delta(x+y+z+w-1)
\]
\[
[0^2 - (x+y+w/\xi)\mu^2 + ie]^{-1}
\]
\[
T_{\alpha \gamma \mu}^{(1)}
\]
(23.22)

where
\[
T_{\alpha \gamma \mu}^{(1)} = [l+zp+i\lambda(\lambda-1)q]_\gamma \cdot [(l+zp+i\lambda(\lambda+1)q)_\alpha [l+zp+i\lambda(\lambda+1)q]_\mu - [l+zp+i\lambda(\lambda+1)q]^2 g_{\alpha \mu}].
\]
(23.23)
Linear and cubic terms in $l$ in (23.23) vanish upon symmetric integration. The term independent of $l$ is proportional to $m^2/\mu^2$, so we neglect it. Thus, effectively, we may write

$$ T^{(1)}_{\alpha \gamma \mu} = \frac{1}{4} l^2 \left[ g_{\alpha \gamma} [zp_\mu + \frac{1}{2} (\lambda + 1) q_\mu] + g_{\gamma \mu} [zp_\alpha + \frac{1}{2} (\lambda + 1) q_\alpha] - g_{\alpha \mu} [5zp_\gamma + \frac{1}{2} (5\lambda + 1) q_\gamma] \right] \quad (23.24) $$

$I^{(2)}$ can be worked out analogously. The total contribution of (23.20) to $V_\mu^a$ is

$$ -\frac{6i\epsilon^2}{(2\pi)^4} M_0 \cos \beta \left( 1 - 1/\xi \right) \bar{u} \gamma^a \gamma u \int \frac{d^4l \, dx \, dy \, dz \, dw \, \delta(x+y+z+w-1)}{[l^2-(x+y+w/\xi)\mu^2+i\epsilon]^4} \frac{l^2}{4} T_{\alpha \gamma \mu} \quad (23.25) $$

where, with $\mu = x - y - w$,

$$ T_{\alpha \gamma \mu} = q_{\alpha \mu} \left[ \frac{1}{2} (\mu - 5\lambda) q_\gamma - 4zp_\gamma \right] + g_{\alpha \mu} \left[ \frac{1}{2} (\lambda - 5\mu) q_\alpha - 4zp_\alpha \right] + g_{\alpha \gamma} \left[ 2zp_\mu + \frac{1}{2} (\lambda + \mu) q_\mu \right]. \quad (23.26) $$

Inserting (23.26) into (23.25) and contracting $T_{\alpha \gamma \mu}$ with $\gamma^a \gamma^i$, we obtain for the contribution (23.20) to $V_\mu^a$

$$ -\frac{9 \alpha^2 M_0 \cos \beta}{(2\pi)^4} \left( 1 - 1/\xi \right) \bar{u} q_{\mu} q^\mu u \int \frac{d^4l \, dx \, dy \, dz \, dw \, \delta(x+y+z+w-1)}{[l^2-(x+y+w/\xi)\mu^2+i\epsilon]^4} \quad (23.27) $$

and the contribution to $F_2(0)$ is

$$ -\frac{3\alpha}{2\pi} \frac{m M_0 \cos \beta}{\mu^2} \left( 1 - 1/\xi \right) \bar{u} \gamma a \gamma u \int dx \, dy \, dz \, dw \, \delta(x+y+z+w-1) \left[ \frac{w}{x+y+w/\xi} \right]. \quad (23.28) $$

The integral in (23.28) is

$$ \frac{1}{\xi} \int_0^1 \left[ (1-t) - \frac{t}{\xi} \log \left( 1 + \frac{1-t}{\xi t} \right) \right] \, dt $$

and (23.28) becomes

$$ -\frac{\alpha M_0}{\pi} \frac{m}{\mu^2} \left[ 1 - \frac{1}{2} \frac{\xi}{(\xi-1)} + \frac{1}{2} \frac{\xi}{(\xi-1)^2} \log \xi \right]. \quad (23.29) $$

The remaining term in $V_\mu^a$ [eq. (23.10)] is

$$ -\frac{i\epsilon^2}{(2\pi)^4} M_0 \cos \beta \bar{u} (p+q/2) \gamma^a \gamma u (p+q/2) (1-1/\xi)^2 \times \int d^4k \frac{k_2 a k_2^0 k_1^0 k_4^0 \Gamma_{\delta \beta \mu}}{[(k-p)^2-M_0^2+i\epsilon]^{-1}[k_1^2-\mu^2+i\epsilon]^{-1}[k_2^2-\mu^2+i\epsilon]^{-1}[k_4^2-\mu^2+i\epsilon]^{-1}}. \quad (23.30) $$

From either eqs. (23.19), it follows that

$$ k_2^2 k_4^0 T_{\delta \beta \mu} = 0 \quad (23.31) $$

so there is no contribution from (23.30). Therefore, from (23.18) and (23.29),

$$ F_2^a(0) = \frac{-\alpha m M_0}{\pi} \frac{m}{\mu^2} \left[ 1 - \frac{1}{2} \frac{\xi}{(\xi-1)} + \frac{1}{2} \frac{\xi}{(\xi-1)^2} \log \xi \right]. \quad (23.32) $$
Next we turn to the graphs of figs. 23.3b and 23.3c. The relevant vertices have been worked out in (23.6), (23.7), and (23.8). For instance,

\[
V^b_\mu = -\frac{i e^2}{(2\pi)^4} \int \mathrm{d}^4k \ \bar{u}(p+q/2)N^a_b \ u(p+q/2) \left[ -g_{\alpha\mu} \frac{(1-1/\xi)(k-q/2)_a(k-q/2)_\mu}{(k-q/2)^2 - \mu^2/\xi + i\epsilon} \right] \\
\times \left[ (k-p)^2 - M_0^2 + i\epsilon \right]^{-1} \left[ (k+q/2)^2 - \mu^2/\xi + i\epsilon \right]^{-1} \left[ (k-q/2)^2 - \mu^2 + i\epsilon \right]^{-1} \tag{23.33}
\]

where

\[
N^a_b = \left[ (m \cos \beta - M_0) \frac{1-\gamma_5}{2} + (m - M_0 \cos \beta) \frac{1+\gamma_5}{2} \right] \left[ M_0 + \gamma \cdot (p-k) \right] \left[ \cos \beta \gamma^\alpha \frac{1-\gamma_5}{2} + \gamma^\alpha \frac{1+\gamma_5}{2} \right] = M_0 \cos \beta \gamma \cdot (k-p) \gamma^\alpha - M_0^2 (1+\cos^2 \beta) \gamma^\alpha + \text{parity violating terms} + \text{terms of order } m/M_0 \tag{23.34}
\]

The second term, proportional to \( M_0^2 \), contributes terms of the order \( m/\mu \) times those we are keeping to \( F^b_2(0) \), so we shall neglect it.

Similarly, we take \( V^c_\mu \) to be

\[
V^c_\mu = -\frac{i e^2}{(2\pi)^4} M_0 \cos \beta \int \mathrm{d}^4k \ \bar{u}(p+q/2) \gamma^\alpha \gamma \cdot (k-p) \ u(p-k/2) \\
= \frac{-g_{\alpha\mu} + (1-1/\xi)(k+q/2)_a(k+q/2)_\mu / (k+q/2)^2 - \mu^2/\xi + i\epsilon}{[(k-p)^2 - M_0^2 + i\epsilon] \left[ (k+q/2)^2 - \mu^2/\xi + i\epsilon \right]^{-1} \left[ (k-q/2)^2 - \mu^2/\xi + i\epsilon \right]} \tag{23.35}
\]

First we calculate the terms in (23.33) and (23.35) proportional to \( g_{\alpha\mu} \). This can be done by introducing Feynman parameters into (23.33) just as we did in eq. (23.14), and interchanging them in (23.34) to obtain a common denominator. The calculation is straightforward, with the result that the contribution of these terms to \( F^b_2(0) + F^c_2(0) \) is

\[
\frac{\alpha m M_0 \cos \beta}{\pi \mu^2} \int_0^1 \int_0^{1-x} \frac{x}{(x/\xi + y)} \frac{1}{x \log \left[ 1 + \frac{1-x}{\xi x} \right]} \mathrm{d}y \mathrm{d}x \tag{23.36}
\]

There remain the terms proportional to \((1-1/\xi)\) in (23.33) and (23.35). We can parametrize them just as we did the “crossed” terms in fig. 23.3a to obtain

\[
-\frac{6i e^2}{(2\pi)^4} (1-1/\xi) M_0 \cos \beta \int \mathrm{d}^4k \ \mathrm{d}w \mathrm{d}x \mathrm{d}y \mathrm{d}z \ \delta(w+x+y+z-1) \\
\times \bar{u}(p+q/2) \left[ \frac{\gamma \cdot (k-p) \gamma \cdot k_1}{[w(k_1^2 - \mu^2) + x(k_2^2 - \mu^2/\xi) + y(k_3^2 - \mu^2/\xi) + z((k-p)^2 - M_0^2)] + i\epsilon} \right]^4 \\
+ \frac{\gamma \cdot k_2 \gamma \cdot (k-p)}{[w(k_1^2 - \mu^2) + x(k_2^2 - \mu^2/\xi) + y(k_3^2 - \mu^2/\xi) + z((k-p)^2 - M_0^2)] + i\epsilon} \right]^4 \ u(p-q/2). \tag{23.27}
\]
Define

\[ k = l + zp + \lambda q/2 \quad (\lambda = x - y + w) \]

in the first term, and

\[ k = l + zp + \mu q/2 \quad (\mu = x - y - w) \]

in the second. In the approximations (23.4) and (23.5), the denominators in both terms are the same. Up to terms of order \( m^2/\mu^2 \), (23.37) becomes, after doing the \( l \)-integration,

\[
\frac{-e^2 M_0 \cos \beta}{32\pi^2 \mu^2} \left[ \frac{1}{1-\xi} \right] \frac{dwdxdy dx}{\delta(x+y+z+w-1) \left((12z-4)p_\mu + 3(x-y)q_\mu \right)}
\]

(23.38)

The expression (23.38) is identically zero for all \( \xi \). We conclude that

\[
F_2^\mu(0) + F_2^\nu(0) = -\frac{\alpha}{2\pi} \frac{m M_0 \cos \beta}{\mu^2} \left[ \xi \left(\frac{\xi}{\xi-1} \right) - \frac{1}{2} \left(\xi-1\right)^2 \log \xi \right].
\]

(23.39)

Finally, we compute the contribution of fig. 23.3d. The \( A_\mu \) vertex is the usual charged scalar meson electromagnetic vertex, and the other two are given by (23.7). Thus

\[
V_\mu^d = -\frac{i e^2}{(2\pi)^2} \mu^2 \left[ (m \cos \beta - M_0) \frac{(1+\gamma_s)}{2} + (m - M_0 \cos \beta) \frac{(1-\gamma_s)}{2} \right] \gamma \cdot (p-k) + M_0
\]

\times \left[ (m \cos \beta - M_0) \frac{(1-\gamma_s)}{2} + (m - M_0 \cos \beta) \frac{(1+\gamma_s)}{2} \right].
\]

(23.40)

Therefore

\[
V_\mu^d = -\frac{2ie^2 m^3 \cos \beta}{(2\pi)^2} \mu^2 \int d^4k \frac{k_\mu [k_\beta - \mu^2/\xi + i\epsilon]}{\xi-1} [k_2^2 - \mu^2/\xi + i\epsilon]^{-1} \left[(p-k)^2 - M_0^2 + i\epsilon\right]^{-1}
\]

+ terms smaller by at least \( m/M_0 \).

Provided \( M_0^2 \ll \mu^2/\xi \), the integral in (23.42) is of order \( \mu^2 \), so that, in our approximation,

\[
V_\mu^d = F_2^d(0) = 0.
\]

(23.43)

From (23.32), (23.39), and (23.43), we conclude that the leading term in \( F_2(0) \) is

\[
F_2(0) = F_2^\mu(0) + F_2^\nu(0) + F_2^\pi(0) + F_2^d(0) = -\frac{\alpha}{\pi} \frac{m M_0}{\mu^2}
\]

(23.44)

which is independent of \( \xi \) as expected.
This result is independent of our limitation on $\xi$. In the Landau gauge limit, $\xi \to \infty$, the integral in (23.42) is evidently of order $M_0^2$ instead of $\mu^2$, and in that case

$$F_2^d(0) = \frac{-\alpha}{4\pi} \frac{m M_0}{\mu^2} \quad (\xi \to \infty).$$

(23.45)

Careful treatment of the graphs in figs. 23.3b and 23.3c shows that $F_2^b(0) + F_2^c(0)$ also has the value (23.45) in that limit. The calculation leading to the expression (23.22) for $F_2^d(0)$ is correct in this limit, so that

$$F_2^a(0) = \frac{-\alpha}{2\pi} \frac{m M_0}{\mu^2} \quad (\xi \to \infty).$$

(23.46)

The sum is still given by (23.44), independently of the approximation (23.5).

What is the experimental situation? Using eqs. (9.14) and (9.27) we can rewrite (23.44) as

$$F_2(0) = -G m M_*/2\pi^2 \sqrt{2} \sin^2 \beta$$

(23.47)

where $M_*$ is the mass of the charged heavy muon.

Electromagnetic corrections to $F_2(0)$ have been calculated up to sixth order in quantum electrodynamics, with the result

$$F_2^{\text{QED}}(0) = (116582 \pm 1) \times 10^{-8}.$$  

(23.48)

Hadron corrections to the photon propagator have been estimated to add $(6.5 \pm 0.5) \times 10^{-8}$ to this value, so that, neglecting weak corrections, the theoretical prediction is

$$F_2^{\text{Th}}(0) = (116589 \pm 2) \times 10^{-8}.$$  

(23.49)

The most recent available experimental figure is

$$F_2^{\text{Exp}}(0) = (116618 \pm 32) \times 10^{-8}$$  

(23.50)

so that the theoretical value is well within the present experimental error without adding any weak corrections.

The weak correction calculated in eq. (23.47) has the value

$$F_2^{\text{w}}(0) = -4.5 \times 10^{-9} (M_*/m)/\sin^2 \beta.$$  

(23.51)

We know that $M_*/m > M_K/m = 4.7$, so that $|F_2^{\text{W}}(0)|$ in the heavy muon model is at least $2.14 \times 10^{-8}$. Let us take two standard deviations as a reasonable lower limit for the true experimental value for $F_2(0)$, so that $F_2^{\text{Th}}(0) + F_2^{\text{W}}(0) > 116542 \times 10^{-8}$. Then $F_2^{\text{W}}(0)$ must be less than $47 \times 10^{-8}$, and $(M_*/m)/\sin^2 \beta$ must be less than 100. Thus

$$M_* < 10 \text{ GeV}.$$  

(23.52)

Clearly, a more accurate measurement of $F_2(0)$ could put a much lower upper bound on the mass of the heavy muon.
Bibliography

An exact calculation of $F_2(q^2)$ in the $R_T$ gauge in the Weinberg model and in both heavy-lepton models discussed in section 9, has been performed by

The magnetic moment was also calculated in the Weinberg model using the U-gauge by
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and in the Georgi-Glashow model, also in the U-gauge, by

These U-gauge calculations are simpler, since all the graphs containing charged scalar mesons are absent, but the remaining graphs are not unambiguously convergent, and must be treated carefully.

Most of the recent calculations of higher order corrections of weak interactions are reviewed by

The numerical experimental and theoretical values for $F_2(0)$ were taken from

Is it to be concluded that to be on intimate terms with nature has a soothing influence, while the passion to penetrate the mystery lying behind appearances provokes expenditure of nervous energy which ultimately wears out the body and soul?

Germain Bazin