Ettore Majorana: Unpublished Research Notes on Theoretical Physics
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“But, then, there are geniuses like Galileo and Newton. Well, Ettore Majorana was one of them...”

Enrico Fermi (1938)
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Without listing his works, all of which are highly notable both for the originality of the methods utilized as well as for the importance of the results achieved, we limit ourselves to the following:

In modern nuclear theories, the contribution made by this researcher to the introduction of the forces called ‘Majorana forces’ is universally recognized as the one, among the most fundamental, that permits us to theoretically comprehend the reasons for nuclear stability. The work of Majorana today serves as a basis for the most important research in this field.

In atomic physics, the merit of having resolved some of the most intricate questions on the structure of spectra through simple and elegant considerations of symmetry is due to Majorana.

Lastly, he devised a brilliant method that permits us to treat the positive and negative electron in a symmetrical way, finally eliminating the necessity to rely on the extremely artificial and unsatisfactory hypothesis of an infinitely large electrical charge diffused in space, a question that had been tackled in vain by many other scholars [4].

With this justification, the judging committee of the 1937 competition for a new full professorship in theoretical physics at Palermo, chaired by Enrico Fermi (and including Enrico Persico, Giovanni Polvani and Antonio Carrelli), suggested the Italian Minister of National Education should appoint Ettore Majorana “independently of the competition rules, as full professor of theoretical physics in a university of the Italian kingdom\(^1\) because of his high and well-deserved reputation” [4]. Evidently, to gain such a high reputation the few papers that the Italian scientist had chosen to publish were enough. It is interesting to note that proper light was shed by Fermi on Majorana’s symmetrical approach to electrons and antielectrons (today climaxing in its application to neutrinos and antineutrinos) and on its ability to eliminate the hypothesis\(^1\)

\(^1\)Which happened to be the University of Naples.
known as the “Dirac sea”, a hypothesis that Fermi defined as “extremely artificial and unsatisfactory”, despite the fact that in general it had been uncritically accepted. However, one of the most important works of Majorana, the one that introduced his “infinite-components equation” was not mentioned: it had not been understood yet, even by Fermi and his colleagues.

Bruno Pontecorvo [2], a younger colleague of Majorana at the Institute of Physics in Rome, in a similar way recalled that “some time after his entry into Fermi’s group, Majorana already possessed such an erudition and had reached such a high level of comprehension of physics that he was able to speak on the same level with Fermi about scientific problems. Fermi himself held him to be the greatest theoretical physicist of our time. He often was astounded ....”

Majorana’s fame rests solidly on testimonies like these, and even more on the following ones.

At the request of Edoardo Amaldi [1], Giuseppe Cocconi wrote from CERN (18 July 1965):

In January 1938, after having just graduated, I was invited, essentially by you, to come to the Institute of Physics at the University of Rome for six months as a teaching assistant, and once I was there I would have the good fortune of joining Fermi, Gilberto Bernardini (who had been given a chair at Camerino University a few months earlier) and Mario Ageno (he, too, a new graduate) in the research of the products of disintegration of $\mu$ “mesons” (at that time called mesotrons or yukons), which are produced by cosmic rays....

A few months later, while I was still with Fermi in our workshop, news arrived of Ettore Majorana’s disappearance in Naples. I remember that Fermi busied himself with telephoning around until, after some days, he had the impression that Ettore would never be found.

It was then that Fermi, trying to make me understand the significance of this loss, expressed himself in quite a peculiar way; he who was so objectively harsh when judging people. And so, at this point, I would like to repeat his words, just as I can still hear them ringing in my memory: ‘Because, you see, in the world there are various categories of scientists: people of a secondary or tertiary standing, who do their best but do not go very far. There are also those of high standing, who come to discoveries of great importance, fundamental for the development of science’ (and here I had the impression that he placed himself in that category). ‘But then there are geniuses like Galileo and Newton. Well, Ettore was one of them. Majorana had what no one else in the world had ...’.

Fermi, who was rather severe in his judgements, again expressed himself in an unusual way on another occasion. On 27 July 1938 (after
Majorana’s disappearance, which took place on 26 March 1938), writing from Rome to Prime Minister Mussolini to ask for an intensification of the search for Majorana, he stated: “I do not hesitate to declare, and it would not be an overstatement in doing so, that of all the Italian and foreign scholars that I have had the chance to meet, Majorana, for his depth of intellect, has struck me the most” [4].

But, nowadays, some interested scholars may find it difficult to appreciate Majorana’s ingeniousness when basing their judgement only on his few published papers (listed below), most of them originally written in Italian and not easy to trace, with only three of his articles having been translated into English [9, 10, 11, 12, 28] in the past. Actually, only in 2006 did the Italian Physical Society eventually publish a book with the Italian and English versions of Majorana’s articles [13].

Anyway, Majorana has also left a lot of unpublished manuscripts relating to his studies and research, mainly deposited at the Domus Galilaeana in Pisa (Italy), which help to illuminate his abilities as a theoretical physicist, and mathematician too.

The year 2006 was the 100th anniversary of the birth of Ettore Majorana, probably the brightest Italian theoretician of the twentieth century, even though to many people Majorana is known mainly for his mysterious disappearance, in 1938, at the age of 31. To celebrate such a centenary, we had been working—among others—on selection, study, typographical setting in electronic form and translation into English of the most important research notes left unpublished by Majorana: his so-called Quaderni (booklets); leaving aside, for the moment, the notable set of loose sheets that constitute a conspicuous part of Majorana’s manuscripts. Such a selection is published for the first time, with some understandable delay, in this book. In a previous volume [15], entitled Ettore Majorana: Notes on Theoretical Physics, we analogously published for the first time the material contained in different Majorana booklets—the so-called Volumetti, which had been written by him mainly while studying physics and mathematics as a student and collaborator of Fermi. Even though Ettore Majorana: Notes on Theoretical Physics contained many highly original findings, the preparation of the present book remained nevertheless a rather necessary enterprise, since the research notes publicited in it are even more (and often exceptionally) interesting, revealing more fully Majorana’s genius. Many of the results we will cover on the hundreds of pages that follow are novel and even today, more than seven decades later, still of significant importance for contemporary theoretical physics.
Historical prelude

For nonspecialists, the name of Ettore Majorana is frequently associated with his mysterious disappearance from Naples, on 26 March 1938, when he was only 31; afterwards, in fact, he was never seen again.

But the myth of his “disappearance” [4] has contributed to nothing but the fame he was entitled to, for being a genius well ahead of his time.

Ettore Majorana was born on 5 August 1906 at Catania, Sicily (Italy), to Fabio Majorana and Dorina Corso. The fourth of five sons, he had a rich scientific, technological and political heritage: three of his uncles had become vice-chancellors of the University of Catania and members of the Italian parliament, while another, Quirino Majorana, was a renowned experimental physicist, who had been, by the way, a former president of the Italian Physical Society.

Ettore’s father, Fabio, was an engineer who had founded the first telephone company in Sicily and who went on to become chief inspector of the Ministry of Communications. Fabio Majorana was responsible for the education of his son in the first years of his school-life, but afterwards Ettore was sent to study at a boarding school in Rome. Eventually, in 1921, the whole family moved from Catania to Rome. Ettore finished high school in 1923 when he was 17, and then joined the Faculty of Engineering of the local university, where he excelled, and counted Giovanni Gentile Jr., Enrico Volterra, Giovanni Enriques and future Nobel laureate Emilio Segrè among his friends.

In the spring of 1927 Orso Mario Corbino, the director of the Institute of Physics at Rome and an influential politician (who had succeeded in elevating to full professorship the 25-year-old Enrico Fermi, just with the intention of enabling Italian physics to make a quality jump) launched an appeal to the students of the Faculty of Engineering, inviting the most brilliant young minds to study physics. Segrè and Edoardo Amaldi rose to the challenge, joining Fermi and Franco Rasetti’s group, and telling them of Majorana’s exceptional gifts. After some encouragement from Segrè and Amaldi, Majorana eventually decided to meet Fermi in the autumn of that year.

The details of Majorana and Fermi’s first meeting were narrated by Segrè [3], Rasetti and Amaldi. The first important work written by Fermi in Rome, on the statistical properties of the atom, is today known as the Thomas–Fermi method. Fermi had found that he needed the solution to a nonlinear differential equation characterized by unusual boundary conditions, and in a week of assiduous work he had calculated the solution with a little hand calculator. When Majorana met Fermi for the first time, the latter spoke about his equation, and showed his
numerical results. Majorana, who was always very sceptical, believed Fermi’s numerical solution was probably wrong. He went home, and solved Fermi’s original equation in analytic form, evaluating afterwards the solution’s values without the aid of a calculator. Next morning he returned to the Institute and sceptically compared the results which he had written on a little piece of paper with those in Fermi’s notebook, and found that their results coincided exactly. He could not hide his amazement, and decided to move from the Faculty of Engineering to the Faculty of Physics. We have indulged ourselves in the foregoing anecdote since the pages on which Majorana solved Fermi’s differential equation were found by one of us (S.E.) years ago. And recently [22] it was explicitly shown that he followed that night two independent paths, the first of them leading to an Abel equation, and the second one resulting in his devising a method still unknown to mathematics. More precisely, Majorana arrived at a series solution of the Thomas–Fermi equation by using an original method that applies to an entire class of mathematical problems. While some of Majorana’s results anticipated by several years those of renowned mathematicians or physicists, several others (including his final solution to the equation mentioned) have not been obtained by anyone else since. Such facts are further evidence of Majorana’s brilliance.

Majorana’s published articles

Majorana published few scientific articles: nine, actually, besides his sociology paper entitled “Il valore delle leggi statistiche nella fisica e nelle scienze sociali” (“The value of statistical laws in physics and the social sciences”), which was, however, published not by Majorana but (posthumously) by G. Gentile Jr., in *Scientia* (36:55–56, 1942), and much later was translated into English. Majorana switched from engineering to physics studies in 1928 (the year in which he published his first article, written in collaboration with his friend Gentile) and then went on to publish his works on theoretical physics for only a few years, practically only until 1933. Nevertheless, even his published works are a mine of ideas and techniques of theoretical physics that still remain largely unexplored. Let us list his nine published articles, which only in 2006 were eventually reprinted together with their English translations [13]:


While still an undergraduate, in 1928 Majorana published his first paper, (1), in which he calculated the splitting of certain spectroscopic terms in gadolinium, uranium and caesium, owing to the spin of the electrons. At the end of that same year, Fermi invited Majorana to give a talk at the Italian Physical Society on some applications of the Thomas–Fermi model [23] (attention to which was drawn by F. Guerra and N. Robotti). Then on 6 July 1929, Majorana was awarded his master’s degree in physics, with a dissertation having as a subject “The quantum theory of radioactive nuclei”.

By the end of 1931 the 25-year-old physicist had published two articles, (2) and (4), on the chemical bonds of molecules, and two more papers, (3) and (5), on spectroscopy, one of which, (3), anticipated results later obtained by a collaborator of Samuel Goudsmith on the “Auger effect” in helium. As Amaldi has written, an in-depth examination of these works leaves one struck by their quality: they reveal both deep knowledge of the experimental data, even in the minutest detail, and an uncommon ease, without equal at that time, in the use of the symmetry properties of the quantum states to qualitatively simplify problems and choose the most suitable method for their quantitative resolution.

In 1932, Majorana published an important paper, (6), on the nonadiabatic spin-flip of atoms in a magnetic field, which was later extended by Nobel laureate Rabi in 1937, and by Bloch and Rabi in 1945. It established the theoretical basis for the experimental method used to reverse the spin also of neutrons by a radio-frequency field, a method that
is still practised today, for example, in all polarized-neutron spectrometers. That paper contained an independent derivation of the well-known Landau–Zener formula (1932) for nonadiabatic transition probability. It also introduced a novel mathematical tool for representing spherical functions or, rather, for representing spinors by a set of points on the surface of a sphere (Majorana sphere), attention to which was drawn not long ago by Penrose and collaborators [29] (and by Leonardi and coworkers [30]). In the present volume the reader will find some additions (or modifications) to the above-mentioned published articles.

However, the most important 1932 paper is that concerning a relativistic field theory of particles with arbitrary spin, (7). Around 1932 it was commonly believed that one could write relativistic quantum equations only in the case of particles with spin 0 or 1/2. Convinced of the contrary, Majorana—as we have known for a long time from his manuscripts, constituting a part of the Quaderni finally published here—began constructing suitable quantum-relativistic equations for higher spin values (1, 3/2, etc.); and he even devised a method for writing the equation for a generic spin value. But still he published nothing, until he discovered that one could write a single equation to cover an infinite family of particles of arbitrary spin (even though at that time the known particles could be counted on one hand). To implement his programme with these “infinite-components” equations, Majorana invented a technique for the representation of a group several years before Eugene Wigner did. And, what is more, Majorana obtained the infinite-dimensional unitary representations of the Lorentz group that would be rediscovered by Wigner in his 1939 and 1948 works. The entire theory was reinvented in a Soviet series of articles from 1948 to 1958, and finally applied by physicists years later. Sadly, Majorana’s initial article remained in the shadows for a good 34 years until Fradkin [28], informed by Amaldi, realized what Majorana many years earlier had accomplished. All the scientific material contained in (and in preparation for) this publication of Majorana’s works is illuminated by the manuscripts published in the present volume.

At the beginning of 1932, as soon as the news of the Joliot–Curie experiments reached Rome, Majorana understood that they had discovered the “neutral proton” without having realized it. Thus, even before the official announcement of the discovery of the neutron, made soon afterwards by Chadwick, Majorana was able to explain the structure and stability of light atomic nuclei with the help of protons and neutrons,

2Starting in 1974, some of us [21] published and reevaluated only a few of the pages devoted in Majorana’s manuscripts to the case of a Dirac-like equation for the photon (spin-1 case).
antedating in this way also the pioneering work of D. Ivanenko, as both Segré and Amaldi have recounted. Majorana’s colleagues remember that even before Easter he had concluded that protons and neutrons (indistinguishable with respect to the nuclear interaction) were bound by the “exchange forces” originating from the exchange of their spatial positions alone (and not also of their spins, as Heisenberg would propose), so as to produce the $\alpha$ particle (and not the deuteron) as saturated with respect to the binding energy. Only after Heisenberg had published his own article on the same problem was Fermi able to persuade Majorana to go for a 6-month period, in 1933, to Leipzig and meet there his famous colleague (who would be awarded the Nobel prize at the end of that year); and finally Heisenberg was able to convince Majorana to publish his results in the paper “Über die Kerntheorie”. Actually, Heisenberg had interpreted the nuclear forces in terms of nucleons exchanging spinless electrons, as if the neutron were formed in practice by a proton and an electron, whereas Majorana had simply considered the neutron as a “neutral proton”, and the theoretical and experimental consequences were quickly recognized by Heisenberg. Majorana’s paper on the stability of nuclei soon became known to the scientific community—a rare event, as we know—thanks to that timely “propaganda” made by Heisenberg himself, who on several occasions, when discussing the “Heisenberg–Majorana” exchange forces, used, rather fairly and generously, to point out more Majorana’s than his own contributions [33]. The manuscripts published in the present book refer also to what Majorana wrote down before having read Heisenberg’s paper. Let us seize the present opportunity to quote two brief passages from Majorana’s letters from Leipzig. On 14 February 1933, he wrote to his mother (the italics are ours): “The environment of the physics institute is very nice. I have good relations with Heisenberg, with Hund, and with everyone else. I am writing some articles in German. The first one is already ready...” [4]. The work that was already ready is, naturally, the cited one on nuclear forces, which, however, remained the only paper in German. Again, in a letter dated 18 February, he told his father (our italics): “I will publish in German, after having extended it, also my latest article which appeared in Il Nuovo Cimento” [4].

But Majorana published nothing more, either in Germany—where he had become acquainted, besides with Heisenberg, with other renowned scientists, including Ehrenfest, Bohr, Weisskopf and Bloch—or after his return to Italy, except for the article (in 1937) of which we are about to speak. It is therefore important to know that Majorana was engaged in writing other papers: in particular, he was expanding his article about the infinite-components equations. His research activity during the years 1933–1937 is testified by the documents presented in this volume, and
particularly by a number of unpublished scientific notes, some of which are reproduced here: as far as we know, it focused mainly on field theory and quantum electrodynamics. As already mentioned, in 1937 Majorana decided to compete for a full professorship (probably with the only desire to have students); and he was urged to demonstrate that he was still actively working in theoretical physics. Happily enough, he took from a drawer his writing on the symmetrical theory of electrons and antielectrons, publishing it that same year under the title “Symmetric theory of electrons and positrons”. This paper—at present probably the most famous of his—was initially noticed almost exclusively for having introduced the Majorana representation of the Dirac matrices in real form. But its main consequence is that a neutral fermion can be identical with its antiparticle. Let us stress that such a theory was rather revolutionary, since it was at variance with what Dirac had successfully assumed in order to solve the problem of negative energy states in quantum field theory. With rare daring, Majorana suggested that neutrinos, which had just been postulated by Pauli and Fermi to explain puzzling features of radioactive $\beta$ decay, could be particles of this type. This would enable the neutrino, for instance, to have mass, which may have a bearing on the phenomena of neutrino oscillations, later postulated by Pontecorvo.

It may be stressed that, exactly as in the case of other writings of his, the “Majorana neutrino” too started to gain prominence only decades later, beginning in the 1950s; and nowadays expressions such as Majorana spinors, Majorana mass and even “majorons” are fashionable. It is moreover well known that many experiments are currently devoted the world over to checking whether the neutrinos are of the Dirac or the Majorana type. We have already said that the material published by Majorana (but still little known, despite everything) constitutes a potential gold mine for physics. Many years ago, for example, Bruno Touschek noticed that the article entitled “Symmetric theory of electrons and positrons” implicitly contains also what he called the theory of the “Majorana oscillator”, described by the simple equation $q + \omega^2 q = \varepsilon \delta(t)$, where $\varepsilon$ is a constant and $\delta$ is the Dirac function [4]. According to Touschek, the properties of the Majorana oscillator are very interesting, especially in connection with its energy spectrum; but no literature seems to exist on it yet.

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3As we said, from the existing manuscripts it appears that Majorana had formulated also the essential lines of his paper (9) during the years 1932–1933.
An account of the unpublished manuscripts

The largest part of Majorana’s work was left unpublished. Even though the most important manuscripts have probably been lost, we are now in possession of (1) his M.Sc. thesis on “The quantum theory of radioactive nuclei”; (2) five notebooks (the Volumetti) and 18 booklets (the Quaderni); (3) 12 folders with loose papers; and (4) the set of his lecture notes for the course on theoretical physics given by him at the University of Naples. With the collaboration of Amaldi, all these manuscripts were deposited by Luciano Majorana (Ettore’s brother) at the Domus Galilaeana in Pisa. An analysis of those manuscripts allowed us to ascertain that they, except for the lectures notes, appear to have been written approximately by 1933 (even the essentials of his last article, which Majorana proceeded to publish, as we already know, in 1937, seem to have been ready by 1933, the year in which the discovery of the positron was confirmed). Besides the material deposited at the Domus Galilaeana, we are in possession of a series of 34 letters written by Majorana between 17 March 1931 and 16 November 1937, in reply to his uncle Quirino—a renowned experimental physicist and a former president of the Italian Physical Society—who had been pressing Majorana for help in the theoretical explanation of his experiments. Such letters have recently been deposited at Bologna University, and have been published in their entirety by Dragoni [8]. They confirm that Majorana was deeply knowledgeable even about experimental details. Moreover, Ettore’s sister, Maria, recalled that, even in those years, Majorana—who had reduced his visits to Fermi’s institute, starting from the beginning of 1934 (that is, just after his return from Leipzig)—continued to study and work at home for many hours during the day and at night. Did he continue to dedicate himself to physics? From one of those letters of his to Quirino, dated 16 January 1936, we find a first answer, because we learn that Majorana had been occupied “for some time, with quantum electrodynamics”; knowing Majorana’s love for understatements, this no doubt means that during 1935 he had performed profound research at least in the field of quantum electrodynamics.

This seems to be confirmed by a recently retrieved text, written by Majorana in French [25], where he dealt with a peculiar topic in quantum electrodynamics. It is instructive, as to that topic, to quote directly from Majorana’s paper.

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4In the past, one of us (E.R.) was able to publish only short passages of them, since they are rather technical; see [4].
Let us consider a system of \( p \) electrons and set the following assumptions:

1) the interaction between the particles is sufficiently small, allowing us to speak about individual quantum states, so that one may regard the quantum numbers defining the configuration of the system as good quantum numbers; 2) any electron has a number \( n > p \) of inner energy levels, while any other level has a much greater energy. One deduces that the states of the system as a whole may be divided into two classes. The first one is composed of those configurations for which all the electrons belong to one of the inner states. Instead, the second one is formed by those configurations in which at least one electron belongs to a higher level not included in the above-mentioned \( n \) levels. We shall also assume that it is possible, with a sufficient degree of approximation, to neglect the interaction between the states of the two classes. In other words, we will neglect the matrix elements of the energy corresponding to the coupling of different classes, so that we may consider the motion of the \( p \) particles, in the \( n \) inner states, as if only these states existed. Our aim becomes, then, translating this problem into that of the motion of \( n - p \) particles in the same states, such new particles representing the holes, according to the Pauli principle.

Majorana, thus, applied the formalism of field quantization to Dirac’s hole theory, obtaining a general expression for the quantum electrodynamics Hamiltonian in terms of anticommuting “hole quantities”. Let us point out that in justifying the use of anticommutators for fermionic variables, Majorana commented that such a use “cannot be justified on general grounds, but only by the particular form of the Hamiltonian. In fact, we may verify that the equations of motion are better satisfied by these relations than by the Heisenberg ones.” In the second (and third) part of the same manuscript, Majorana took into consideration also a reformulation of quantum electrodynamics in terms of a photon wavefunction, a topic that was particularly studied in his Quaderni (and is reproduced here). Majorana, indeed, reformulated quantum electrodynamics by introducing a real-valued wavefunction for the photon, corresponding only to directly observable degrees of freedom.

In some other manuscripts, probably prepared for a seminar at Naples University in 1938 [24], Majorana set forth a physical interpretation of quantum mechanics that anticipated by several years the Feynman approach in terms of path integrals. The starting point in Majorana’s notes was to search for a meaningful and clear formulation of the concept of quantum state. Afterwards, the crucial point in the Feynman formulation of quantum mechanics (namely that of considering not only the paths corresponding to classical trajectories, but all the possible paths joining an initial point with the final point) was really introduced by Majorana, after a discussion about an interesting example of a harmonic oscillator. Let us also emphasize the key role played by the
symmetry properties of the physical system in the Majorana analysis, a feature quite common in his papers.

Do any other unpublished scientific manuscripts of Majorana exist? The question, raised by his answer to Quirino and by his letters from Leipzig to his family, becomes of greater importance when one reads also his letters addressed to the National Research Council of Italy (CNR) during that period. In the first one (dated 21 January 1933), he asserts: “At the moment, I am occupied with the elaboration of a theory for the description of arbitrary-spin particles that I began in Italy and of which I gave a summary notice in Il Nuovo Cimento ....” [4]. In the second one (dated 3 March 1933) he even declares, referring to the same work: “I have sent an article on nuclear theory to Zeitschrift für Physik. I have the manuscript of a new theory on elementary particles ready, and will send it to the same journal in a few days” [4]. Considering that the article described above as a “summary notice” of a new theory was already of a very high level, one can imagine how interesting it would be to discover a copy of its final version, which went unpublished. (Is it still, perhaps, in the Zeitschrift für Physik archives? Our search has so far ended in failure.)

A few of Majorana’s other ideas which did not remain concealed in his own mind have survived in the memories of his colleagues. One such reminiscence we owe to Gian-Carlo Wick. Writing from Pisa on 16 October 1978, he recalls:

The scientific contact [between Ettore and me], mentioned by Segré, happened in Rome on the occasion of the ‘A. Volta Congress’ (long before Majorana’s sojourn in Leipzig). The conversation took place in Heitler’s company at a restaurant, and therefore without a blackboard ...; but even in the absence of details, what Majorana described in words was a ‘relativistic theory of charged particles of zero spin based on the idea of field quantization’ (second quantization). When much later I saw Pauli and Weisskopf’s article [Helv. Phys. Acta 7 (1934) 709], I remained absolutely convinced that what Majorana had discussed was the same thing ... [4, 26].

Teaching theoretical physics

As we have seen, Majorana contributed significantly to theoretical research which was among the frontier topics in the 1930s, and, indeed, in the following decades. However, he deeply thought also about the basics, and applications, of quantum mechanics, and his lectures on theoretical physics provide evidence of this work of his.
As realized only recently [34], Majorana had a genuine interest in advanced physics teaching, starting from 1933, just after he obtained, at the end of 1932, the degree of libero docente (analogous to the German Privatdozent title). As permitted by that degree, he requested to be allowed to give three subsequent annual free courses at the University of Rome, between 1933 and 1937, as testified by the lecture programmes proposed by him and still present in Rome University’s archives. Such documents also refer to a period of time that was regarded by his colleagues as Majorana’s “gloomy years”. Although it seems that Majorana never delivered these three courses, probably owing to lack of appropriate students, the topics chosen for the lectures appear very interesting and informative.

The first course (academic year 1933–1934) proposed by Majorana was on mathematical methods of quantum mechanics. The second course (academic year 1935–1936) proposed was on mathematical methods of atomic physics. Finally, the third course (academic year 1936–1937) proposed was on quantum electrodynamics.

Majorana could actually lecture on theoretical physics only in 1938 when, as recalled above, he obtained his position as a full professor in Naples. He gave his lectures starting on 13 January and ending with his disappearance (26 March), but his activity was intense, and his interest in teaching was very high. For the benefit of his students, and perhaps

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5The programme for it contained the following topics: (1) unitary geometry, linear transformations, Hermitian operators, unitary transformations, and eigenvalues and eigenvectors; (2) phase space and the quantum of action, modifications of classical kinematics, and general framework of quantum mechanics; (3) Hamiltonians which are invariant under a transformation group, transformations as complex quantities, noncompatible systems, and representations of finite or continuous groups; (4) general elements on abstract groups, representation theorems, the group of spatial rotations, and symmetric groups of permutations and other finite groups; (5) properties of the systems endowed with spherical symmetry, orbital and intrinsic momenta, and theory of the rigid rotator; (6) systems with identical particles, Fermi and Bose–Einstein statistics, and symmetries of the eigenfunctions in the centre-of-mass frames; (7) Lorentz group and spinor calculus, and applications to the relativistic theory of the elementary particles.

6The corresponding subjects were matrix calculus, phase space and the correspondence principle, minimal statistical sets or elementary cells, elements of quantum dynamics, statistical theories, general definition of symmetry problems, representations of groups, complex atomic spectra, kinematics of the rigid body, diatomic and polyatomic molecules, relativistic theory of the electron and the foundations of electrodynamics, hyperfine structures and alternating bands, and elements of nuclear physics.

7The main topics were relativistic theory of the electron, quantization procedures, field quantities defined by commutability and anticommutability laws, their kinematic equivalence with sets with an undetermined number of objects obeying Bose–Einstein or Fermi statistics, respectively, dynamical equivalence, quantization of the Maxwell–Dirac equations, study of relativistic invariance, the positive electron and the symmetry of charges, several applications of the theory, radiation and scattering processes, creation and annihilation of opposite charges, and collisions of fast electrons.
also for writing a book, he prepared careful lecture notes [17, 18]. A recent analysis [36] showed that Majorana’s 1938 course was very innovative for that time, and this has been confirmed by the retrieval (in September 2004) of a faithful transcription of the whole set of Majorana’s lecture notes (the so-called Moreno document) comprising the six lectures not included in the original collection [19].

The first part of his course on theoretical physics dealt with the phenomenology of atomic physics and its interpretation in the framework of the old Bohr–Sommerfeld quantum theory. This part has a strict analogy with the course given by Fermi in Rome (1927–1928), attended by Majorana when a student. The second part started, instead, with classical radiation theory, reporting explicit solutions to the Maxwell equations, scattering of solar light and some other applications. It then continued with the theory of relativity: after the presentation of the corresponding phenomenology, a complete discussion of the mathematical formalism required by that theory was given, ending with some applications such as the relativistic dynamics of the electron. Then, there followed a discussion of important effects for the interpretation of quantum mechanics, such as the photoelectric effect, Thomson scattering, Compton effects and the Franck–Hertz experiment. The last part of the course, more mathematical in nature, treated explicitly quantum mechanics, both in the Schrödinger and in the Heisenberg formulations. This part did not follow the Fermi approach, but rather referred to personal previous studies, getting also inspiration from Weyl’s book on group theory and quantum mechanics.

A brief sketch of Ettore Majorana: Notes on Theoretical Physics

In Ettore Majorana: Notes on Theoretical Physics we reproduced, and translated, Majorana’s Volumetti: that is, his study notes, written in Rome between 1927 and 1932. Each of those neatly organized booklets, prefaced by a table of contents, consisted of about 100–150 sequentially numbered pages, while a date, penned on its first blank page, recorded the approximate time during which it was completed. Each Volumetto was written during a period of about 1 year. The contents of those notebooks range from typical topics covered in academic courses to topics at the frontiers of research: despite this unevenness in the level of sophistication, the style is never obvious. As an example, we can recall Majorana’s study of the shift in the melting point of a substance when it is placed in a magnetic field, or his examination of heat propagation
using the “cricket simile”. As to frontier research arguments, we can recall two examples: the study of quasi-stationary states, anticipating Fano’s theory, and the already mentioned Fermi theory of atoms, reporting analytic solutions of the Thomas–Fermi equation with appropriate boundary conditions in terms of simple quadratures. He also treated therein, in a lucid and original manner, contemporary physics topics such as Fermi’s explanation of the electromagnetic mass of the electron, the Dirac equation with its applications and the Lorentz group.

Just to give a very short account of the interesting material in the Volumetti, let us point out the following.

First of all, we already mentioned that in 1928, when Majorana was starting to collaborate (still as a university student) with the Fermi group in Rome, he had already revealed his outstanding ability in solving involved mathematical problems in original and clear ways, by obtaining an analytical series solution of the Thomas–Fermi equation. Let us recall once more that his whole work on this topic was written on some loose sheets, and then diligently transcribed by the author himself in his Volumetti, so it is contained in Ettore Majorana: Notes on Theoretical Physics. From those pages, the contribution of Majorana to the relevant statistical model is also evident, anticipating some important results found later by leading specialists. As to Majorana’s major finding (namely his methods of solutions of that equation), let us stress that it remained completely unknown until very recently, to the extent that the physics community ignored the fact that nonlinear differential equations, relevant for atoms and for other systems too, can be solved semianalytically (see Sect. 7 of Volumetto II). Indeed, a noticeable property of the method invented by Majorana for solving the Thomas–Fermi equation is that it may be easily generalized, and may then be applied to a large class of particular differential equations. Several generalizations of his method for atoms were proposed by Majorana himself: they were rediscovered only many years later. For example, in Sect. 16 of Volumetto II, Majorana studied the problem of an atom in a weak external electric field, that is, the problem of atomic polarizability, and obtained an expression for the electric dipole moment for a (neutral or arbitrarily ionized) atom. Furthermore, he also started applying the statistical method to molecules, rather than single atoms, by studying the case of a diatomic molecule with identical nuclei (see Sect. 12 of Volumetto II). Finally, he considered the second approximation for the potential inside the atom, beyond the Thomas–Fermi approximation, by generalizing the statistical model of neutral atoms to those ionized \( n \) times, the case \( n = 0 \) included (see Sect. 15 of Volumetto II). As recently pointed out by one of us (S.E.) [23], the approach used by Majorana to this end is
rather similar to the one now adopted in the renormalization of physical quantities in modern gauge theories.

As is well documented, Majorana was among the first to study nuclear physics in Rome (we already know that in 1929 he defended an M.Sc. thesis on such a subject). But he continued to do research on similar topics for several years, till his famous 1933 theory of nuclear exchange forces. For \((\alpha,p)\) reactions on light nuclei, whose experimental results had been interpreted by Chadwick and Gamov, in 1930 Majorana elaborated a dynamical theory (in Sect. 28 of \textit{Volumetto IV}) by describing the energy states associated with the superposition of a continuous spectrum and one discrete level \cite{35}. Actually, Majorana provided a complete theory for the artificial disintegration of nuclei bombarded by \(\alpha\) particles (with and without \(\alpha\) absorption). He approached this question by considering the simplest case, with a single unstable state of a nucleus and an \(\alpha\) particle, which spontaneously decays by emitting an \(\alpha\) particle or a proton. The explicit expression for the total cross-section was also given, rendering his approach accessible to experimental checks. Let us emphasize that the peculiarity of Majorana’s theory was the introduction of quasi-stationary states, which were considered by U. Fano in 1935 (in a quite different context), and widely used in condensed matter physics about 20 years later.

In Sect. 30 of \textit{Volumetto II}, Majorana made an attempt to find a relation between the fundamental constants \(e, \ h\) and \(c\). The interest in this work resides less in the particular mechanical model adopted by Majorana (which led, indeed, to the result \(e^2 \simeq hc\) far from the true value, as noticed by the Majorana himself) than in the interpretation adopted for the electromagnetic interaction, in terms of particle exchange. Namely, the space around charged particles was regarded as quantized, and electrons interacted by exchanging particles; Majorana’s interpretation substantially coincides with that introduced by Feynman in quantum electrodynamics after more than a decade, when the space surrounding charged particles would be identified with the quantum electrodynamics vacuum, while the exchanged particles would be assumed to be photons.

Finally, one cannot forget the pages contained in \textit{Volumetti III} and \textit{V} on group theory, where Majorana showed in detail the relationship between the representations of the Lorentz group and the matrices of the (special) unitary group in two dimensions. In those pages, aimed also at extending Dirac’s approach, Majorana deduced the \textit{explicit} form of the transformations of every bilinear quantity in the spinor fields. Certainly, the most important result achieved by Majorana on this subject is his discovery of the \textit{infinite-dimensional} unitary representations.
of the Lorentz group: he set forth the *explicit* form of them too (see Sect. 8 of *Volumetto V*, besides his published article (7)). We have already recalled that such representations were rediscovered by Wigner only in 1939 and 1948, and later, in 1948–1958, were eventually studied by many authors. People such as van der Waerden recognized the importance, also mathematical, of such a Majorana result, but, as we know, it remained unnoticed till Fradkin’s 1966 article mentioned above.

**This volume: Majorana’s research notes**

The material reproduced in *Ettore Majorana: Notes on Theoretical Physics* was a paragon of order, conciseness, essentiality and originality, so much so that those notebooks can be partially regarded as an innovative text of theoretical physics, even after about 80 years, besides being another gold mine of theoretical, physical and mathematical ideas and hints, stimulating and useful for modern research too.

But Majorana’s most remarkable scientific manuscripts—namely his *research notes*—are represented by a host of loose papers and by the *Quaderni*: and this book reproduces a selection of the latter. But the manuscripts with Majorana’s research notes, at variance with the *Volumetti*, rarely contain any introductions or verbal explanations.

The topics covered in the *Quaderni* range from classical physics to quantum field theory, and comprise the study of a number of applications for atomic, molecular and nuclear physics. Particular attention was reserved for the Dirac theory and its generalizations, and for quantum electrodynamics.

The Dirac equation describing spin-1/2 particles was mostly considered by Majorana in a *Lagrangian framework* (in general, the canonical formalism was adopted), obtained from a least action principle (see Chap. 1 in the present volume). After an interesting preliminary study of the problem of the vibrating string, where Majorana obtained a (classical) Dirac-like equation for a two-component field, he went on to consider a semiclassical relativistic theory for the electron, within which the Klein–Gordon and the Dirac equations were deduced starting from a semiclassical Hamilton–Jacobi equation. Subsequently, the field equations and their properties were considered in detail, and the quantization of the (free) Dirac field was discussed by means of the standard formalism, with the use of annihilation and creation operators. Then, the electromagnetic interaction was introduced into the Dirac equation, and the superposition of the Dirac and Maxwell fields was studied in a very personal and original way, obtaining the expression for the quantized
Hamiltonian of the interacting system after a normal-mode decomposition.

Real (rather than complex) Dirac fields, published by Majorana in his famous paper, (9), on the symmetrical theory of electrons and positrons, were considered in the Quaderni in various places (see Sect. 1.6), by two slightly different formalisms, namely by different decompositions of the field. The introduction of the electromagnetic interaction was performed in a quite characteristic manner, and he then obtained an explicit expression for the total angular momentum, carried by the real Dirac field, starting from the Hamiltonian.

Some work, as well, at the basis of Majorana’s important paper (7) can be found in the present Quaderni (see Sect. 1.7 of this volume). We have already seen, when analysing the works published by Majorana, that in 1932 he constructed Dirac-like equations for spin 1, 3/2, 2, etc. (discovering also the method, later published by Pauli and Fiertz, for writing down a quantum-relativistic equation for a generic spin value). Indeed, in the Quaderni reproduced here, Majorana, starting from the usual Dirac equation for a four-component spinor, obtains explicit expressions for the Dirac matrices in the cases, for instance, of six-component and 16-component spinors. Interestingly enough, at the end of his discussion, Majorana also treated the case of spinors with an odd number of components, namely of a five-component field.

With regard to quantum electrodynamics too, Majorana dealt with it in a Lagrangian and Hamiltonian framework, by use of a least action principle. As is now done, the electromagnetic field was decomposed in plane-wave operators, and its properties were studied within a full Lorentz-invariant formalism by employing group-theoretical arguments. Explicit expressions for the quantized Hamiltonian, the creation and annihilation operators for the photons as well as the angular momentum operator were deduced in several different bases, along with the appropriate commutation relations. Even leaving aside, for a moment, the scientific value those results had especially at the time when Majorana achieved them, such manuscripts have a certain importance from the historical point of view too: they indicate Majorana’s tendency to tackle topics of that kind, nearer to Heisenberg, Born, Jordan and Klein’s, than to Fermi’s.

As we were saying, and as already pointed out in previous literature [21], in the Quaderni one can find also various studies, inspired by an idea of Oppenheimer, aimed at describing the electromagnetic field within a Dirac-like formalism. Actually, Majorana was interested in describing the properties of the electromagnetic field in terms of a real wavefunction for the photon (see Sects. 2.2, 2.10), an approach that
went well beyond the work of contemporary authors. Other noticeable investigations of Majorana concerned the introduction of an intrinsic time delay, regarded as a universal constant, into the expressions for electromagnetic retarded fields (see Sect. 2.14), or studies on the modification of Maxwell’s equations in the presence of magnetic monopoles (see Sect. 2.15).

Besides purely theoretical work in quantum electrodynamics, some applications as well were carefully investigated by Majorana. This is the case of free electron scattering (reported in Sect. 2.12), where Majorana gave an explicit expression for the transition probability, and the coherent scattering, of bound electrons (see Sect. 2.13). Several other scattering processes were also analysed (see Chap. 6) within the framework of perturbation theory, by the adoption of Dirac’s or of Born’s method.

As mentioned above, the contribution by Majorana to nuclear physics which was most known to the scientific community of his time is his theory in which nuclei are formed by protons and neutrons, bound by an exchange force of a particular kind (which corrected Heisenberg’s model). In the present Quaderni (see Chap. 7), several pages were devoted to analysing possible forms of the nucleon potential inside a given nucleus, determining the interaction between neutrons and protons. Although general nuclei were often taken into consideration, particular care was given by Majorana to light nuclei (deuteron, α particle, etc.). As will be clear from what is published in this volume, the studies performed by Majorana were, at the same time, preliminary studies and generalizations of what had been reported by him in his well-known publication (8), thus revealing a very rich and personal way of thinking. Notice also that, before having understood and thought of all that led him to the paper mentioned, (8), Majorana had seriously attempted to construct a relativistic field theory for nuclei as composed of scalar particles (see Sect. 7.6), arriving at a characteristic description of the transitions between different nuclei.

Other topics in nuclear physics were broached by Majorana (and were presented in the Volumetti too): we shall only mention, here, the study of the energy loss of β particles when passing through a medium, when he deduced the Thomson formula by classical arguments. Such work too might a priori be of interest for a correct historical reconstruction, when confronted with the very important theory on nuclear β decay elaborated by Fermi in 1934.

The largest part of the Quaderni is devoted, however, to atomic physics (see Chap. 3), in agreement with the circumstance that it was the main research topic tackled by the Fermi group in Rome in 1928–
1933. Indeed, also the articles published by Majorana in those years deal with such a subject; and echoes of those publications can be found, of course, in the present Quaderni, showing that, especially in the case of article (5) on the incomplete $P'$ triplets, some interesting material did not appear in the published papers (see Sect. 3.18).

Several expressions for the wavefunctions and the different energy levels of two-electron atoms (and, in particular, of helium) were discovered by Majorana, mainly in the framework of a variational method aimed at solving the relevant Schrödinger equation. Numerical values for the corresponding energy terms were normally summarized by Majorana in large tables, reproduced in this book. Some approximate expressions were also obtained by him for three-electron atoms (and, in particular, for lithium), and for alkali metals; including the effect of polarization forces in hydrogen-like atoms.

In the present Quaderni, the problem of the hyperfine structure of the energy spectra of complex atoms was moreover investigated in some detail, revealing the careful attention paid by Majorana to the existing literature. The generalization, for a non-Coulombian atomic field, of the Landé formula for the hyperfine splitting was also performed by Majorana, together with a relativistic formula for the Rydberg corrections of the hyperfine structures. Such a detailed study developed by Majorana constituted the basis of what was discussed by Fermi and Segrè in a well-known 1933 paper of theirs on this topic, as acknowledged by those authors themselves.

A small part of the Quaderni was devoted to various problems of molecular physics (see Sect. 4.3). Majorana studied in some detail, for example, the helium molecule, and then considered the general theory of the vibrational modes in molecules, with particular reference to the molecule of acetylene, $\text{C}_2\text{H}_2$ (which possesses peculiar geometric properties).

Rather important are some other pages (see Sects. 5.3, 5.4, 5.5), where the author considered the problem of ferromagnetism in the framework of Heisenberg's model with exchange interactions. However, Majorana's approach in this study was, as always, original, since it followed neither Heisenberg's nor the subsequent van Vleck formulation in terms of a spin Hamiltonian. By using statistical arguments, instead, Majorana evaluated the magnetization (with respect to the saturation value) of the ferromagnetic system when an external magnetic field acts on it, and the phenomenon of spontaneous magnetization. Several examples of ferromagnetic materials, with different geometries, were analysed by him as well.
A number of other interesting questions, even dealing with topics that Majorana had encountered during his academic studies at Rome University (see Chaps. 8, 9), can be found in these Quaderni. This is the case, for example, of the electromagnetic and electrostatic mass of the electron (a problem that was considered by Fermi in one of his 1924 known papers), or of his studies on tensor calculus, following his teacher Levi-Civita. We cannot discuss them here, however, our aim being that of drawing the attention of the reader to a few specific points only. The discovery of the large number of exceedingly interesting and important studies that were undertaken by Majorana, and written by him in these Quaderni, is left to the reader’s patience.

About the format of this volume

As is clear from what we have discussed already, Majorana used to put on paper the results of his studies in different ways, depending on his opinion about the value of the results themselves. The method used by Majorana for composing his written notes was sometimes the following. When he was investigating a certain subject, he reported his results only in a Quaderno. Subsequently, if, after further research on the topic considered, he reached a simpler and conclusive (in his opinion) result, he reported the final details also in a Volumetto. Therefore, in his preliminary notes we find basically mere calculations, and only in some rare cases can an elaborated text, clearly explaining the calculations, be found in the Quaderni. In other words, a clear exposition of many particular topics can be found only in the Volumetti.

The 18 Quaderni deposited at the Domus Galilaeana are booklets of approximately of 15 cm × 21 cm, endowed with a black cover and a red external boundary, as was common in Italy before the Second World War. Each booklet is composed of about 200 pages, giving a total of about 2,800 pages. Rarely, some pages were torn off (by Majorana himself), while blank pages in each Quaderno are often present. In a few booklets, extra pages written by the author were put in.

An original numbering style of the pages is present only in Quaderno 1 (in the centre at the top of each page). However, all the Quaderni have nonoriginal numbering (written in red ink) at the top-left corner of their odd pages. Blank pages too were always numbered. Interestingly enough, even though original numbering by Majorana in general is not present, nevertheless sometimes in a Quaderno there appears an original reference to some pages of that same booklet. Some other strange cross-references, not easily understandable to us, appear (see below) in several
booklets. Some of them refer, probably, to pages of the *Volumetti*, but we have been unable to interpret the remaining ones.

As was evident also from a previous catalogue of the unpublished manuscripts, prepared long ago by Baldo, Mignani and Recami [14], often the material regarding the same subject was not written in the *Quaderni* in a sequential, logical order: in some cases, it even appeared in the reverse order.

The major part of the *Quaderni* contains calculations without explanations, even though, in few cases, an elaborated text is fortunately present.

At variance with what is found for the *Volumetti*, in the *Quaderni* no date appears, except for *Quaderno* 16 (“1929–1930”), 17 (“started on 20 June 1932”) and, probably, 7 (“about year 1928”). Therefore, the actual dates of composition of the manuscripts may be inferred only from a detailed comparison of the topics studied therein with what is present in the *Volumetti* and in the published literature, including Majorana’s published papers. Some additional information comes from some cross-references explicitly penned by the author himself, referring either to his *Quaderni* or to his *Volumetti*. In a few cases, references to some of the existing literature are explicitly introduced by Majorana.

Since no consequential or time order is present in the present *Quaderni*, in this book we have grouped the material by subject, and grouped the topics into four (large) parts. To identify the correspondence between what is reproduced by us in a given section and the material present in the original manuscripts, we have added a “code” to each section (or, in some cases, subsection). For instance, the code Q11p138 means that section contains material present in *Quaderno* 11, starting from page 138.

Of course, we have also reported, in a second index (to be found at the end of this Preface, after the Bibliography), the complete list of the subjects present in the 18 *Quaderni*. If a particular subject is reproduced also in the present volume, this is indicated by the mere presence of the corresponding “code”.

We have made a major effort in carefully checking and typing all equations and tables, and, even more, in writing down a brief presentation of the argument exploited in each subsection. In addition, we have inserted among Majorana’s calculations a minimum number of words, when he had left his formalism without any text, trying to facilitate the reading of Majorana’s research notebooks, but limiting as much as possible the insertion of any personal comments of ours. Our hope is to have rendered the intellectual treasures, contained in the *Quaderni*, accessible for the first time to the widest audience. With such an aim,
we have had frequent recourse to more modern notations for the mathematical symbols. For example, the Laplacian operator has been written $\nabla^2$ by us, instead of $\Delta_2$; the gradient has been denoted by $\nabla$, instead of grad; and the vector product is represented by $\times$, instead of $\wedge$; and so on. Analogously, we have treated the scalar product between vectors. In some cases, when the corresponding vectorial quantities were operators, we have retained the original Majorana notation, $(a, b)$, which is still used in many mathematical books.

The figures appearing in the Quaderni have been reproduced anew, without the use of photographic or scanning devices, but they are otherwise true in form to the original drawings. The same holds for tables; several tables had gaps, since in those cases Majorana for some reason did not perform the corresponding calculations. Other minor corrections performed by us, mainly related to typos in the original manuscripts, have been explicitly pointed out in suitable footnotes. More precisely, all changes with respect to the original, introduced by us in the present English version, have been pointed out by means of footnotes. Many additional footnotes have been introduced, whenever the interpretation of some procedures, or the meaning of particular parts, required some more words of presentation. Footnotes which are not present in the original manuscript are denoted by the symbol @. Moreover, all the additions we have made ourselves in the present volume are written, as a rule, in italics, while the original text written by Majorana always appears in Roman characters.

At the end of this Preface, we attach a short Bibliography. Far from being exhaustive, it provides just some references about the topics touched upon in this Preface.
Acknowledgements

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The Editors
Biographical papers, written by witnesses who knew Ettore Majorana, are the following:


Accurate biographical information, completed by the reproduction of many documents, is to be found in the following book (where almost all the relevant documents existing by 2002—discovered or collected by that author—appeared for the first time):


See also:


Scientific published articles by Majorana have been discussed and/or translated into English in the following papers:


A preliminary catalogue of the unpublished papers by Majorana first appeared [5] as well as in:

The English translation of the *Volumetti* appeared as:


The original Italian version, was published in:


The anastatic reproduction of the original notes for the lectures delivered by Majorana at the University of Naples (during the first months of 1938) is in:


The complete set of the lecture notes (including the so-called Moreno document) was published in:


See also:


An English translation of (only) his notes for his inaugural lecture appeared as:


Other previously unknown scientific manuscripts by Majorana have been reevaluated (and/or published with comments) in the following articles:


Some scientific papers elaborating on several intuitions by Majorana are the following:


Further scientific papers can be found in:


Further historical studies on Majorana’s work may be found in the following recent papers:


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PART I
1

DIRAC THEORY

1.1. VIBRATING STRING

Starting from the problem of the vibrating string (which is studied in the framework of the canonical formalism), Majorana obtained a (classical) Dirac-like equation for a two-component field $u = (u_1, u_2)$, where Pauli matrices $\sigma$ appear.

$$-\frac{1}{2} \delta \int \left[ \left( \frac{\partial q}{\partial t} \right)^2 - \left( \frac{\partial q}{\partial x} \right)^2 \right] \, d\tau = 0,$$

$$\ddot{q} = \frac{\partial^2 q}{\partial x^2}, \quad p = \frac{\partial q}{\partial t},$$

$$H = \frac{1}{2} \int \left[ p^2 + \left( \frac{\partial q}{\partial x} \right)^2 \right] \, dx,$$

$$(q_1, p_1) \quad (q_2, p_2) \quad (q_3, p_3) \ldots,$$

$$H = \frac{1}{2} \sum_{\lambda} (\lambda^2 q_{\lambda}^2 + p_{\lambda}^2).$$

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 = \left( \frac{1}{c} \frac{\partial}{\partial t} + \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \right)$$

$$\times \left( \frac{1}{c} \frac{\partial}{\partial t} - \sigma_x \frac{\partial}{\partial x} - \sigma_y \frac{\partial}{\partial y} - \sigma_z \frac{\partial}{\partial z} \right),$$

$$\left( \frac{1}{c} \frac{\partial}{\partial t} - \sigma_x \frac{\partial}{\partial x} \sigma_y \frac{\partial}{\partial y} \sigma_z \frac{\partial}{\partial z} \right) u = 0,$$

$$u = (u_1, u_2),$$

$$\frac{\partial u}{\partial t} = c \left( \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \right) u,$$
\[
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

\[
\frac{1}{c} \frac{\partial u_1}{\partial t} = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) u_2 + \frac{\partial}{\partial z} u_1,
\]

\[
\frac{1}{c} \frac{\partial u_2}{\partial t} = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u_1 - \frac{\partial}{\partial z} u_2,
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) u_1 = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) u_2,
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) u_2 = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u_1.
\]

\[
x_0 = i ct,
\]

\[
x_1 = x,
\]

\[
x_2 = y,
\]

\[
x_3 = z,
\]

\[
i \left( \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_3} \right) u_1 = \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) u_2,
\]

\[
i \left( \frac{\partial}{\partial x_0} - i \frac{\partial}{\partial x_3} \right) u_2 = \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) u_1.
\]

1.2. A SEMICLASSICAL THEORY FOR THE ELECTRON

1.2.1 Relativistic Dynamics

In the following, the relativistic equations of motion for an electron in a force field \( F \) are considered in a non-usual way, by separating the radial \( F_r \) and the transverse component \( F_t \) (with respect to the particle velocity \( \beta c \)) of the force. Expressions for the time derivative of the charge density \( \rho \) and current density \( i \), which satisfy the continuity equation, are obtained.
DIRAC THEORY

charge + e
mass m

\( \rho, \quad i_x = \rho \beta_x, \quad i_y = \rho \beta_y, \quad i_z = \rho \beta_z; \)

\( \beta_x = v_x/c, \quad \beta_y = v_y/c, \quad \beta_z = v_z/c; \)

\( \beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} = v/c. \)

\[
\frac{d}{dt} \left( \frac{mv_x}{\sqrt{1 - \beta^2}} \right) = eF_x,
\]

\[
\frac{d}{dt} \left( \frac{mv_y}{\sqrt{1 - \beta^2}} \right) = eF_y,
\]

\[
\frac{d}{dt} \left( \frac{mv_z}{\sqrt{1 - \beta^2}} \right) = eF_z.
\]

\( k = \frac{e}{mc}. \)

\[
\frac{d}{dt} \frac{\beta}{\sqrt{1 - \beta^2}} = \frac{1}{k} F,
\]

\[
\frac{d}{dt} \frac{\beta}{\sqrt{1 - \beta^2}} = \frac{\dot{\beta}}{\sqrt{1 - \beta^2}} + \frac{(\beta \cdot \dot{\beta}) \beta}{(1 - \beta^2)^{3/2}} = \frac{1}{\sqrt{1 - \beta^2}} \left( \dot{\beta} + \frac{\beta \cdot \dot{\beta}}{1 - \beta^2} \right),
\]

\[
\frac{1}{\sqrt{1 - \beta^2}} \dot{\beta} + \frac{1}{(1 - \beta^2)^{3/2}} (\beta \cdot \dot{\beta}) \beta = \frac{1}{k} F.
\]

\[
\frac{1}{k} F \cdot \beta = \frac{1}{(1 - \beta^2)^{3/2}} (\beta \cdot \dot{\beta}),
\]

\[
\frac{1}{k} F \times \beta = \frac{1}{\sqrt{1 - \beta^2}} \dot{\beta} \times \beta;
\]

\[
\beta_r = (1 - \beta^2)^{3/2} \frac{1}{k} F_r,
\]

\[
\beta_t = \sqrt{1 - \beta^2} \frac{1}{k} F_t;
\]
\[ \dot{\beta} = \dot{\beta}_r + \dot{\beta}_t, \]
\[ F = F_r + F_t. \]

\[ F_r = \left( F_x \beta_x + F_y \beta_y + F_z \beta_z \right) \frac{\beta_x}{\beta^2}, \quad \left( F_x \beta_x + F_y \beta_y + F_z \beta_z \right) \frac{\beta_y}{\beta^2}, \]

\[ \left( F_x \beta_x + F_y \beta_y + F_z \beta_z \right) \frac{\beta_z}{\beta^2}, \]

\[ F_t = \left( F_x - (F_x \beta_x + F_y \beta_y + F_z \beta_z) \frac{\beta_x}{\beta^2}, \quad F_y - (F_x \beta_x + F_y \beta_y + F_z \beta_z) \frac{\beta_y}{\beta^2}, \right. \]

\[ \left. F_z - (F_x \beta_x + F_y \beta_y + F_z \beta_z) \frac{\beta_z}{\beta^2} \right). \]

\[ \dot{\beta}_x = \frac{\sqrt{1 - \beta^2}}{k} \left[ F_x - (F_x \beta_x + F_y \beta_y + F_z \beta_z) \beta_x \right] = \frac{d}{dt} \beta_x, \]

\[ \dot{\beta}_y = \frac{\sqrt{1 - \beta^2}}{k} \left[ F_y - (F_x \beta_x + F_y \beta_y + F_z \beta_z) \beta_y \right] = \frac{d}{dt} \beta_y, \]

\[ \dot{\beta}_z = \frac{\sqrt{1 - \beta^2}}{k} \left[ F_z - (F_x \beta_y + F_y \beta_y + F_z \beta_z) \beta_z \right] = \frac{d}{dt} \beta_z. \]

\[ \frac{\partial \rho}{\partial t} + c \left( \frac{\partial i_x}{\partial x} + \frac{\partial i_y}{\partial y} + \frac{\partial i_z}{\partial z} \right) = 0; \]

\[ \frac{d \rho}{dt} = \frac{\partial \rho}{\partial t} + c \left( \beta_x \frac{\partial \rho}{\partial x} + \beta_y \frac{\partial \rho}{\partial y} + \beta_z \frac{\partial \rho}{\partial z} \right); \]

\[ \frac{d \rho}{dt} = c \left( \beta_x \frac{\partial \rho}{\partial x} + \beta_y \frac{\partial \rho}{\partial y} + \beta_z \frac{\partial \rho}{\partial z} - \frac{\partial i_x}{\partial x} - \frac{\partial i_y}{\partial y} - \frac{\partial i_z}{\partial z} \right); \]

\[ \frac{\partial i_x}{\partial t} = \frac{d i_x}{dt} - c \left( \beta_x \frac{\partial i_x}{\partial x} + \beta_y \frac{\partial i_y}{\partial y} + \beta_z \frac{\partial i_z}{\partial z} \right); \]

\[ \frac{d i_x}{dt} = \frac{d}{dt} (\rho \beta_x) = \beta_x \frac{d \rho}{dt} + \rho \frac{d \beta_x}{dt} \]

\[ = \beta_x \cdot c \left( \beta_x \frac{\partial \rho}{\partial x} + \beta_y \frac{\partial \rho}{\partial y} + \beta_z \frac{\partial \rho}{\partial z} - \frac{\partial i_x}{\partial x} - \frac{\partial i_y}{\partial y} - \frac{\partial i_z}{\partial z} \right) \]

\[ + \rho \frac{\sqrt{1 - \beta^2}}{k} \left[ F_x - (F_x \beta_x + F_y \beta_y + F_z \beta_z) \beta_x \right]. \]
1.2.2 Field Equations

The author began now to study the field equations for an electron in an electromagnetic potential \((\varphi, C)\) by following two different approaches. In the first part, he “tries” with a semiclassical Hamilton-Jacobi equation corresponding to the relativistic expression for the energy-momentum relation, by imposing the constraint of a positive value for the energy. From appropriate correspondence relations, he then deduced a Klein-Gordon equation for the field \(\psi\) and, on introducing the Pauli matrices, the Dirac equations for the electron 4-component wavefunction. Some (mathematical) consequences of the formalism adopted (mainly related to the charge-current density) were also analyzed.

In the second part, Majorana focused his attention on the standard formalism for the Dirac equation, again discussing in detail the expressions for the Dirac charge-current density \((\rho, i)\) and some peculiar constraints on Lorentz-invariant field quantities. He introduced and studied the consequences of several ansatz leading to Dirac-like equations for the electron.

\[
-\left(-\frac{1}{c}\frac{\partial S}{\partial t} + \frac{e}{c} \varphi\right)^2 + \sum_x \left(\frac{\partial S}{\partial x} + \frac{e}{c} C_x\right)^2 + m^2 c^2 = 0;
\]

\[
-\frac{1}{c}\frac{\partial S}{\partial t} + \frac{e}{c} \varphi > 0.
\]

\[
\psi = Ae^{2\pi i S/\hbar}, \quad A = |\psi|.
\]

\[
\frac{\partial \psi}{\partial x} = \left(\frac{\partial A}{\partial x} + A \frac{2\pi i \partial S}{h \partial x}\right) e^{2\pi i S/\hbar} = \left(\frac{1}{A} \frac{\partial A}{\partial x} + \frac{2\pi i \partial S}{h \partial x}\right) \psi
\]

\[
\frac{\partial^2 \varphi}{\partial x^2} = \left(\frac{\partial^2 A}{\partial x^2} + 2 \frac{\partial A}{\partial x} \frac{2\pi i \partial S}{h \partial x} + A \frac{2\pi i \partial S}{h \partial x^2} - A \frac{4\pi^2 \partial^2 S}{h^2 \partial x^2}\right) e^{2\pi i S/\hbar}
\]

\[
= \left(\frac{1}{A} \frac{\partial^2 A}{\partial x^2} + \frac{2 \pi i \partial S}{A h \partial x} + \frac{2\pi i \partial^2 S}{h \partial x^2} - \frac{4\pi^2 \partial^2 S}{h^2 \partial x^2}\right) \psi
\]

Versuchsweise: \(^1\)

\(^1\) This German word means “tentatively”, and refers to the successive assumptions. Note, however, that in the original paper the cited word is written as “versucherweiser”. 

\[
\begin{align*}
\left\{
\frac{\partial S}{\partial x} &= \frac{h}{2\pi i} \frac{1}{\psi} \frac{\partial \psi}{\partial x}; \\
\frac{\partial S}{\partial t} &= \frac{h}{2\pi i} \frac{1}{\psi} \frac{\partial \psi}{\partial t}; \\
\frac{\partial S}{\partial x} &= -\frac{h}{2\pi i} \frac{1}{\psi} \frac{\partial \psi}{\partial x}; \\
\frac{\partial S}{\partial t} &= -\frac{h}{2\pi i} \frac{1}{\psi} \frac{\partial \psi}{\partial t}.
\end{align*}
\]

\[- \left[ \left( -\frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \right]^2 + \sum_x \left[ \left( \frac{h}{2\pi i} \frac{\partial}{\partial x} + \frac{e}{c} C_x \right) \psi \right]^2 + m^2 c^2 \psi^2 = 0. \quad (B)\]

Approximate condition:
\[\bar{\psi} \left( -\frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \psi + \psi \left( \frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} + e\varphi \right) \bar{\psi} > 0.\]

In exact form:
\[- \left( -\frac{1}{c} \frac{\partial S}{\partial t} + \frac{e}{c} \varphi \right)^2 + \sum_x \left( \frac{\partial S}{\partial x} + \frac{e}{c} C_x \right)^2 + m^2 c^2 = 0, \quad (A)\]

\[|\psi| = 1; \quad (C)\]

\[\psi = e^{2\pi i S/h}, \quad \frac{\partial \psi}{\partial x} = \frac{2\pi i}{h} \frac{\partial S}{\partial x} \psi.\]

\[(A) \equiv (B) + (C).\]

\[\psi_0 = \sin \frac{2\pi}{h} S, \quad \psi_1 = \cos \frac{2\pi}{h} S;\]
\[\frac{\partial \psi_0}{\partial x} = \frac{2\pi}{h} \frac{\partial S}{\partial x} \cos \frac{2\pi}{h} S, \quad \frac{\partial \psi_1}{\partial x} = -\frac{2\pi}{h} \frac{\partial S}{\partial x} \sin \frac{2\pi}{h} S;\]
\[\frac{h}{2\pi} \frac{\partial \psi_0}{\partial x} = \frac{\partial S}{\partial x} \psi_1, \quad \frac{h}{2\pi} \frac{\partial \psi_1}{\partial x} = -\frac{\partial S}{\partial x} \psi_0,\]
\[
\frac{\partial S}{\partial x} = \frac{1}{\psi_1} \frac{h \partial \psi_0}{2\pi \partial x} = -\frac{1}{\psi_0} \frac{h \partial \psi_1}{2\pi \partial x}.
\]

\[
\delta \int \left\{ \left( \frac{1}{c} \frac{\partial \varphi_0}{\partial t} - \frac{e}{c} \varphi \psi \right) \left( \frac{1}{c} \frac{\partial \varphi_1}{\partial t} + \frac{e}{c} \varphi \psi \right) + \sum_x \left( \frac{h}{2\pi} \frac{\partial \psi_0}{\partial x} + \frac{e}{c} C_x \psi \right) \left( \frac{h}{2\pi} \frac{\partial \psi_1}{\partial x} - \frac{e}{c} C_x \psi \right) + m^2 c^2 \psi_0 \psi_1 \right\} \, d\tau = 0
\]
\( (d\tau = dV dt). \)

\[
\frac{h}{2\pi} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial \psi_0}{\partial t} - \frac{e}{c} \varphi \psi \right) + \frac{e}{c} \varphi \left( \frac{1}{c} \frac{\partial \psi_1}{\partial t} + \frac{e}{c} \varphi \psi \right) - \sum_x \left[ \frac{h}{2\pi} \frac{\partial}{\partial x} \left( \frac{h}{2\pi} \frac{\partial \psi_0}{\partial x} + \frac{e}{c} C_x \varphi \right) - \frac{e}{c} C_x \left( \frac{2}{2\pi} \frac{\partial \psi_1}{\partial x} - \frac{e}{c} C_x \psi \right) \right]
\]
\[+ m^2 c^2 \psi_0 = 0. \]

\[
\left[ -\frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} + \frac{e}{c} \varphi + \rho_3 \sigma \cdot \left( \frac{h}{2\pi i} \nabla + \frac{e}{c} C \right) + \rho_1 mc \right] \psi = 0,
\]

\[
\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} ; 
\]
\[A = (\psi_1, \psi_2), \quad B = (\psi_3, \psi_4). \]

\[
\left[ -\frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} + \frac{e}{c} \varphi + \sigma \cdot \left( \frac{h}{2\pi i} \nabla + \frac{e}{c} C \right) \right] A + mcB = 0, \]

\[
\left[ -\frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} + \frac{e}{c} \varphi - \sigma \cdot \left( \frac{h}{2\pi i} \nabla + \frac{e}{c} C \right) \right] B + mcA = 0.
\]

\[
\rho = \tilde{A}A + \tilde{B}B = \bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2 + \bar{\psi}_3 \psi_3 + \bar{\psi}_4 \psi_4, \]
\[
i_x = \tilde{A} \sigma_x A + \tilde{B} \sigma_x B = -\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1 + \bar{\psi}_3 \psi_4 + \bar{\psi}_4 \psi_3, \]
\[
i_y = \tilde{A} \sigma_y A + \tilde{B} \sigma_y B = i(\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1 - \bar{\psi}_3 \psi_4 + \bar{\psi}_4 \psi_3), \]
\[
i_z = \tilde{A} \sigma_z A + \tilde{B} \sigma_z B = -\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2 + \bar{\psi}_3 \psi_3 - \bar{\psi}_4 \psi_4.
\]

\[\text{Note that, more appropriately, it should be written } d^4\tau = d^3V dt, \text{ since } d\tau \text{ denotes the 4-dimensional volume element, while } drmV \text{ is the 3-dimensional space volume element.} \]
$$\psi_1, \psi_2 \sim -\bar{\psi}_4, +\bar{\psi}_3,$$
$$\psi_3, \psi_4 \sim \bar{\psi}_2, -\bar{\psi}_1.$$

Versuchsweise:

$$\begin{align*}
\psi_3 &= k \bar{\psi}_2, \\
\psi_4 &= -k \bar{\psi}_1;
\end{align*}$$

$$\begin{align*}
\psi_1 &= -(1/k) \bar{\psi}_4, \\
\psi_2 &= (1/k) \bar{\psi}_3;
\end{align*}$$

$$k = k(x, y, r, t),$$

$$\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4 = 0.$$

$$\begin{align*}
\left( -\frac{1}{c} \frac{\partial}{2\pi i \partial t} + \frac{e}{c} \varphi \right) \psi_1 + \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e}{c} \left( C_x - i C_y \right) \right] \psi_2 \\
+ \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_1 + mc \psi_3 = 0,
\end{align*}$$

$$\begin{align*}
\left( -\frac{1}{c} \frac{\partial}{2\pi i \partial t} + \frac{e}{c} \varphi \right) \psi_2 + \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{e}{c} \left( C_x + i C_y \right) \right] \psi_1 \\
- \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_2 + mc \psi_4 = 0,
\end{align*}$$

$$\begin{align*}
\left( -\frac{1}{c} \frac{\partial}{2\pi i \partial t} + \frac{e}{c} \varphi \right) \psi_3 - \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e}{c} \left( C_x - i C_y \right) \right] \psi_4 \\
- \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_3 + mc \psi_1 = 0,
\end{align*}$$

$$\begin{align*}
\left( -\frac{1}{c} \frac{\partial}{2\pi i \partial t} + \frac{e}{c} \varphi \right) \psi_4 - \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{e}{c} \left( C_x + i C_y \right) \right] \psi_3 \\
+ \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_4 + mc \psi_2 = 0.
\end{align*}$$

$$k = k(x, y, r, t)$$
\[
\left(-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} + \frac{\varphi}{c}\right) \psi_1 + \left[\frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) + \frac{\varphi}{c} (C_x - i C_y)\right] \psi_2 \\
+ \left(\frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{\varphi}{c} \right) \psi_1 + kmc \psi_2 = 0,
\]
\[
\left(-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} + \frac{\varphi}{c}\right) \psi_2 + \left[\frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) + \frac{\varphi}{c} (C_x + i C_y)\right] \psi_1 \\
- \left(\frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{\varphi}{c} \right) \psi_2 - kmc \psi_1 = 0,
\]
\[
\left(-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} + \frac{\varphi}{c}\right) (k \overline{\psi}_2) - \left[\frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) + \frac{\varphi}{c} (C_x - i C_y)\right] (-k \overline{\psi}_1) \\
- \left(\frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{\varphi}{c} \right) (k \overline{\psi}_2) + mc \psi_1 = 0,
\]
\[
\left(-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} + \frac{\varphi}{c}\right) (-k \overline{\psi}_1) - \left[\frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) + \frac{\varphi}{c} (C_x + i C_y)\right] (k \overline{\psi}_2) \\
+ \left(\frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{\varphi}{c} \right) (-k \overline{\psi}_1) + mc \psi_2 = 0.
\]

without field\(^3\): \( k = \pm 1; \quad \psi_3 = \overline{\psi}_2; \quad \psi_4 = -\overline{\psi}_1; \quad \varphi, C = 0
\]
\[
\begin{align*}
-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} \psi_1 + \frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \psi_2 + \frac{\hbar}{2\pi i} \frac{\partial}{\partial r} \psi_1 + mc \overline{\psi}_2 &= 0, \\
-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} \psi_2 + \frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \psi_1 - \frac{\hbar}{2\pi i} \frac{\partial}{\partial r} \psi_2 - mc \overline{\psi}_1 &= 0, \\
-\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} \overline{\psi}_2 + \frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \overline{\psi}_1 - \frac{\hbar}{2\pi i} \frac{\partial}{\partial r} \overline{\psi}_2 + mc \psi_1 &= 0, \\
+\frac{1}{c} \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} \overline{\psi}_1 - \frac{\hbar}{2\pi i} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \overline{\psi}_2 - \frac{\hbar}{2\pi i} \frac{\partial}{\partial r} \overline{\psi}_1 + mc \psi_2 &= 0.
\end{align*}
\]

For real \( u_1, u_2, u_3, u_4 \):

\(^3\)@ This interesting side note is present in the original manuscript: we can use \( \pm m \) in place of \( k = \pm 1 \): \( k = 1 \) corresponds to \( m \) and \( k = -1 \) corresponds to \( -m \).
\( k = 1 : \)

\[
\psi_1 = \frac{u_1 + iu_2}{\sqrt{2}}, \quad \psi_2 = \frac{u_3 + iu_4}{\sqrt{2}},
\]

\[
\psi_3 = \frac{u_3 - iu_4}{\sqrt{2}}, \quad \psi_4 = \frac{-u_1 + iu_2}{\sqrt{2}};
\]

\( k = -1 : \)

\[
\psi_1 = \frac{u_1 + iu_2}{\sqrt{2}}, \quad \psi_2 = \frac{u_3 + iu_4}{\sqrt{2}},
\]

\[
\psi_3 = \frac{-u_3 + iu_4}{\sqrt{2}}, \quad \psi_4 = \frac{u_1 - iu_2}{\sqrt{2}}.
\]

\[
\rho = u_1^2 + u_2^2 + u_3^2 + u_4^2,
\]

\[
i_x = -(2u_1u_3 + 2u_2u_4),
\]

\[
i_y = -(2u_1u_4 - 2u_2u_3),
\]

\[
i_z = -(u_1^2 + u_2^2 - u_3^2 - u_4^2).
\]

\[
\frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} u_1 - \frac{h}{2\pi} \frac{\partial}{\partial x} u_3 - \frac{h}{2\pi} \frac{\partial}{\partial y} u_4 - \frac{h}{2\pi} \frac{\partial}{\partial z} u_1 - mc u_4 = 0,
\]

\[
\frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} u_2 - \frac{h}{2\pi} \frac{\partial}{\partial x} u_4 + \frac{h}{2\pi} \frac{\partial}{\partial y} u_3 - \frac{h}{2\pi} \frac{\partial}{\partial z} u_2 - mc u_3 = 0,
\]

\[
\frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} u_3 - \frac{h}{2\pi} \frac{\partial}{\partial x} u_1 + \frac{h}{2\pi} \frac{\partial}{\partial y} u_2 + \frac{h}{2\pi} \frac{\partial}{\partial z} u_3 + mc u_2 = 0,
\]

\[
\frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} u_4 - \frac{h}{2\pi} \frac{\partial}{\partial x} u_2 - \frac{h}{2\pi} \frac{\partial}{\partial y} u_1 + \frac{h}{2\pi} \frac{\partial}{\partial z} u_4 + mc u_1 = 0.
\]

\[
\frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} u = \left[ \frac{h}{2\pi} \left( \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} \right) + \delta mc \right] u.
\]

\[
\gamma_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
\gamma_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\gamma_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
\delta = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

\( \gamma_1 = \rho_1, \quad \gamma_2 = -\sigma_y \rho_2, \quad \gamma_3 = \rho_3, \quad \delta = -i\sigma_x \rho_2. \)
For \( u = u(r,t) \):

\[
\begin{align*}
\frac{\hbar}{2\pi} \left( \frac{1}{c} \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) u_1 &= mc u_4, \\
\frac{\hbar}{2\pi} \left( \frac{1}{c} \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) u_2 &= mc u_3, \\
\frac{\hbar}{2\pi} \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) u_3 &= -mc u_2, \\
\frac{\hbar}{2\pi} \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) u_4 &= -mc u_1;
\end{align*}
\]

\[
\begin{align*}
u_1 &= \lambda_1 R e^{\frac{2\pi i}{\hbar}(-at + bz)}, \\
u_2 &= \lambda_2 R e^{\frac{2\pi i}{\hbar}(-at + bz)}, \\
u_3 &= \lambda_3 R e^{\frac{2\pi i}{\hbar}(-at + bz)}, \\
u_1 &= \lambda_4 R e^{\frac{2\pi i}{\hbar}(-at + bz)}.
\end{align*}
\]

\[
\begin{align*}
-i \left( \frac{a}{c} + b \right) \lambda_1 &= mc \lambda_4, \\
-i \left( \frac{a}{c} + b \right) \lambda_2 &= mc \lambda_3, \\
-i \left( \frac{a}{c} - b \right) \lambda_3 &= -mc \lambda_2, \\
-i \left( \frac{a}{c} - b \right) \lambda_4 &= -mc \lambda_1;
\end{align*}
\]

\[
\begin{align*}
a^2 &= mc^2 + b^2, \\
\frac{\lambda_4}{\lambda_1} &= -\frac{i}{mc} \left( \frac{a}{c} + b \right) = \frac{\lambda_3}{\lambda_2}.
\end{align*}
\]

\[
\rho = u^\dagger L_0 u, \quad i_x = u^\dagger L_1 u, \quad i_y = u^\dagger L_2 u, \quad i_z = u^\dagger L_3 u;
\]

\[
L_0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad L_1 = -\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} = -\gamma_1,
\]
\[
L_2 = -
\begin{vmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{vmatrix} = -\gamma_2,
L_3 = -
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{vmatrix} = -\gamma_3.
\]

\[
\rho^2 = (u_1^2 + u_2^2 + u_3^2 + u_4^2)^2
= u_1^4 + u_2^4 + u_3^4 + u_4^4 + 2u_1^2u_2^2 + 2u_1^2u_3^2 + 2u_1^2u_4^2 + 2u_2^2u_3^2 + 2u_2^2u_4^2 + 2u_3^2u_4^2,
\]

\[
i^2_x = 4(u_1u_3 + u_2u_4)^2 = 4u_1^2u_3^2 + 4u_2^2u_4^2 + 8u_1u_2u_3u_4,
\]

\[
i^2_y = 4(u_1u_4 - u_2u_3)^2 = 4u_1^2u_4^2 + 4u_2^2u_3^2 - 8u_1u_2u_3u_4,
\]

\[
i^2_z = (u_1^2 + u_2^2 - u_3^2 - u_4^2)^2,
= u_1^4 + u_2^4 + u_3^4 + u_4^4 + 2u_1^2u_2^2 - 2u_1^2u_3^2 - 2u_1^2u_4^2 - 2u_2^2u_3^2 - 2u_2^2u_4^2 - 2u_3^2u_4^2;
\]

\[
\rho^2 - i^2_x - i^2_y - i^2_z = 0.
\]

\[
(\overline{\psi}_1\psi_1 + \overline{\psi}_2\psi_2 + \overline{\psi}_3\psi_3 + \overline{\psi}_4\psi_4)^2 = \overline{\psi}_1^2\psi_1^2 + \overline{\psi}_2^2\psi_2^2 + \overline{\psi}_3^2\psi_3^2 + \overline{\psi}_4^2\psi_4^2
+ 2\overline{\psi}_1\overline{\psi}_2\psi_1\psi_2 + 2\overline{\psi}_1\overline{\psi}_3\psi_1\psi_3 + 2\overline{\psi}_1\overline{\psi}_4\psi_1\psi_4 + 2\overline{\psi}_2\overline{\psi}_3\psi_2\psi_3
+ \psi_2\overline{\psi}_4\psi_2\psi_4 + 2\psi_3\overline{\psi}_4\psi_3\psi_4,
\]

\[
(-\overline{\psi}_1\psi_1 - \overline{\psi}_2\psi_1 + \overline{\psi}_3\psi_1 - \overline{\psi}_4\psi_1)^2 = \overline{\psi}_1^2\psi_1^2 + \overline{\psi}_2^2\psi_1^2 + \overline{\psi}_3^2\psi_1^2 + \overline{\psi}_4^2\psi_1^2
+ 2\overline{\psi}_1\overline{\psi}_2\psi_1\psi_2 - 2\overline{\psi}_1\overline{\psi}_3\psi_1\psi_3 - 2\overline{\psi}_1\overline{\psi}_4\psi_1\psi_4 - 2\overline{\psi}_2\overline{\psi}_3\psi_2\psi_3
- 2\overline{\psi}_2\overline{\psi}_4\psi_2\psi_4 + 2\overline{\psi}_3\overline{\psi}_4\psi_3\psi_4,
\]

\[
(-\overline{\psi}_1\psi_2 - \overline{\psi}_2\psi_2 + \overline{\psi}_3\psi_2 - \overline{\psi}_4\psi_2)^2 = -\overline{\psi}_1^2\psi_2^2 - \overline{\psi}_2^2\psi_2^2 - \overline{\psi}_3^2\psi_2^2 - \overline{\psi}_4^2\psi_2^2
+ 2\overline{\psi}_1\overline{\psi}_2\psi_1\psi_2 + 2\overline{\psi}_1\overline{\psi}_3\psi_1\psi_3 - 2\overline{\psi}_1\overline{\psi}_4\psi_1\psi_4 - 2\overline{\psi}_2\overline{\psi}_3\psi_2\psi_3
+ 2\overline{\psi}_2\overline{\psi}_4\psi_2\psi_4 + 2\overline{\psi}_3\overline{\psi}_4\psi_3\psi_4,
\]

\[
(-\overline{\psi}_1\psi_3 - \overline{\psi}_2\psi_3 + \overline{\psi}_3\psi_3 - \overline{\psi}_4\psi_3)^2 = \overline{\psi}_1^2\psi_3^2 + \overline{\psi}_2^2\psi_3^2 + \overline{\psi}_3^2\psi_3^2 + \overline{\psi}_4^2\psi_3^2
- 2\overline{\psi}_1\overline{\psi}_2\psi_1\psi_2 - 2\overline{\psi}_1\overline{\psi}_3\psi_1\psi_3 + 2\overline{\psi}_1\overline{\psi}_4\psi_1\psi_4 + 2\overline{\psi}_2\overline{\psi}_3\psi_2\psi_3
- 2\overline{\psi}_2\overline{\psi}_4\psi_2\psi_4 - 2\overline{\psi}_3\overline{\psi}_4\psi_3\psi_4.
\]

\[
\rho^2 - i^2_x = 4\overline{\psi}_1\overline{\psi}_2\psi_1\psi_2 + 4\overline{\psi}_1\overline{\psi}_3\psi_1\psi_3 + 4\overline{\psi}_1\overline{\psi}_4\psi_1\psi_4 + 4\overline{\psi}_2\overline{\psi}_3\psi_2\psi_3 + 4\overline{\psi}_2\overline{\psi}_4\psi_2\psi_4 + 4\overline{\psi}_3\overline{\psi}_4\psi_3\psi_4,
\]

\[
i^2_x + i^2_y = 4\overline{\psi}_1\overline{\psi}_2\psi_1\psi_2 - 4\overline{\psi}_1\overline{\psi}_4\psi_2\psi_4 - 4\overline{\psi}_2\overline{\psi}_3\psi_1\psi_3 - 4\overline{\psi}_2\overline{\psi}_4\psi_1\psi_4 + 4\overline{\psi}_3\overline{\psi}_4\psi_3\psi_4.
\]
\[ \begin{align*}
\rho^2 - i^2_x - i^2_y - i^2_r &= 4\overline{\psi}_1\psi_3\psi_1\psi_3 + 4\overline{\psi}_2\psi_4\psi_2\psi_4 + 4\overline{\psi}_1\psi_4\psi_2\psi_3 \\
&
+ 4\overline{\psi}_2\psi_3\psi_1\psi_4 \\
&= 4(\psi_1\psi_3 + \psi_2\psi_4)(\psi_1\overline{\psi}_3 + \psi_2\overline{\psi}_4) = \overline{Q}Q;
\end{align*} \]

\[ \overline{Q} = 2(\psi_1\psi_3 + \psi_2\psi_4), \quad Q = (\psi_1\overline{\psi}_3 + \psi_2\overline{\psi}_4). \]

\[ \begin{bmatrix}
\left(\frac{W}{c} + \frac{e}{c} \varphi\right) + \rho_3 \sum_x \sigma_x \left(p_x + \frac{e}{c} C_x\right) + \rho_1 mc
\end{bmatrix} \psi = 0. \]

\[ \delta \int \overline{\psi} \begin{bmatrix}
\left(\frac{W}{c} + \frac{e}{c} \varphi\right) + \rho_3 \sum_x \sigma_x \left(p_x + \frac{e}{c} C_x\right) + \rho_1 mc
\end{bmatrix} \psi \ d\tau = 0; \]

\[ d\tau = dV \ dt. \]

\[ \psi_1\overline{\psi}_3 + \psi_2\overline{\psi}_4 - \overline{\psi}_1\psi_3 - \overline{\psi}_4\psi_2 = 0. \]

\[ \delta \int \begin{bmatrix}
\overline{\psi} \begin{bmatrix}
\left(\frac{W}{c} + \frac{e}{c} \varphi\right) + \rho_3 \sum_x \sigma_x \left(p_x + \frac{e}{c} C_x\right) + \rho_1 mc
\end{bmatrix} \psi \\
+ \lambda i(\psi_1\psi_3 + \psi_2\psi_4 - \psi_1\overline{\psi}_3 - \psi_2\overline{\psi}_4)
\end{bmatrix} \ d\tau = 0. \]

\[ \begin{vmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & i \\
-i & 0 & 0 & 0 \\
0 & -i & 0 & 0
\end{vmatrix} = -\rho_2. \]

\[ \delta \int \overline{\psi} \begin{bmatrix}
\left(\frac{W}{c} + \frac{e}{c} \varphi\right) + \rho_3 \sum_x \sigma_x \left(p_x + \frac{e}{c} C_x\right) + \rho_1 mc - \lambda \rho_2
\end{bmatrix} \psi \ d\tau = 0. \]

\[ \begin{cases}
\left[\frac{W}{c} + \frac{e}{c} \varphi + \rho_3 \sum_x \sigma_x \left(p_x + \frac{e}{c} C_x\right) + \rho_1 mc\right] \varphi = \lambda \rho_2 \psi, \\
\overline{\psi} \rho_2 \psi = 0.
\end{cases} \]

\[ \rho_3 \sigma_x = \alpha_x, \quad \rho_3 \sigma_y = \alpha_y, \quad \rho_3 \sigma_z = \alpha_z, \quad \rho_1 = \alpha_4, \quad \rho_2 = \alpha_5; \]
\[ \alpha_i \alpha_k + \alpha_k \alpha_i = 2 \delta_{ik}; \]
\[ \alpha = (\alpha_x, \alpha_y, \alpha_z). \]

\[ \left[ \frac{W}{c} + \frac{e}{c} \varphi + \alpha \cdot \left( p + \frac{e}{c} C \right) + \alpha_4 mc \right] \psi = \alpha_5 \lambda \psi, \quad \bar{\psi} \alpha_5 \psi = 0. \]

\[ - \frac{W}{c} \psi = \left[ \frac{e}{c} \varphi + \alpha \cdot \left( p + \frac{e}{c} C \right) + \alpha_4 mc - \alpha_5 \lambda \right] \psi, \]
\[ - \bar{\psi} \alpha_5 \frac{W}{c} \psi = \frac{e}{c} \varphi \bar{\psi} \alpha_5 \psi - \sum_x \bar{\psi} \alpha_x \alpha_5 \left( p_x + \frac{e}{c} C_x \right) \psi - \bar{\psi} \alpha_4 \alpha_5 mc \psi - \lambda \bar{\psi}. \]

\[ A = (\psi_1, \psi_2), \quad B(\psi_3, \psi_4). \]

\[ \left[ \frac{W}{c} + \frac{e}{c} \varphi - \sigma \cdot \left( p + \frac{e}{c} C \right) \right] A + mc B = - \lambda i B, \]
\[ \tilde{B} A - \tilde{A} B = 0. \]

\[ \left[ \frac{W}{c} + \frac{e}{c} \varphi - \sigma \cdot \left( p + \frac{e}{c} C \right) \right] B + mc A = \lambda i B. \]
\[
\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4 - \bar{\psi}_3 \psi_1 - \bar{\psi}_4 \psi_2 = 0.
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \bar{\psi}_1 - \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - \frac{e}{c} (C_x + i C_y) \right] \bar{\psi}_2
\]

\[- \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} - \frac{e}{c} C_z \right) \bar{\psi}_1 + mc \bar{\psi}_3 = \lambda \bar{\psi}_3,
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \bar{\psi}_2 - \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) - \frac{e}{c} (C_x - i C_y) \right] \bar{\psi}_1
\]

\[+ \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} - \frac{e}{c} C_z \right) \bar{\psi}_2 + mc \bar{\psi}_4 = \lambda i \bar{\psi}_4,
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \bar{\psi}_3 + \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - \frac{e}{c} (C_x + i C_y) \right] \bar{\psi}_4
\]

\[+ \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} - \frac{e}{c} C_z \right) \bar{\psi}_3 + mc \bar{\psi}_1 = -\lambda i \bar{\psi}_1,
\]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \bar{\psi}_4 + \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) - \frac{e}{c} (C_x - i C_y) \right] \bar{\psi}_3
\]

\[- \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} - \frac{e}{c} C_z \right) \bar{\psi}_4 + mc \bar{\psi}_2 = -\lambda i \bar{\psi}_2.
\]
\[ -\bar{\psi}_4 \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_2 - mc \bar{\psi}_4 \psi_4 - \lambda \bar{\psi}_4 \psi_4 + \text{complex conjugate terms.} \]

\[
\delta \int \bar{\psi} \left[ \frac{W}{c} + \frac{e}{c} \varphi + \rho_3 \sum_x \sigma_x \left( p + \frac{e}{c} C_x \right) + (\cos \lambda \rho_1 + \sin \lambda \rho_2) mc \right] \psi = 0.
\]

\[
\left( -\frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \psi_1 + \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x - iC_y) \right] \psi_2 + \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_1 + e^{-i\lambda mc} \psi_3 = 0,
\]

\[
\left( -\frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \psi_2 + \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x + iC_y) \right] \psi_1 - \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_2 + e^{-i\lambda mc} \psi_4 = 0,
\]

\[
\left( -\frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \psi_3 - \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x - iC_y) \right] \psi_4 - \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_3 + e^{i\lambda mc} \psi_1 = 0,
\]

\[
\left( -\frac{1}{c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right) \psi_4 - \left[ \frac{\hbar}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x + iC_y) \right] \psi_3 + \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_4 + e^{i\lambda mc} \psi_2 = 0.
\]

\[ \bar{\psi}(-\sin \lambda \rho_1 + \cos \lambda \rho_2)\psi = 0. \]

\[
\rho_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},
\]
\[
\begin{align*}
\cos \lambda \rho_1 + \sin \lambda \rho_2 &= \begin{vmatrix}
0 & 0 & e^{-i\lambda} & 0 \\
0 & 0 & 0 & e^{-i\lambda} \\
e^{i\lambda} & 0 & 0 & 0 \\
0 & e^{i\lambda} & 0 & 0
\end{vmatrix}, \\
- \sin \lambda \rho_1 + \cos \lambda \rho_2 &= \begin{vmatrix}
0 & 0 & -ie^{-i\lambda} & 0 \\
0 & 0 & 0 & -ie^{-i\lambda} \\
ie^{i\lambda} & 0 & 0 & 0 \\
ie^{i\lambda} & 0 & 0 & 0
\end{vmatrix}.
\end{align*}
\]

\[
\tilde{\psi}(-\sin \lambda \rho_1 + \cos \lambda \rho_2) \psi = (1/i) \left( e^{-i\lambda} \bar{\psi}_1 \psi_3 + e^{-i\lambda} \bar{\psi}_2 \psi_4 - e^{i\lambda} \bar{\psi}_3 \psi_1 - e^{i\lambda} \bar{\psi}_4 \psi_2 \right) = 0.
\]

\[
e^{-i\lambda} (\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4) - e^{i\lambda} (\bar{\psi}_3 \psi_1 + \bar{\psi}_4 \psi_2) = 0.
\]

\[
\frac{1}{c} \frac{h}{2\pi i} \frac{\partial}{\partial t} \left( e^{-i\lambda} \bar{\psi}_1 \psi_3 + e^{-i\lambda} \bar{\psi}_2 \psi_4 - e^{i\lambda} \bar{\psi}_3 \psi_1 - e^{i\lambda} \bar{\psi}_4 \psi_2 \right)
\]

\[
= -\frac{1}{c} \frac{h}{2\pi} \left( e^{-i\lambda} \bar{\psi}_1 \psi_3 + e^{-i\lambda} \bar{\psi}_2 \psi_4 + e^{i\lambda} \bar{\psi}_3 \psi_2 + e^{i\lambda} \bar{\psi}_4 \psi_2 \right) \frac{\partial \lambda}{\partial t} + D + \overline{D},
\]

\[
D = e^{-i\lambda} \left\{ \bar{\psi}_1 \frac{e}{c} \varphi_3 - \psi_1 \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x - i C_y) \right] \psi_4 \\
- \bar{\psi}_1 \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_3 + e^{i\lambda mc} \bar{\psi}_1 \psi_1 \right\} + e^{-i\lambda} \left\{ \bar{\psi}_2 \frac{e}{c} \varphi_4 - \psi_2 \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x + i C_y) \right] \psi_3 \\
+ \bar{\psi}_2 \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_4 + e^{i\lambda mc} \bar{\psi}_2 \psi_2 \right\} + e^{i\lambda} \left\{ \bar{\psi}_3 \frac{e}{c} \varphi_1 + \bar{\psi}_3 \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x - i C_y) \right] \psi_2 \\
+ \bar{\psi}_3 \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_1 + e^{-i\lambda mc} \bar{\psi}_3 \psi_3 \right\} + e^{i\lambda} \left\{ \bar{\psi}_4 \frac{e}{c} \varphi_2 + \bar{\psi}_4 \left[ \frac{h}{2\pi i} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) + \frac{e}{c} (C_x + i C_y) \right] \psi_1 \\
- \bar{\psi}_4 \left( \frac{h}{2\pi i} \frac{\partial}{\partial z} + \frac{e}{c} C_z \right) \psi_2 + e^{-i\lambda mc} \bar{\psi}_4 \psi_4 \right\}
\right\}.
\]
\[
\frac{e}{c} C_x \left[ e^{-i\lambda} (\psi_1 \psi_4 + \psi_2 \psi_3) + e^{i\lambda} (\bar{\psi}_3 \psi_2 + \bar{\psi}_4 \psi_1) \right] \\
+ \frac{e}{c} C_y \left[ i e^{-i\lambda} (\psi_1 \psi_4 - \psi_2 \psi_3) - i e^{i\lambda} (\bar{\psi}_4 \psi_1 - \bar{\psi}_3 \psi_2) \right] \\
- \frac{e}{c} C_z \left[ e^{-i\lambda} (\psi_1 \psi_3 - \psi_2 \psi_4) + e^{i\lambda} (\bar{\psi}_3 \psi_1 - \bar{\psi}_4 \psi_2) \right] \\
+ mc \left( \psi_1 \psi_1 + \psi_2 \psi_2 - \psi_3 \psi_3 - \psi_4 \psi_4 \right) \\
- \frac{\hbar}{2\pi i} \left[ \left( \frac{\partial}{\partial x} \psi_4 + \frac{\partial}{\partial x} \psi_3 \right) e^{-i\lambda} + \left( \frac{\partial}{\partial x} \psi_2 + \frac{\partial}{\partial x} \psi_1 \right) e^{i\lambda} \right] \\
+ \frac{\hbar}{2\pi i} \left[ \left( \frac{\partial}{\partial y} \psi_4 - \frac{\partial}{\partial y} \psi_3 \right) e^{-i\lambda} + \left( \frac{\partial}{\partial y} \psi_2 - \frac{\partial}{\partial y} \psi_1 \right) e^{i\lambda} \right] \\
- \frac{\hbar}{2\pi i} \left[ \left( \frac{\partial}{\partial z} \psi_1 - \frac{\partial}{\partial z} \psi_4 \right) e^{-i\lambda} + \left( \frac{\partial}{\partial z} \psi_3 - \frac{\partial}{\partial z} \psi_2 \right) e^{i\lambda} \right].
\]

\[
\beta = \begin{vmatrix} \\
0 & 0 & e^{-i\lambda} & 0 \\
0 & 0 & 0 & e^{-i\lambda} \\
e^{-i\lambda} & 0 & 0 & 0 \\
0 & e^{-i\lambda} & 0 & 0 \\
\end{vmatrix}, \\
\gamma = \begin{vmatrix} \\
0 & 0 & -ie^{-i\lambda} & 0 \\
0 & 0 & 0 & -ie^{-i\lambda} \\
ie^{i\lambda} & 0 & 0 & 0 \\
0 & e^{i\lambda} & 0 & 0 \\
\end{vmatrix}.
\]

\[
\psi = \begin{vmatrix} \\
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\end{vmatrix}, \quad \psi^\dagger = \begin{vmatrix} \\
\psi_1, \psi_2, \psi_3, \psi_4 \\
\end{vmatrix}, \\
\bar{\psi} = \begin{vmatrix} \\
\bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_3 \bar{\psi}_4 \\
\end{vmatrix}.
\]

\[
\beta = \beta(\lambda), \quad \gamma = \gamma(\lambda); \\
\beta = \cos \lambda \rho_1 + \sin \lambda \rho_2, \quad \gamma = -\sin \lambda \rho_1 + \cos \lambda \rho_2; \\
\beta\gamma = \gamma\beta = 0, \quad \beta^2 = \gamma^2 = 1.
\]

\[
\bar{\psi}\gamma\psi = 0.
\]

---

4. Note that some things in the last three square brackets (the x, y, z-derivatives and the indices 1, 2, 3, 4 of the ψ components) should be slightly corrected. However, at variance with what is usually done by us, we choose to leave unchanged the expressions appearing in the original manuscript.
\[ 0 = -\frac{1}{c} \frac{\hbar}{2\pi} \tilde{\psi} \beta \psi \frac{\partial \lambda}{\partial t} - 2e \frac{\hbar}{c} \tilde{\psi} \beta \sigma \cdot C \psi - \tilde{\psi} \beta \sigma \cdot p \psi + \psi^\dagger \beta \sigma \cdot \tilde{p} \psi. \]

\[ \tilde{\psi} \beta \psi = e^{-i\lambda} (\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4) + e^{i\lambda} (\bar{\psi}_3 \psi_1 + \bar{\psi}_4 \psi_2) = 2e^{-i\lambda} (\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4). \]

\[ \frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} \left( e^{-i\lambda} \bar{\psi}_1 \psi_3 + e^{-i\lambda} \bar{\psi}_2 \psi_4 + e^{i\lambda} \bar{\psi}_3 \psi_1 + e^{i\lambda} \bar{\psi}_4 \psi_2 \right) \]
\[ = \frac{1}{c} \frac{\hbar}{2\pi} \left( -i e^{-i\lambda} \bar{\psi}_1 \psi_3 - i e^{-i\lambda} \bar{\psi}_2 \psi_4 + i e^{i\lambda} \bar{\psi}_3 \psi_1 + i e^{i\lambda} \bar{\psi}_4 \psi_1 \right) \frac{\partial \lambda}{\partial t} \]
\[ + L + \overline{L}, \]

\[ L = i e^{-i\lambda} \left\{ \frac{-e}{c} \phi \psi_3 - \frac{h}{2\pi} \left[ \frac{e}{c} \frac{\partial}{\partial x} \right] - i \frac{e}{c} \frac{\partial}{\partial y} \right\} \]
\[ + i e^{-i\lambda} \left\{ \ldots \right\} \]
\[ + i e^{i\lambda} \left\{ \ldots \right\} \]
\[ + i e^{i\lambda} \left\{ \ldots \right\} \]
\[ = i \frac{e}{c} \phi \left[ e^{-i\lambda} (\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4) + e^{i\lambda} (\bar{\psi}_3 \psi_1 + \bar{\psi}_4 \psi_2) \right] \]
\[ + i \frac{e}{c} C_x \left[ e^{-i\lambda} (\bar{\psi}_1 \psi_4 + \bar{\psi}_2 \psi_3) - e^{i\lambda} (\bar{\psi}_3 \psi_2 + \bar{\psi}_4 \psi_1) \right] \]
\[ + i \frac{e}{c} C_y \left[ \ldots \right] \]
\[ \pm i \frac{e}{c} C_z \left[ \ldots \right] \]
\[ - \frac{h}{2\pi} \left[ (\bar{\psi}_1 \frac{\partial}{\partial x} \psi_4 + \bar{\psi}_2 \frac{\partial}{\partial x} \psi_3) e^{-i\lambda} - (\bar{\psi}_3 \frac{\partial}{\partial x} \psi_2 + \bar{\psi}_4 \frac{\partial}{\partial x} \psi_1) e^{i\lambda} \right] \]
\[ - \frac{h}{2\pi} \left[ \ldots \right] \]
\[ - \frac{h}{2\pi} \left[ \ldots \right]. \]
\[
\frac{1}{c} \frac{\hbar}{2\pi} \frac{\partial}{\partial t} (\bar{\psi} \beta \psi) = 2 \frac{e}{c} \bar{\psi} \gamma \sigma \cdot C \psi + \bar{\psi} \gamma \sigma \cdot p \psi - \psi^* \gamma \sigma \cdot p \bar{\psi}.
\]

\[
e^{-i\lambda}(\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4) - e^{i\lambda}(\bar{\psi}_3 \psi_1 + \bar{\psi}_4 \psi_2) = 0.
\]

\[
e^{i\lambda} = \sqrt{\frac{\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4}{\psi_3 \psi_1 + \psi_4 \psi_2}}; \quad e^{-i\lambda}(\bar{\psi}_1 \psi_3 + \bar{\psi}_2 \psi_4) > 0;
\]

provided that not all \( \psi_i \) be zero \( (\psi_1 = \psi_2 = \psi_3 = \psi_4 = 0) \) at the same time.

### 1.3. QUANTIZATION OF THE DIRAC FIELD

The canonical quantization of a Dirac field \( \psi \) is here considered (starting from a Lagrangian density \( L \)), by introducing the field variables \( P, \bar{P} \) conjugate to \( \psi, \bar{\psi} \). After imposing the commutation rules, the Hamiltonian \( H \) was deduced, and an expression for the energy \( W \) was obtained in terms of the annihilation and creation operators \( a, b \). The quantities \( n_i \) are number operators.

\[
WA = VA - c \sigma \cdot p B - mc^2 A,
\]

\[
WB = VB - c \sigma \cdot p A - mc^2 B.
\]

\[
W_0 B_0 = \left( V + \frac{p^2}{2m} + mc^2 \right) B_0,
\]

\[
A_0 = - \frac{\sigma \cdot p B_0}{2mc}.
\]

\[
W = - \frac{\hbar}{2\pi i} \frac{\partial}{\partial t} p_x = \frac{\hbar}{2\pi i} \frac{\partial}{\partial x}
\]

\[
L = \frac{1}{2m} \left\{ \left( -\frac{W}{c} + \frac{e}{c} \varphi \right) \bar{\psi} \left( \frac{W}{c} + \frac{e}{c} \varphi \right) \psi + \sum_x \left( -p_x + \frac{e}{c} A_x \right) \bar{\psi} \left( p_x + \frac{e}{c} A_x \right) \psi + m^2 c^2 \bar{\psi} \psi \right\}.
\]
\[ \psi, \quad P = \left( -\frac{W}{c} + \frac{e}{c} \varphi \right) \overline{\psi}; \]
\[ \overline{\psi}, \quad \overline{P} = \left( \frac{W}{c} + \frac{e}{c} \varphi \right) \psi. \]

\[ \psi(q) \psi(q') - \psi(q') \psi(q) = 0, \quad P(q) P(q') - P(q') P(q) = 0, \]
\[ \psi(q) \overline{\psi}(q') - \overline{\psi}(q') \psi(q) = 0, \quad P(q) \overline{P}(q') - \overline{P}(q') P(q) = 0, \]
\[ \overline{\psi}(q) \overline{\psi}(q') - \overline{\psi}(q') \overline{\psi}(q) = 0, \quad \overline{P}(q) \overline{P}(q') - \overline{P}(q') \overline{P}(q) = 0. \]

\[ \psi(q) P(q') - P(q') \psi(q) = \delta(q - q') \, 2mc, \]
\[ \psi(q) \overline{P}(q') - \overline{P}(q') \psi(q) = 0, \]
\[ \overline{\psi}(q) P(q') - P(q') \overline{\psi}(q) = 0, \quad \overline{\psi}(q) \overline{P}(q') - \overline{P}(q') \overline{\psi}(q') = -\delta(q - q') \, 2mc. \]

\[ H = \frac{1}{2m} \left\{ \left( -\frac{W}{c} + \frac{e}{c} \varphi \right) \overline{\psi} \frac{W}{c} \psi + \left( \frac{W}{c} + \frac{e}{c} \varphi \right) \psi \frac{W}{c} \overline{\psi} \right\} - L \]
\[ = \frac{1}{2m} \left\{ P \left( \overline{P} - \frac{e}{c} \varphi \overline{\psi} \right) + \overline{P} \left( P - \frac{e}{c} \varphi \psi \right) - PP \right. \]
\[ + \sum_x \left( -p_x + \frac{e}{c} A_x \right) \overline{\psi} \left( p_x + \frac{e}{c} A_x \right) \psi + \frac{m^2 c^2}{2} \overline{\psi} \psi \]
\[ = \frac{1}{2m} \left\{ \overline{P} P - \frac{e}{c} \varphi (P \psi + \overline{P} \overline{\psi}) \right. \]
\[ + \sum_x \left( -p_x + \frac{e}{c} A_x \right) \overline{\psi} \left( p_x + \frac{e}{c} A_x \right) \psi + \frac{m^2 c^2}{2} \overline{\psi} \psi \} \].

\[ a, \overline{a}; \quad b, \overline{b}. \]

\[ ab - ba = 2mc, \]
\[ \overline{a} \overline{b} - b \overline{a} = -2mc. \]

\[ n = \frac{1}{2\sqrt{mc}} \left\{ \frac{1}{\sqrt{4/m^2 c^2 + p^2}} \overline{b} + \sqrt{4/m^2 c^2 + p^2} a \right\}, \]
\[ n' = \frac{1}{2\sqrt{mc}} \left\{ \frac{1}{\sqrt{4/m^2 c^2 + p^2}} b - \sqrt{4/m^2 c^2 + p^2} \overline{a} \right\}. \]
\[
1 + n_1 + n_2 = \frac{1}{2mc} \left[ \frac{1}{\sqrt{m^2 c^2 + p^2}} \bar{b} + \sqrt{m^2 c^2 + p^2 \bar{a}} \right], \\
n_1 - n_2 = \frac{1}{2mc} [\bar{a}b + ab]; \\
n_1 = \frac{1}{4mc} \left\{ \frac{1}{\sqrt{m^2 c^2 + p^2}} b + \frac{4}{\sqrt{m^2 c^2 + p^2 \bar{a}}} \right\} \\
\times \left\{ \frac{1}{\sqrt{m^2 c^2 + p^2}} \bar{b} + \frac{4}{\sqrt{m^2 c^2 + p^2 a}} \right\}, \\
n_2 = \frac{1}{4mc} \left\{ \frac{1}{\sqrt{m^2 c^2 + p^2}} \bar{b} - \frac{4}{\sqrt{m^2 c^2 + p^2 a}} \right\} \\
\times \left\{ \frac{1}{\sqrt{m^2 c^2 + p^2}} b - \frac{4}{\sqrt{m^2 c^2 + p^2 \bar{a}}} \right\}.
\]

\[
\psi = \sum a_i f_i, \quad P = \sum b_i \bar{f}_i; \\
\bar{\psi} = \sum \bar{a}_i \bar{f}_i, \quad \bar{P} = \sum \bar{b}_i f_i.
\]

\[
W = \frac{1}{2m} \left\{ \sum_i \bar{b}_i b_i + \sum_i (m^2 c^2 + p_i^2) \bar{a}_i a_i \\
- \frac{e}{c} \sum_{i,k} \int \bar{f}_i(q) f_k(q) \varphi(q) \, dq \cdot (b_i a_k + b_k a_i) \\
+ \frac{e}{c} \sum_{i,k} \int \bar{f}_i(q) f_k(q) (p_i + p_k) \cdot A \, dq \cdot \bar{a}_i a_k \\
+ \frac{e^2}{c^2} \sum_{i,k} \int \bar{f}_i(q) f_k(q) A^2 \, dq \right\}. \\
\]

\[
a_i = \sqrt[4]{\frac{m^2 c^2}{m^2 c^2 + p_i^2}} (u_i - \bar{v}_i), \\
b_i = mc \sqrt[4]{\frac{m^2 c^2 + p_i^2}{m^2 c^2}} (\bar{u}_i + v_i);
\]
\[
\begin{align*}
\bar{a}_i a_k &= mc \sqrt[4]{m^2 c^2 + p_i^2} \left( \bar{u}_i u_k - \bar{v}_i v_k - \bar{u}_i \bar{v}_k + v_i u_k \right), \\
b_i a_k &= mc \sqrt[4]{m^2 c^2 + p_k^2} \left( \bar{u}_i u_k - \bar{v}_i v_k - \bar{u}_i \bar{v}_k + v_i u_k \right).
\end{align*}
\]

### 1.4. INTERACTING DIRAC FIELDS

In the following pages, the author again studied the problem of the electromagnetic interaction of a Dirac field \( \psi \); the electromagnetic scalar and vector potentials are denoted with \( \varphi \) and \( C \), respectively. After some explicit passages on the (interacting) Dirac equation (see Sect. 1.4.1), Majorana considered in some detail also the Maxwell equations for the electromagnetic field (see Sect. 1.4.2). The starting point are the field equations deduced from a variational principle, and the role of the gauge constraints is particularly pointed out. The superposition of Dirac and Maxwell fields was, then, studied using again a canonical formalism (see Sect. 1.4.3); choosing appropriate state variables and conjugate momenta, the quantization of both the Dirac and the Maxwell field was carried out. An expression for the Hamiltonian of the interacting system was deduced and, finally, normal mode decomposition was well introduced (see Sect. 1.4.3.1). This part ends with some explicit matrix expressions for the Dirac operators in particular representations (see Sect. 1.4.3.2).

#### 1.4.1 Dirac Equation

\[
\left[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) + \alpha_x \left( p_x + \frac{e}{c} C_x \right) + \alpha_y \left( p_y + \frac{e}{c} C_y \right) + \alpha_z \left( p_z + \frac{e}{c} C_z \right) + \beta mc \right] \psi = 0;
\]

\[
\begin{align*}
\alpha_x &= \rho_1 \sigma_x, & \alpha_y &= \rho_1 \sigma_y, & \alpha_z &= \rho_1 \sigma_z, & \beta &= \rho_3;
\end{align*}
\]

\[
-\frac{1}{e} \rho = \psi \bar{\psi}, \quad -\frac{1}{e} i_x = -\psi \alpha_x \bar{\psi}, \quad -\frac{1}{e} i_y = -\psi \alpha_y \bar{\psi}, \quad -\frac{1}{e} i_z = -\psi \alpha_z \bar{\psi};
\]
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\[ \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \]

\[ \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

\[ P_0 = \frac{W}{c} + \frac{e}{c} \varphi, \quad P_x = p_x + \frac{e}{c} C_x, \quad P_y = p_y + \frac{e}{c} C_y, \quad P_z = p_z + \frac{e}{c} C_z. \]

\[ F = (P_x, P_y, P_z), \quad \alpha = (\alpha_x, \alpha_y, \alpha_z). \]

\[ [P_0 + \alpha \cdot F + \beta mc] \psi = 0. \]

\[ (P_0 + mc)\psi_1 + (P_x - iP_y)\psi_4 + P_z\psi_3 = 0, \]

\[ (P_0 + mc)\psi_2 + (P_x + iP_y)\psi_3 - P_z\psi_4 = 0, \]

\[ (P_0 - mc)\psi_3 + (P_x - iP_y)\psi_2 + P_z\psi_1 = 0, \]

\[ (P_0 - mc)\psi_4 + (P_x + iP_y)\psi_1 - P_z\psi_2 = 0. \]

\[ \left( \frac{W}{c} + mc \right) \psi_1 + (p_x - ip_y)\psi_4 + p_z\psi_3 \]

\[ + \frac{e}{c} [\varphi \psi_1 + (C_x - iC_y)\psi_4 + C_z\psi_3] = 0, \]
\[
\left( \frac{W}{c} + mc \right) \psi_2 + (p_x + ip_y) \psi_3 - p_z \psi_4 \\
+ \frac{e}{c} [\varphi \psi_2 + (C_x + iC_y) \psi_3 - C_z \psi_4] = 0,
\]

\[
\left( \frac{W}{c} - mc \right) \psi_3 + (p_x - ip_y) \psi_2 + p_z \psi_1 \\
+ \frac{e}{c} [\varphi \psi_3 + (C_x - iC_y) \psi_2 + C_z \psi_1] = 0,
\]

\[
\left( \frac{W}{c} - mc \right) \psi_4 + (p_x + ip_y) \psi_1 - p_z \psi_2 \\
+ \frac{e}{c} [\varphi \psi_4 + (C_x + iC_y) \psi_1 - C_z \psi_3] = 0;
\]

\[
\left( - \frac{W}{c} + mc \right) \overline{\psi}_1 - (p_x + ip_y) \overline{\psi}_4 - p_z \overline{\psi}_3 \\
+ \frac{e}{c} [\varphi \overline{\psi}_1 + (C_x + iC_y) \overline{\psi}_4 + C_z \overline{\psi}_3] = 0,
\]

\[
\left( - \frac{W}{c} + mc \right) \overline{\psi}_2 - (p_x - ip_y) \overline{\psi}_3 + p_z \overline{\psi}_4 \\
+ \frac{e}{c} [\varphi \overline{\psi}_2 + (C_x - iC_y) \overline{\psi}_3 - C_z \overline{\psi}_4] = 0,
\]

\[
\left( - \frac{W}{c} - mc \right) \overline{\psi}_3 - (p_x + ip_y) \overline{\psi}_2 - p_z \overline{\psi}_1 \\
+ \frac{e}{c} [\varphi \overline{\psi}_3 + (C_x + iC_y) \overline{\psi}_2 + C_z \overline{\psi}_1] = 0,
\]

\[
\left( - \frac{W}{c} - mc \right) \overline{\psi}_4 - (p_x - ip_y) \overline{\psi}_1 + p_z \overline{\psi}_2 \\
+ \frac{e}{c} [\varphi \overline{\psi}_4 + (C_x - iC_y) \overline{\psi}_1 - C_z \overline{\psi}_2] = 0.
\]

\[
\begin{align*}
 u_0 &= \overline{\psi}_1 \psi_1 + \overline{\psi}_2 \psi_2 + \overline{\psi}_3 \psi_3 + \overline{\psi}_4 \psi_4, \\
 u_x &= - (\overline{\psi}_1 \psi_4 + \overline{\psi}_2 \psi_3 + \overline{\psi}_3 \psi_2 + \overline{\psi}_4 \psi_1), \\
 u_y &= i (\overline{\psi}_1 \psi_4 - \overline{\psi}_2 \psi_3 + \overline{\psi}_3 \psi_2 - \overline{\psi}_4 \psi_1), \\
 u_z &= - (\overline{\psi}_1 \psi_3 - \overline{\psi}_2 \psi_4 + \overline{\psi}_3 \psi_1 - \overline{\psi}_4 \psi_2).
\end{align*}
\]

### 1.4.2 Maxwell Equations

\[
\begin{align*}
 x_0 &= i t, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z; \\
 S_0 &= i \rho, \quad S_1 = \rho \frac{v_x}{c}, \quad S_2 = \rho \frac{v_y}{c}, \quad S_3 = \rho \frac{v_z}{c};
\end{align*}
\]
\[ \phi_0 = i \varphi, \quad \phi_1 = C_x, \quad \phi_2 = C_y, \quad \phi_3 = C_z; \]
\[ F_{ik} = \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k}. \]

\[ F_{01} = iE_x, \quad F_{23} = H_x, \]
\[ F_{02} = iE_y, \quad F_{31} = H_y, \]
\[ F_{03} = iE_z, \quad F_{12} = H_z. \]

The Maxwell equations are:

\[ \sum_k \frac{\partial F_{ik}}{\partial x_k} = 4\pi S_i; \] \[ \text{I} \]
\[ \frac{\partial F_{ik}}{\partial x_i} + \frac{\partial F_{kl}}{\partial x_l} + \frac{\partial F_{li}}{\partial x_k} = 0. \] \[ \text{II} \]

\[ 4\pi S_i = \sum_k \frac{\partial F_{ik}}{\partial x_k} = \frac{\partial}{\partial x_i} \sum_k \frac{\partial \phi_k}{\partial x_k} - \sum_k \frac{\partial^2 \phi_i}{\partial x_k} \]
\[ = \frac{\partial}{\partial x_i} \nabla \cdot \phi - \nabla^2 \phi_i, \]
\[ 4\pi S = \nabla \times \nabla \cdot \phi - \nabla^2 \phi. \]

Additional constraint:

\[ \nabla \cdot \phi = 0; \]
\[ \nabla^2 \phi + 4\pi S = 0. \]

Variational approach:

\[ \delta \int \sum_{i<k} F^2_{ik} d\tau = \delta \int \sum_k \left[ \left( \frac{\partial \phi_k}{\partial x_i} \right)^2 - \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_i}{\partial x_k} \right] d\tau \]
\[ = -2 \int \sum_k \left[ \nabla^2 \phi_k - \frac{\partial}{\partial x_k} \nabla \cdot \phi \right] \delta \phi_k \]
\[ = 2 \int \sum_k \left( \frac{\partial}{\partial x_k} \nabla \cdot \phi - \nabla^2 \phi_k \right) \delta \phi_k; \]
\[
\delta \int S \cdot \phi \, d\tau = \int \sum_k S_k \delta \phi_k;
\]

\[
\delta \int \left[ -S \cdot \phi + \frac{1}{8\pi} \sum_{i<k} F^2_{ik} \right] d\tau = -\sum_k \left[ S_k + \frac{1}{4\pi} \nabla^2 \phi_k - \frac{1}{4\pi} \frac{\partial}{\partial x_k} \nabla \cdot \phi \right] \delta \phi_k.
\]

\[
\delta \int \left[ +S \cdot \phi - \frac{1}{8\pi} \sum_{i<k} F^2_{ik} \right] d\tau = 0,
\]

\[
4\pi S + \nabla^2 \phi - \nabla (\nabla \cdot \phi) = 0.
\]

The Maxwell equations are obtained from:

\[
\delta \int \left[ +S \cdot \phi - \frac{1}{8\pi} \sum \left( \frac{\partial \phi_k}{\partial x_i} \right)^2 \right] d\tau = 0;
\]

\[
\nabla^2 \phi + 4\pi S = 0,
\]

\[
\nabla \cdot \phi = 0.
\]

### 1.4.3 Maxwell-Dirac Theory

\[
\left[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) + \alpha \cdot \left( p + \frac{e}{c} C \right) + \beta mc \right] = M;
\]

\[
M \psi = 0.
\]

The Dirac equation is obtained from:

\[
\delta \int \bar{\psi} M \psi \, d\tau = 0;
\]

\[
(\delta \bar{\psi}) M \psi + \bar{\psi} M \delta \psi = 2 \text{ Re} \left[ (\delta \bar{\psi}) M \psi \right] = 0,
\]

\[
M \psi = 0.
\]
In
\[ \delta \int \left[ \bar{\psi} M \psi - \frac{1}{8\pi} \sum_{i<k} F_{ik}^2 \right] d\tau = 0, \]
the Dirac equation
\[ M \psi = 0 \]
is obtained from a variation of the variables \( \psi \), while the Maxwell equations
\[ -4\pi S - \nabla^2 \phi + \nabla (\nabla \cdot \phi) = 0 \]
come from a variation of \( \phi \).

Eichinvarianz: \( \psi = 0 \).

State variables:
\[ \psi_1, \psi_2, \psi_3, \psi_4; \quad C_x, C_y, C_z; \]
Conjugate momenta:
\[ -\frac{\hbar}{2\pi i} \bar{\psi}_1, \quad -\frac{\hbar}{2\pi i} \bar{\psi}_2, \quad -\frac{\hbar}{2\pi i} \bar{\psi}_3, \quad -\frac{\hbar}{2\pi i} \bar{\psi}_4; \]
\[ P_x = -\frac{E_x}{4\pi c}, \quad P_y = -\frac{E_y}{4\pi c}, \quad P_z = -\frac{E_z}{4\pi c}. \]
\[ E = \frac{1}{c} \frac{\partial C}{\partial t}, \quad H = \nabla \times C; \]
\[ \varphi = 0, \]
\[ \nabla \cdot C = 0. \]

\[ \delta \int \bar{\psi} \left[ +W + c \alpha \cdot \left( p + \frac{e}{c} C \right) + \beta mc^2 \right] \psi \]
\[ - \frac{1}{8\pi} \left[ (\nabla \times C)^2 - \frac{1}{c^2} \left( \frac{\partial C}{\partial t} \right)^2 \right] d\tau = 0. \]

\( ^5 \) This German word means “gauge invariance”; the author uses this property in order to set the potential \( \varphi \) to zero.
\[ P_i(q)C_k(q') - C_k(q')P_i(q) = \frac{\hbar}{2\pi i} \delta(q - q'), \]
\[ \psi_i(q)\overline{\psi}_k(q') + \overline{\psi}_k(q')\psi_i(q) = \delta(q - q'). \]

\[ C = ABA, \]
\[ C_{ik} = \sum A_{ir}B_{rs}A_{sk}, \]
\[ C_{ki} = \sum B_{rs}A_{kr}A_{is} = \sum B_{rs}\overline{A}_{ir}A_{ks}; \]
\[ \frac{\partial C_i}{\partial t} = -cE_i = 4\pi c^2 P_i = C_{ik}. \]

\[ H = \int \left\{ -\tilde{\psi} \left[ c\alpha \cdot (p + \frac{e}{c}C) + \beta mc^2 \right] \psi + \frac{1}{8\pi} |\nabla \times C|^2 + 2\pi c^2 |F|^2 \right\} \, d\tau. \]

1.4.3.1 Normal mode decomposition.

\[ \psi = \sum a_r \psi_r, \quad \overline{\psi} = \sum \overline{a}_r \psi_r; \]
\[ a_r \overline{a}_r + \overline{a}_r a_r = \delta_{rs}. \]

\[ C = \sum q_\nu u_\nu, \quad P = \sum p_\nu u_\nu; \]
\[ p_\nu q_\nu - q_\nu p_\nu = \frac{\hbar}{2\pi}. \]

\[ a_k \tilde{a}_i a_k - \tilde{a}_i a_k a_k = a_k \tilde{a}_i a_k + \tilde{a}_i a_k a_k = \delta_{ik} a_k, \]
\[ a_k \tilde{a}_i b_k - \tilde{a}_i b_k a_k = a_k \tilde{a}_i b_k - \tilde{a}_i a_k b_k = (a_k \tilde{a}_i - \tilde{a}_i a_k) b_k, \]
\[ a_k \tilde{b}_i a_k - \tilde{b}_i a_k a_k = \tilde{b}_i (a_k a_k - a_k a_k). \]

\[ |\nabla \times C|^2 = \sum_{i,k} \left[ \left( \frac{\partial C_i}{\partial x_k} \right)^2 - \frac{\partial C_i}{\partial x_k} \frac{\partial C_k}{\partial x_i} \right]. \]

\[ C_i \sum \frac{\partial c_k}{\partial x_k} = 0, \]

\[ \int \frac{\partial C_i}{\partial x_k} \frac{\partial C_k}{\partial x_i} \, d\tau = - \int C_i \frac{\partial^2 C_k}{\partial x_i \partial x_k} = - \int C_i \frac{\partial}{\partial x_i} \frac{\partial c_k}{\partial x_k}, \]
\[
\sum_{i,k} \int \frac{\partial C_i}{\partial x_i} \frac{\partial C_k}{\partial x_k} d\tau = - \sum_{i,k} \int C_i \frac{\partial}{\partial x_i} C_k = - C \cdot \nabla (\nabla \cdot C) = 0.
\]

\[
\int |\nabla \times C|^2 d\tau = - \int C \nabla^2 C d\tau = \sum q_{\nu}^2 \frac{4 \pi^2 \nu^2}{c^2},
\]

\[
\int |F|^2 d\tau = \sum p_{\nu}^2.
\]

\[
H = - \left[ c \alpha \left( p + \frac{e}{c} \sum q_{\nu} u_{\nu}(q) \right) + \beta mc^2 \right] + \frac{\pi}{2c^2} \sum q_{\nu}^2 \nu^2 + 2\pi c^2 \sum p_{\nu}^2.
\]

\[
\frac{\pi \nu^2}{2c^2} q_{\nu}^2 + 2\pi c^2 p_{\nu}^2 = 2\pi c^2 \left( p_{\nu}^2 + \frac{\nu^2}{4c^2} q_{\nu}^2 \right)
\]

\[
= 2\pi c^2 \left( p_{\nu} - \frac{\nu i}{2c^2} q_{\nu} \right) \left( p_{\nu} + \frac{\nu i}{2c^2} q_{\nu} \right).
\]

\[
c_{\nu} = c \sqrt{\frac{2\pi}{\hbar \nu}} \left( p_{\nu} - \frac{\nu i}{2c^2} q_{\nu} \right), \quad \bar{c}_{\nu} = c \sqrt{\frac{2\pi}{\hbar \nu}} \left( p_{\nu} + \frac{\nu i}{2c^2} q_{\nu} \right);
\]

\[
\tilde{c}_{\nu} c_{\nu} = \frac{W_{\nu}}{\hbar \nu} - \frac{1}{2}, \quad c_{\nu} \bar{c}_{\nu} = \frac{W_{\nu}}{\hbar \nu} + \frac{1}{2},
\]

\[
c_{\nu} \tilde{c}_{\nu} - \tilde{c}_{\nu} c_{\nu} = 1
\]

\[
W_{\nu} = \hbar \nu \left( \tilde{c}_{\nu} c_{\nu} + \frac{1}{2} \right).
\]

### 1.4.3.2 Particular representations of Dirac operators.

\[
\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

\[
\varepsilon^2 = 0, \quad \bar{\varepsilon}^2 = 0, \quad \rho^2 = 1;
\]

\[
\varepsilon \rho + \rho \varepsilon = 0, \quad \varepsilon \rho + \rho \bar{\varepsilon} = 0, \quad \varepsilon \bar{\varepsilon} + \bar{\varepsilon} \varepsilon = 1;
\]

\[
\varepsilon \bar{\varepsilon} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varepsilon \bar{\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

\[
a_r a_s + a_s a_r = \delta_{rs}, \quad a_r a_s + a_s a_r = 0, \quad a_r \bar{a}_s + \bar{a}_s a_r = 0.
\]
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For \( s > r \):

\[
\begin{align*}
    a_r &= \rho_1 \rho_2 \cdots \rho_{r-1} \varepsilon_r, \\
    \bar{a}_r &= \rho_1 \rho_2 \cdots \rho_{r-1} \bar{\varepsilon}_r, \\
    a_s &= \rho_1 \rho_2 \cdots \rho_r \rho_{r+1} \cdots \rho_{s-1} \varepsilon_s, \\
    \bar{a}_s &= \rho_1 \rho_2 \cdots \rho_r \rho_{r+1} \cdots \rho_{s-1} \bar{\varepsilon}_s, \\
    a_r a_s &= -\rho_r \rho_{r+1} \cdots \rho_s \varepsilon_r \varepsilon_s, \\
    a_s a_r &= \rho_r \rho_{r+1} \cdots \rho_s \bar{\varepsilon}_r \bar{\varepsilon}_s, \\
    a_r a_s + a_s a_r &= 0, \\
    \bar{a}_r \bar{a}_s + \bar{a}_s \bar{a}_r &= 0, \\
    a_r \bar{a}_s &= -\rho_r \cdots \rho_{s-1} \varepsilon_r \bar{\varepsilon}_s, \\
    \bar{a}_s a_r &= \rho_r \cdots \rho_{s-1} \bar{\varepsilon}_s \varepsilon_r, \\
    a_r \bar{a}_r &= \varepsilon_r \bar{\varepsilon}_r, \\
    \bar{a}_r a_r &= \bar{\varepsilon}_r \varepsilon_r, \\
    a_r \bar{a}_r + \bar{a}_r a_r &= 1.
\end{align*}
\]

\( c \bar{c} - \bar{c} c = 1, \)

\( c \bar{c} = r. \)

\[
\begin{align*}
    c_{r-1,r} &= \sqrt{r}, \\
    \bar{c}_{r,r-1} &= \sqrt{r}, \\
    c_{rs} &= \delta_{r+1,s} \sqrt{s}, \\
    \bar{c}_{rs} &= \delta_{r-1,s} \sqrt{r}; \\
    (c \bar{c})_{rs} &= \sum_t c_{rt} c_{ts} = t \delta_{r+1,t} \delta_{t-1,s} = t \delta_{rs} = (r + 1) \delta_{rs}, \\
    (\bar{c} c)_{rs} &= \sum_t \bar{c}_{rt} c_{ts} = \sqrt{r} \sqrt{s} \delta_{r-1,t} \delta_{t+1,s} = r \delta_{rs}.
\end{align*}
\]

\( c \bar{c} - \bar{c} c = 1. \)

\[
c = \begin{bmatrix}
    0 & \sqrt{1} & 0 & 0 & 0 \\
    0 & 0 & \sqrt{2} & 0 & 0 \\
    0 & 0 & 0 & \sqrt{3} & 0 \\
    0 & 0 & 0 & 0 & \sqrt{4} \\
    0 & 0 & 0 & 0 & 0 \\
    \cdots
\end{bmatrix}, \quad \bar{c} = \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    \sqrt{1} & 0 & 0 & 0 & 0 \\
    0 & \sqrt{2} & 0 & 0 & 0 \\
    0 & 0 & \sqrt{3} & 0 & 0 \\
    0 & 0 & 0 & \sqrt{4} & 0 \\
    \cdots
\end{bmatrix};
\]
\[
\tilde{c}c = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 4
\end{bmatrix}, \quad c\tilde{c} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

\[
\epsilon = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad \bar{\epsilon} = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}, \quad \rho = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, \quad \epsilon\bar{\epsilon} = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
a_1 = \epsilon_1, \quad \bar{a}_1 = \bar{\epsilon}_1, \\
a_2 = \rho_1\bar{\epsilon}_2, \quad \bar{a}_2 = \rho_1\bar{\epsilon}_2.
\]

\[
a_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{a}_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
a_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{a}_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

\[
a = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & \sqrt{2} \\
0 & 0 & 0
\end{bmatrix}, \quad \bar{a} = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & \sqrt{2} & 0
\end{bmatrix},
\]

\[
\bar{a}a = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}, \quad a\bar{a} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix};
\]

\[
a^2 = \begin{bmatrix}
0 & 0 & \sqrt{2} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \bar{a}^2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\sqrt{2} & 0 & 0
\end{bmatrix};
\]

\[
a\bar{a} + \bar{a}a = \begin{bmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 2
\end{bmatrix}.
\]
1.5. SYMMETRIZATION

Inserted in the discussion of the Maxwell-Dirac theory (see Sect. 1.4.3), we find a page where the (anti-)symmetrization of Dirac fields, describing spin-1/2 particles, was considered.

\[ \psi = \sum a_r \psi_r, \]
\[ \varphi = \varphi(n_r), \]

with \( n_r = 0, 1. \)

1. \( \sum n_r = 1; n_s \) is different from zero:
\[ \varphi = \varphi(s) = c_s; \]
\[ \varphi \sim \sum c_s \psi_s(q). \]

2. \( \sum n_r = 2; n_s, n_t \) are different from zero \((s < t)\):
\[ \varphi = \varphi(s, t) = c_{st}; \]
\[ \varphi \sim \sum_{s<t} c_{st} \frac{\psi_s(q_1)\psi_t(q_2) - \psi_t(q_2)\psi_s(q_1)}{\sqrt{2}}. \]

3. \( \sum n_r = n; n_{i_1}, n_{i_2}, \ldots, n_{i_n} \) are different from zero \((i_1 < i_2 < i_3 < \ldots < i_n)\):
\[ \varphi = \varphi(i_1, i_2, \ldots i_n); \]
\[ \varphi \sim \frac{1}{\sqrt{n!}} \sum_p (-1)^p P_q \psi_{i_1}(q_1)\psi_{i_2}(q_2) \cdots \psi_{i_n}(q_n). \]

1.6. PRELIMINARIES FOR A DIRAC EQUATION IN REAL TERMS

What is reported in the following appears to be a preliminary study for Majorana’s article on a Symmetrical theory of electrons and positrons [Nuovo Cim. 14 (1937) 171], where he put forth the known Majorana representation for spin-1/2 fields. The Dirac equation and its consequences were considered using slightly different formalisms (different decompositions of the wave function \( \psi \)). An expression was obtained for the total angular momentum carried by the field \( \psi \), starting from the Hamiltonian. In some places, the interaction with the electromagnetic potential
(φ, A) was included as well in a somewhat interesting fashion. Note, however, that real fields (that is: directly related to the Majorana representation mentioned above) were considered only in very few points in the following pages.

1.6.1 First Formalism

\[ \alpha_x = \rho_1 \sigma_x, \quad \alpha_y = \rho_3, \quad \alpha_z = \rho_1 \sigma_z, \]
\[ \beta = -\rho_1 \sigma_y. \]

Without field (That is, without interaction with the electromagnetic field), and for \( U = \bar{U} \), we have:

\[ \left\{ \frac{W}{c} + (\alpha, p) + \beta mc \right\} U = 0. \]

For \( \psi = U + iV \):

\[ \left\{ \frac{W}{c} + (\alpha, p) + \beta mc \right\} U + \frac{e}{c} \varphi + (\alpha, A) V = 0, \]
\[ \left\{ \frac{W}{c} + (\alpha, p) + \beta mc \right\} V - \frac{e}{c} \varphi + (\alpha, A) U = 0. \]

\[ \beta' = -i\beta; \quad \mu = \frac{2\pi mc}{\hbar}; \quad \epsilon = \frac{2\pi e}{hc} \left( = \frac{1}{137e} \right). \]

\[ \left\{ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right\} U + \epsilon \varphi + (\alpha, A) V = 0, \]
\[ \left\{ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right\} V - \epsilon \varphi + (\alpha, A) U = 0. \]

\[ \delta \int \left\{ V^* \left\{ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right\} U + \frac{1}{2} \epsilon V^* \varphi + (\alpha, A) \right\} + \frac{1}{2} \epsilon U^*[\varphi + (\alpha, A)] \right\} \, dq \, dt = 0. \]

\[ \psi = U + iV, \quad \tilde{\psi} = U^* - iV^*. \]
\[
\left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] U + \epsilon [\varphi + (\alpha, A)] V = 0, \\
\left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] V - \epsilon [\varphi + (\alpha, A)] U = 0.
\]

\[\delta \int \frac{\hbar c}{2\pi} \left\{ U^* \left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] U \\
+ V^* \left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] V \\
+ \epsilon U^* [\varphi + (\alpha, A)] V - \epsilon V^* [\varphi + (\alpha, A)] U \right\} \, dq \, dt = 0.
\]

\[\text{[The footnote continues on the next page]}\]
\[ U_i(q)U_k(q') + U_k(q')U_i(q) = \frac{1}{2} \delta_{ik} \delta(q - q'), \]
\[ U_i(q)V_k(q') + V_k(q')U_i(q) = 0, \]
\[ V_i(q)V_k(q') + V_k(q')V_i(q) = \frac{1}{2} \delta_{ik} \delta(q - q'). \]

### 1.6.2 Second Formalism

\[
\left[ \frac{W}{c} + \rho_1(\sigma, p) + \rho_3mc \right] \psi = 0. 
\]

\[ A = (\psi_1, \psi_2), \quad B = (\psi_3, \psi_4): \]
\[
\begin{align*}
\left( \frac{W}{c} + mc \right) A + (\sigma, p) B &= 0, \\
\left( \frac{W}{c} - mc \right) B + (\sigma, p) A &= 0.
\end{align*}
\]

\[ A = - \left( \frac{W}{c} + mc \right)^{-1} (\sigma, p) B, \]

\[ B = - \left( \frac{W}{c} - mc \right)^{-1} (\sigma, p) A. \]

\[ \varepsilon = \sqrt{m^2c^2 + p^2}. \]

\[ \frac{W}{c} = \pm \varepsilon. \]

\[
\sum_k A_{ik} \dot{q}_k = -\frac{1}{2} \sum_s B_{is} q_s + \frac{1}{2} \sum_r B_{ri} q_r = -\sum_k B_{ik} q_k, \]

\[ q_r q_s + q_s q_r = +\frac{h}{4\pi} \sum_l A^{-1}_{sl} \delta_{ir} = +\frac{h}{4\pi} A^{-1}_{rs}. \]
1) \( \frac{W}{c} = \varepsilon: \)

\[
\begin{align*}
A &= - (\varepsilon + mc)^{-1} (\sigma, p) B,
\tilde{A} &= - [ (\varepsilon + mc)^{-1} p B ] , \sigma.
\end{align*}
\]

\[
\tilde{A} A = \begin{pmatrix}
( (\varepsilon + mc)^{-1} p B ) , ( (\varepsilon + mc)^{-1} p B ) \\
+i [ (\varepsilon + mc)^{-1} p x B ] ( (\varepsilon + mc)^{-1} p y \sigma z B ) \\
-i [ (\varepsilon + mc)^{-1} p y B ] ( (\varepsilon + mc)^{-1} p z \sigma x B ) \\
+i [ (\varepsilon + mc)^{-1} p z B ] ( (\varepsilon + mc)^{-1} p x \sigma y B ) \\
-i [ (\varepsilon + mc)^{-1} p x B ] ( (\varepsilon + mc)^{-1} p z \sigma y B ) \\
- i [ (\varepsilon + mc)^{-1} p y B ] ( (\varepsilon + mc)^{-1} p z \sigma x B ) 
\end{pmatrix}
\]

\[
\int \tilde{A} A \, dq = \int \tilde{B} (\varepsilon + mc)^{-2} p^2 B \, dq = \int \tilde{B} (\varepsilon + mc)^{-1} (\varepsilon - mc) B \, dq,
\]

\[
\int (\tilde{A} A + \tilde{B} B) \, dq = \int \tilde{B} \frac{2\varepsilon}{\varepsilon + mc} B \, dq.
\]

2) \( \frac{W}{c} = \varepsilon: \)

\[
\begin{align*}
B &= (\varepsilon + mc)^{-1} (\sigma, p) A,
\end{align*}
\]

\[
\int \tilde{B} B \, dq = \int \tilde{A} (\varepsilon + mc)^{-1} (\varepsilon - mc) A \, dq,
\]

\[
\int (\tilde{A} A + \tilde{B} B) \, dq = \int \tilde{A} \frac{2\varepsilon}{\varepsilon + mc} A \, dq.
\]

\[
\begin{align*}
A &= \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} A' - \frac{(\sigma, p)}{\sqrt{2\varepsilon (\varepsilon + mc)}} B',
B &= \frac{(\sigma, p)}{\sqrt{2\varepsilon (\varepsilon + mc)}} A' + \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} B'.
\end{align*}
\]
\[ \int (\tilde{A}A + \tilde{B}B) dq = \int (\tilde{A}'A + \tilde{B}'B) dq. \]

\[ A' = \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} A + \frac{(\sigma, p)}{\sqrt{2\varepsilon(\varepsilon + mc)}} B, \]

\[ B' = \frac{(\sigma, p)}{\sqrt{2\varepsilon(\varepsilon + mc)}} A + \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} B. \]

### 1.6.3 Angular Momentum

\[ \psi = (A, B), \quad \psi' = (A', B'). \]

\[ H = -c\rho_1(\sigma, p) - \rho_3 mc^2 - e\varphi - \rho_1(\sigma, eU). \]

\[ \int \tilde{\psi} H \psi \ dq = \int \left\{ -c\tilde{A}(\sigma, p)B - c\tilde{B}(\sigma, p)A - mc^2 \tilde{A}A \\
+ mc^2 \tilde{B}B - e\tilde{A}\varphi A - e\tilde{B}\varphi B \\
- e\tilde{A}(\sigma, U)B - e\tilde{B}(\sigma, U)A \right\} dq \\
= \int \tilde{\psi} H_0 \psi \ dq + \int \tilde{\psi} H_1 \psi \ dq. \]

\[ H = H_0 + H_1, \]

\[ H_0 = -c\rho_1(\sigma, p) - \rho_3 mc^2, \quad H_1 = -e\varphi - \rho_1(\sigma, eU). \]

\[ \int \tilde{\psi} H_0 \psi \ dq = \int \left\{ -e\tilde{A}(\sigma, p)B - c\tilde{B}(\sigma, p)A \\
- mc^2 \tilde{A}A + mc^2 \tilde{B}B \right\} dq = c \int (\tilde{B}'\varepsilon B' - \tilde{A}'\varepsilon A') dq. \]

\[ N_x = \frac{1}{2} \left( x \frac{H_0}{c} + \frac{H_0}{c} x \right) = x \frac{H_0}{c} - \frac{h}{4\pi i} \rho_1 \sigma_x, \]

\[ x\varepsilon - \varepsilon x = -\frac{h}{2\pi} \frac{p_x}{\varepsilon}. \]
\[
\int \tilde{\psi} N_x \psi \, dq = \int \tilde{\psi}' N'_x \psi \, dq \\
= \int (\tilde{B}' x \varepsilon B - \tilde{A}' x \varepsilon A) \, dq \\
- \frac{\hbar}{4\pi i} \int \left\{ \tilde{A}' \frac{p_x}{\varepsilon} A' - \tilde{B}' \frac{p_x}{\varepsilon} B' + \tilde{A}' \sigma_x B + \tilde{B}' \sigma_x A \\
- \tilde{A}' \frac{p_x(\sigma, p)}{\varepsilon(\varepsilon + mc)} B - \tilde{B}' \frac{p_x(\sigma, p)}{\varepsilon(\varepsilon + mc)} A \right\} \, dq \\
+ \frac{\hbar}{2\pi i} \int \tilde{A}' \left\{ \frac{(\varepsilon - mc)mcp_x}{4\varepsilon^3} - \sigma_x \frac{(\sigma, p)}{2\varepsilon} + \frac{(\varepsilon - mc)(2\varepsilon + mc)p_x}{4\varepsilon^3(\varepsilon + mc)} B \\
+ \frac{m^2 c^2 p_x}{4\varepsilon^3} + \frac{mcs_x(\sigma, p)}{2\varepsilon(\varepsilon + mc)} \right\} A' \, dq \\
+ \frac{\hbar}{2\pi i} \int \tilde{A}' \left\{ \frac{mcp_x(\sigma, p)}{4\varepsilon^3} + \sigma_x \frac{(\varepsilon - mc)}{2\varepsilon} - \frac{(2\varepsilon + mc)p_x(\varepsilon - mc)(\sigma, p)}{4\varepsilon^3(\varepsilon + mc)} \\
- \frac{m^2 c^2 p_x(\sigma, p)}{4\varepsilon^3(\varepsilon + mc)} + \frac{mcs_x}{2\varepsilon} \right\} \frac{1}{4\varepsilon^3(\varepsilon + mc)} \right\} \\
+ \frac{\hbar}{2\pi i} \int \tilde{B}' \{\ldots\} A' \, dq + \frac{\hbar}{2\pi i} \int \tilde{B}' \{\ldots\} B' \, dq \\
= \int (\tilde{B}' x \varepsilon B - \tilde{A}' x \varepsilon A) \, dq + \int \frac{\hbar}{2\pi i} \left\{ -\tilde{A}' \left[ \frac{mcp_x + \varepsilon s_x(\sigma, p)}{2\varepsilon(\varepsilon + mc)} \right] A' \\
+ \tilde{B}' \left[ \frac{mcp_x + \varepsilon s_x(\sigma, p)}{2\varepsilon(\varepsilon + mc)} \right] \right\} B' \, dq. \\
\]

\[
N'_x = -\rho_3 \left[ \frac{x \varepsilon + \frac{\hbar mcp_x + \varepsilon s_x(\sigma, p)}{4\pi i}}{\varepsilon(\varepsilon + mc)} \right] \\
= -\rho_3 \left[ \frac{x \varepsilon + \frac{\hbar p_x}{4\pi i} \frac{\varepsilon}{\varepsilon + mc} + \frac{\hbar p_y \sigma_z - p_z \sigma_y}{4\pi}} \right]. \\
\]

\[
\frac{H'_0}{c} = -\rho_3 \varepsilon. 
\]
\[
\int \tilde{\psi} \psi dq = \int (\tilde{A}'xA' + \tilde{B}'xB') dq
\]
\[
+ \frac{\hbar}{2\pi i} \int \tilde{A}' \left[ \frac{mc(p_x - mc)}{4\varepsilon^3} + \frac{x}{2\varepsilon(\varepsilon + mc)} \right] A' dq
\]
\[
+ \frac{\hbar}{2\pi i} \int \tilde{A}' \left[ \frac{x}{2\varepsilon(\varepsilon + mc)} \right] B' dq
\]
\[
+ \frac{\hbar}{2\pi i} \int \tilde{B}' \left[ -\frac{x}{2\varepsilon} \left( \frac{\sigma \cdot \mathbf{p}_x}{\varepsilon + mc} \right) \right] A' dq
\]
\[
+ \frac{\hbar}{2\pi i} \int \tilde{B}' \left[ \frac{x}{2\varepsilon(\varepsilon + mc)} \right] B' dq.
\]

\[
x' = x + \frac{\hbar}{2\pi} \frac{p_y \sigma_z - p_z \sigma_y}{2\varepsilon(\varepsilon + mc)} + \frac{\hbar}{2\pi} \rho_2 \left( \frac{\sigma_x}{2\varepsilon} - \frac{(\sigma \cdot \mathbf{p}_x)}{2\varepsilon^2(\varepsilon + mc)} \right).
\]

\[
N' \times = \frac{1}{2} \left( \frac{x'}{c} \frac{H_0'}{c} + \frac{H_0'}{c} x \right) = -\rho_3 x \varepsilon - \frac{\hbar}{4\pi i} \rho_3 \frac{p_x}{\varepsilon}
\]
\[
- \frac{\hbar}{2\pi} \rho_3 \frac{p_y \sigma_z - p_z \sigma_y}{2(\varepsilon + mc)}
\]
\[
= -\rho_3 \left\{ \varepsilon \varepsilon + \frac{\hbar}{4\pi i} \frac{p_x}{\varepsilon} + \frac{\hbar}{4\pi} \frac{p_y \sigma_z - p_z \sigma_y}{\varepsilon + mc} \right\}.
\]
\[ N'_x N'_y - N'_y N'_x = \frac{\hbar}{2\pi i} (x p_y - y p_x) + \frac{\hbar^2}{4\pi^2 i \varepsilon + mc} \varepsilon \sigma_z \]
\[ + \frac{\hbar^2}{8\pi^2 i} \frac{i (p_y p_z \sigma_y + p_z^2 \sigma_z + p_z p_x \sigma_x)}{(\varepsilon + mc)^2} \]
\[ + \frac{\hbar^2}{8\pi^2 i} \frac{-p_y^2 \sigma_z + p_y p_z \sigma_y + p_x p_z \sigma_x - p_z^2 \sigma_z}{(\varepsilon + mc)^2} \]
\[ = \frac{\hbar}{2\pi i} (x p_y - y p_x) + \frac{\hbar^2}{8\pi^2 i} \sigma_z \]
\[ + \frac{\hbar^2}{8\pi^2 i} \left[ \frac{\mathbf{(\sigma, p)} p_z}{(\varepsilon + mc)^2} - \frac{\mathbf{(\sigma, p)} p_z}{(\varepsilon + mc)} \right] \]
\[ = \frac{\hbar}{2\pi i} (x p_y - y p_x) + \frac{\hbar^2}{8\pi^2 i} \sigma_z \]
\[ = \frac{\hbar}{2\pi i} \left[ x p_y - y p_x + \frac{\hbar}{4\pi} \sigma_x \right]. \]

[8]

---

8 Here, the following insert appears in the original manuscript, reporting what follows:

For a relativistic Hamiltonian system described by the variables \( q, p, t, W \):

\[ Z = 0 \]

(for example: \( Z = -W + H(p, q, t) \)).

\[ dq_i : dp_i : dt : dW = \frac{\partial Z}{\partial p_i} : \frac{\partial Z}{\partial q_i} : -\frac{\partial Z}{\partial W} : \frac{\partial Z}{\partial t}. \]

For the states:

\[ S = S(p, q, W, t), \]
\[ ZS = 0. \]

\[ \sum_i \frac{\partial S}{\partial q_i} \frac{\partial Z}{\partial p_i} - \sum_i \frac{\partial S}{\partial p_i} \frac{\partial Z}{\partial q_i} - \frac{\partial S}{\partial t} \frac{\partial Z}{\partial W} + \frac{\partial S}{\partial W} \frac{\partial Z}{\partial t} = 0, \]
\[ [S, Z] = 0. \]

For example:

\[ S = S_0(p, q, t) \delta(-W + H), \]
\[ H = H(p, q, t); \]
\[ \sum_i \frac{\partial S_0}{\partial q_i} \frac{\partial H}{\partial p_i} - \sum_i \frac{\partial S_0}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial S_0}{\partial t} = 0. \]
**Plane-Wave Expansion**

For the Dirac field:

\[ H = H_0 + H_1, \quad H' = H'_0 + H'_1; \]

\[ H_0 = -c\rho_1(\sigma, p) - \rho_3mc^2, \quad H_1 = -e\varphi - e\rho_1(\sigma, U); \]

\[ H'_0 = -\rho_3c\varepsilon, \quad \varepsilon = \sqrt{m^2c^2 + p^2}. \]

\[ \varepsilon = \sqrt{m^2c^2 + h^2\gamma^2}. \]

\[ \psi = (A, B), \quad \psi' = (A', B'): \]

\[ A(q) = \int a(\gamma) e^{2\pi i(\gamma, q)} d\gamma, \quad a(\gamma) = \int A(q) e^{-2\pi i(\gamma, q)} dq; \]

\[ B(q) = \int b(\gamma) e^{2\pi i(\gamma, q)} d\gamma, \quad b(\gamma) = \int B(q) e^{-2\pi i(\gamma, q)} dq; \]

\[ A'(q) = \int a'(\gamma) e^{2\pi i(\gamma, q)} d\gamma, \quad a'(\gamma) = \int A'(q) e^{-2\pi i(\gamma, q)} dq; \]

\[ B'(q) = \int b'(\gamma) e^{2\pi i(\gamma, q)} d\gamma, \quad b'(\gamma) = \int B'(q) e^{-2\pi i(\gamma, q)} dq. \]

\[ a(\gamma) = \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} a'(\gamma) - \frac{h(\sigma, \gamma)}{\sqrt{2\varepsilon(\varepsilon + mc)}} b'(\gamma), \]

\[ b(\gamma) = \frac{h(\sigma, \gamma)}{\sqrt{2\varepsilon(\varepsilon + mc)}} a'(\gamma) + \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} b'(\gamma); \]

\[ a'(\gamma) = \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} a(\gamma) + \frac{h(\sigma, \gamma)}{\sqrt{2\varepsilon(\varepsilon + mc)}} b(\gamma), \]

\[ b'(\gamma) = -\frac{h(\sigma, \gamma)}{\sqrt{2\varepsilon(\varepsilon + mc)}} a(\gamma) + \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} b(\gamma). \]

\[ \chi(\gamma) = (a, b), \quad \chi'(\gamma) = (a', b'): \]

\[ \chi(\gamma) = \left[ \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} - \frac{i h \rho_2(\sigma, \gamma)}{\sqrt{2\varepsilon(\varepsilon + mc)}} \right] \chi'(\gamma), \]

\[ \chi'(\gamma) = \left[ \sqrt{\frac{\varepsilon + mc}{2\varepsilon}} + \frac{i h \rho_2(\sigma, \gamma)}{\sqrt{2\varepsilon(\varepsilon + mc)}} \right] \chi(\gamma). \]
\[ \varepsilon = \sqrt{m^2c^2 + h^2\gamma^2}, \quad \varepsilon' = \sqrt{m^2c^2 + h^2\gamma'^2}. \]

1.6.5 Real Fields

Dirac equation with real fields:

\[
\left[ \frac{W}{c} + \rho_1(\sigma, p) + \rho_3mc \right] \psi = 0.
\]

\[
\psi = \frac{1 - i\rho_2\sigma_y}{\sqrt{2}} \psi', \quad \psi' = \frac{1 + i\rho_2\sigma_y}{\sqrt{2}} \psi.
\]

\[
0 = \frac{1}{2}(1 + i\rho_2\sigma_y) \left[ \frac{W}{c} + \rho_1(\sigma, p) + \rho_3mc \right] (1 - i\rho_3\sigma_y) \psi'
\]

\[
= \left[ \frac{W}{c} + \rho_1\sigma_x p_x + \rho_3 p_y + \rho_1 \sigma_z - \rho_1 \sigma_y \right] \psi' = 0.
\]

1.6.6 Interaction With An Electromagnetic Field

\[
\delta \int \left\{ i\frac{hc}{2\pi} U^* \left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] U \\
+ i\frac{hc}{2\pi} V^* \left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] V \\
+ ieU^* \left[ \varphi + (\alpha, A) \right] V - ieV^* \left[ \varphi + (\alpha, A) \right] U \\
+ \frac{1}{8\pi} (E^2 - H^2) - \frac{1}{8\pi} \left( \frac{1}{c} \dot{\varphi} + \nabla \cdot A \right)^2 \right\} dq \, dt = 0.
\]

\[
\frac{1}{c} \dot{\varphi} + \nabla \cdot A = 0
\]

\[
\left( \nabla^2 \varphi + \frac{1}{c} \nabla \cdot \dot{A} + 4\pi \rho = 0 \right).
\]
\[ \left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \right] U + \frac{2\pi e}{\hbar c} [\varphi + (\alpha, A)] V = 0, \]
\[ \left[ \frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \right] V - \frac{2\pi e}{\hbar c} [\varphi + (\alpha, A)] U = 0. \]

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi + 4\pi ei(U^*V - V^*U) = 0, \]
\[ \left( \frac{1}{c^2} \frac{\partial}{\partial t^2} - \nabla^2 \right) A - 4\pi ei(U^*\alpha V - V^*\alpha V) = 0. \]

\[ \rho = -ei(U^*V - V^*U) = -e \frac{\tilde{\psi}\psi - \psi^*\bar{\psi}}{2}, \]
\[ I = ei(U^*\alpha V - V^*\alpha U) = e \frac{\tilde{\psi}\alpha\psi - \psi^*\alpha\bar{\psi}}{2} \]

([\psi = U + iV].

\[ P_0 = -\frac{1}{4\pi c} \left( \frac{1}{c} \dot{\varphi} + \nabla \cdot A \right), \]
\[ P_x = -\frac{1}{4\pi c} E_x, \]
\[ P_y = -\frac{1}{4\pi c} E_y, \]
\[ P_z = -\frac{1}{4\pi c} E_z. \]

\[ \frac{1}{c} \dot{\varphi} + \nabla \cdot A = 0 : \quad P_0 = 0; \]
\[ \nabla^2 \varphi + \nabla \cdot \dot{A} + 4\pi \rho = 0 : \quad \rho = -c \nabla \cdot F \]

([\mathbf{F} = (P_x, P_y, P_z)]).

\[ H = \int \left\{ \tilde{\psi} [-c(\alpha, p) - \beta mc^2] \psi - (\mathbf{A}, I) + 2\pi e^2 P^2 + \frac{1}{8\pi} |\nabla \times \mathbf{A}|^2 \right\} dq. \]
1.7. DIRAC-LIKE EQUATIONS FOR PARTICLES WITH SPIN HIGHER THAN 1/2

By starting from the known Dirac equation for a 4-component spinor, the author then wrote down the corresponding equations for 16-component, 6-component and 5-component spinors. Explicit expressions for the Dirac matrices for the cases considered were given, thus producing for the first time Dirac-like equations for particle with spin higher than 1/2. In the following we report what found in the Quaderno 4 in the same order as the material appears there; it seems evident, in fact, that the author has obtained the reported results just in this order, i.e., not in the more obvious way from 4-component case to 5-component, to 6-component, to 16-component case.

1.7.1 Spin-1/2 Particles (4-Component Spinors)

\[
\begin{align*}
\left( \frac{W}{c} + \frac{e}{c} A_0 \right) & \rightarrow p_0, \\
\left( p_x + \frac{e}{c} A_x \right) & \rightarrow p_x, \\
\left( p_y + \frac{e}{c} A_y \right) & \rightarrow p_y, \\
\left( p_z + \frac{e}{c} A_z \right) & \rightarrow p_z.
\end{align*}
\]

\[
p_0\psi_1 + p_x\psi_4 - ip_y\psi_4 + p_z\psi_3 + mc \psi_1 = 0,
\]
\[
p_0\psi_2 + p_x\psi_3 + ip_y\psi_3 - p_z\psi_4 + mc \psi_2 = 0,
\]
\[
p_0\psi_3 + p_x\psi_2 - ip_y\psi_2 + p_z\psi_1 - mc \psi_3 = 0,
\]
\[
p_0\psi_4 + p_x\psi_1 + ip_y\psi_1 + p_z\psi_2 - mc \psi_4 = 0.
\]

<table>
<thead>
<tr>
<th></th>
<th>$\psi_1$</th>
<th>$\psi_2$</th>
<th>$\psi_3$</th>
<th>$\psi_4$</th>
</tr>
</thead>
<tbody>
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<td>$\psi_1$</td>
<td>$p_0 + mc$</td>
<td>0</td>
<td>$p_z$</td>
<td>$p_x - ip_y$</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>0</td>
<td>$p_0 + mc$</td>
<td>$p_x + ip_y$</td>
<td>$-p_z$</td>
</tr>
<tr>
<td>$\psi_3$</td>
<td>$p_z$</td>
<td>$p_x - ip_y$</td>
<td>$p_0 - mc$</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_4$</td>
<td>$p_x + ip_y$</td>
<td>$-p_z$</td>
<td>0</td>
<td>$p_0 - mc$</td>
</tr>
</tbody>
</table>
1.7.2 Spin-7/2 Particles (16-Component Spinors)

[See the matrix on page 49.]

Let us set $M = 2m$, $P_0 = p_0 + p'_0$, $Q_0 = p_0 - p'_0$, and so on:

[See the matrix on page 50.]

[See the matrix on page 51.]

[See the matrix on page 52.]

1.7.3 Spin-1 Particles (6-Component Spinors)

\[
\left( \frac{W}{c} + \frac{e}{c} A_0 + mc \right) \psi_1 + \frac{1}{2} \left[ p_x + \frac{e}{c} C_x + i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_2 \\
- \frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_3 - \frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_4 \\
- \frac{1}{2} \left[ p_x + \frac{e}{c} C_x - i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_5 = 0, \\
\frac{1}{2} \left[ p_x + \frac{e}{c} C_x - i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c} A_0 \right) \psi_2 \\
- \frac{1}{2} \left[ p_x + \frac{e}{c} C_x - i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_6 = 0,
\]

9In the following matrices, for obvious editorial reasons, we have introduced the shortened notations: $p_{00} = p_0 \pm mc$, $p'_{00} = p'_0 \pm mc$, $p_{xy} = p_x \pm ip_y$, $p'_{xy} = p'_x \pm ip'_y$, $p_{0z} = p_0 \pm p_z$, $p'_{0z} = p'_0 \pm p'_z$, $P_{00} = P_0 \pm Mc$, $P_{xy} = P_x \pm iP_y$, $P_{0z} = P_0 \pm P_z$, $Q_{0z} = Q_0 \pm Q_z$.

10@ Note that such a matrix was left incomplete by the author.
\[
\begin{array}{cccccccccccccccc}
\text{1} & 11 & p_{q0}^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{-y} & p_{q0}^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{-y} & p_{q0}^+ \\
\text{2} & 21 & 0 & p_{q0}^+ & p_{xy} & -p_z & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{3} & 31 & p_z & - & p_{q0}^+ & 0 & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{4} & 41 & p_{q0}^+ & -p_z & 0 & p_{q0}^+ & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{5} & 51 & 0 & p_{q0}^+ & p_{xy} & p_{q0}^+ & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{6} & 61 & 0 & p_{q0}^+ & p_{xy} & p_{q0}^+ & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{7} & 71 & p_z & p_{q0}^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{8} & 81 & 0 & p_{q0}^+ & p_{xy} & p_{q0}^+ & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{9} & 91 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{10} & 101 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{11} & 111 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{12} & 121 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{13} & 131 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{14} & 141 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{15} & 151 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
\text{16} & 161 & p_z^+ & 0 & p_z & p_{-y} & 0 & \text{p} & p_{xy} & p_{q0}^+ & 0 & p_z & p_{xy} & 0 & \text{p} & p_{xy} & p_{q0}^+ \\
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| 21 + 13 | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) |
| 21 + 14 | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) |
| 21 + 15 | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) | \( Q_{21}^+ \) |

| 31 - 12 | \( P_{3}^+ \) | \( P_{3}^+ \) | \( P_{3}^+ \) | \( P_{3}^+ \) | \( P_{3}^+ \) | \( P_{3}^+ \) |
| 31 - 13 | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) |
| 31 - 14 | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) |
| 31 - 15 | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) | \( Q_{31}^+ \) |

| 41 - 12 | \( P_{4}^+ \) | \( P_{4}^+ \) | \( P_{4}^+ \) | \( P_{4}^+ \) | \( P_{4}^+ \) | \( P_{4}^+ \) |
| 41 - 13 | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) |
| 41 - 14 | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) |
| 41 - 15 | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) | \( Q_{41}^+ \) |

| 51 - 12 | \( P_{5}^+ \) | \( P_{5}^+ \) | \( P_{5}^+ \) | \( P_{5}^+ \) | \( P_{5}^+ \) | \( P_{5}^+ \) |
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| 51 - 15 | \( Q_{51}^+ \) | \( Q_{51}^+ \) | \( Q_{51}^+ \) | \( Q_{51}^+ \) | \( Q_{51}^+ \) | \( Q_{51}^+ \) |

| 61 - 12 | \( P_{6}^+ \) | \( P_{6}^+ \) | \( P_{6}^+ \) | \( P_{6}^+ \) | \( P_{6}^+ \) | \( P_{6}^+ \) |
| 61 - 13 | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) |
| 61 - 14 | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) |
| 61 - 15 | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) | \( Q_{61}^+ \) |
\[-\frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c} A_0 \right) \psi_3 + \frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_6 = 0, \]

\[-\frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c} A_0 \right) \psi_4 + \frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_6 = 0, \]

\[-\frac{1}{2} \left[ p_x + \frac{e}{c} C_x + i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c} A_0 \right) \psi_5 + \frac{1}{2} \left[ p_x + \frac{e}{c} C_x + i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_6 = 0, \]

\[-\frac{1}{2} \left[ p_x + \frac{e}{c} C_x + i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_2 + \frac{1}{2} \left[ p_z + \frac{e}{c} C_z \right] \psi_3 + \frac{1}{2} \left[ p_x + \frac{e}{c} C_x - i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_5 + \left( \frac{W}{c} + \frac{e}{c} A_0 - mc \right) = 0. \]

In first approximation, for \( C_x = C_y = C_z = 0 \):

\[
\psi_1 = 0, \quad \psi_2 = \frac{p_x - ip_y}{2mc} \psi_6, \quad \psi_3 = -\frac{p_z}{2mc} \psi_6, \\
\psi_4 = -\frac{p_z}{2mc} \psi_6; \quad \psi_5 = -\frac{p_x + ip_y}{2mc} \psi_6; \\
\left\{ -\frac{p_x^2 + p_y^2 + p_z^2}{2mc} + \frac{W}{c} + \frac{e}{c} A_0 - mc \right\} \psi_6 = 0, \\
W = mc - eA_0 + \frac{p_x^2 + p_y^2 + p_z^2}{2m}. 
\]
\[
\begin{align*}
\alpha_x &= \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0
\end{pmatrix}, &
\alpha_y &= \begin{pmatrix}
0 & i & 0 & 0 & i & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & 0 & \frac{i}{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & 0 & \frac{i}{2} \\
0 & -\frac{i}{2} & 0 & 0 & -\frac{i}{2} & 0
\end{pmatrix}, \\
\alpha_z &= \begin{pmatrix}
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0
\end{pmatrix}, &
\beta &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
\end{align*}
\]

\[
\begin{pmatrix}
\frac{W}{c} + mc \\
\frac{p_x - ip_y}{2} \\
-\frac{p_x + ip_y}{2} \\
\frac{p_x - ip_y}{2} \\
\frac{p_x + ip_y}{2}
\end{pmatrix}
\begin{pmatrix}
-\frac{p_z}{2} \\
-\frac{p_x}{2} \\
W \\
W \\
W
\end{pmatrix}
+ \begin{pmatrix}
-\frac{p_z}{2} \\
-\frac{p_x}{2} \\
W \\
W \\
W
\end{pmatrix}
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0 \\
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\end{pmatrix}
= 0.
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<tr>
<td>0</td>
<td>0</td>
<td>$\frac{W}{c}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{p_z}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{W}{c}$</td>
<td>0</td>
<td>$\frac{p_z}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{W}{c}$</td>
<td>$\frac{p_x + ip_y}{2}$</td>
</tr>
</tbody>
</table>

$\left| \begin{array}{cccc}
\frac{W}{c} - mc & -\frac{p_x + ip_y}{2} & \frac{p_z}{2} & \frac{p_{xy}}{2} & \frac{W}{c} - mc \\
\frac{W}{c} - mc & -\frac{p_x + ip_y}{2} & \frac{p_z}{2} & \frac{p_{xy}}{2} & \frac{W}{c} - mc \\
\end{array} \right| = 0,$

$2\frac{W^6}{c^6} - 2\frac{W^5}{c^5}mc - \frac{W^4}{c^4}(p_x^2 + p_y^2 + p_z^2) - \frac{W^4}{c^4} \left( \frac{W^2}{c^2} - 2Wmc + m^2c^2 \right) = 0,$

$\frac{W^2}{c^2} - m^2c^2 - (p_x^2 + p_y^2 + p_z^2) = 0.$

### 1.7.4 5-Component Spinors

\[
\left( \frac{W}{c} + \frac{e}{c}A_0 + mc \right) \psi_1 + \frac{1}{2} \left[ p_x + \frac{e}{c}C_x + i \left( p_y + \frac{e}{c}C_y \right) \right] \psi_2
\]

\[-\frac{1}{\sqrt{2}} \left[ p_z + \frac{e}{c}C_z \right] \psi_3 - \frac{1}{2} \left[ p_x + \frac{e}{c}C_x - i \left( p_y + \frac{e}{c}C_y \right) \right] \psi_4 = 0,
\]

\[\frac{1}{2} \left[ p_x + \frac{e}{c}C_x - i \left( p_y + \frac{e}{c}C_y \right) \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c}A_0 \right) \psi_2
\]

\[-\frac{1}{2} \left[ p_x + \frac{e}{c}C_x - i \left( p_y + \frac{e}{c}C_y \right) \right] \psi_5 = 0,
\]

\[-\frac{1}{\sqrt{2}} \left[ p_z + \frac{e}{c}C_z \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c}A_0 \right) \psi_3 + \frac{1}{\sqrt{2}} \left[ p_z + \frac{e}{c}C_z \right] \psi_5 = 0,
\]

\[-\frac{1}{2} \left[ p_x + \frac{e}{c}C_x + i \left( p_y + \frac{e}{c}C_y \right) \right] \psi_1 + \left( \frac{W}{c} + \frac{e}{c}A_0 \right) \psi_4
\]

\[+\frac{1}{2} \left[ p_x + \frac{e}{c}C_x + i \left( p_y + \frac{e}{c}C_y \right) \right] \psi_5 = 0,
\]
\[
\begin{align*}
-\frac{1}{2} \left[ p_x + \frac{e}{c} C_x + i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_2 + \frac{1}{\sqrt{2}} \left[ p_z + \frac{e}{c} C_z \right] \psi_3 \\
+ \frac{1}{2} \left[ p_x + \frac{e}{c} C_x - i \left( p_y + \frac{e}{c} C_y \right) \right] \psi_4 + \left( \frac{W}{c} + \frac{e}{c} A_0 - mc \right) \psi_5 = 0.
\end{align*}
\]

\[
\left( \frac{W}{c} + \frac{e}{c} A_0 \right) + \alpha_x \left( p_x + \frac{e}{c} C_x \right) + \alpha_y \left( p_y + \frac{e}{c} C_y \right) + \alpha_z \left( p_z + \frac{e}{c} C_z \right) + \beta mc = 0.
\]

\[
\alpha_x = \begin{bmatrix}
0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0
\end{bmatrix}, \quad \alpha_y = \begin{bmatrix}
0 & \frac{i}{2} & 0 & \frac{i}{2} & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & \frac{i}{2} \\
0 & 0 & 0 & 0 & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & \frac{i}{2} \\
0 & -\frac{i}{2} & 0 & -\frac{i}{2} & 0
\end{bmatrix},
\]

\[
\alpha_z = \begin{bmatrix}
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0
\end{bmatrix}, \quad \beta = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{bmatrix}.
\]
2

QUANTUM ELECTRODYNAMICS

2.1. BASIC LAGRANGIAN AND HAMILTONIAN FORMALISM FOR THE ELECTROMAGNETIC FIELD

The author studied the dynamics of the electromagnetic field in a lagrangian framework; the Lagrangian density $L$ was deduced from a least action principle and, following a canonical formalism, the Hamiltonian density $H$ was then obtained.

\[ \delta \int L \, ds \, dt = 0, \]
\[ \frac{1}{c} \dot{\phi} + \nabla \cdot A = 0, \]
\[
L = \frac{1}{8\pi} \left\{ -\frac{1}{c^2} \phi^2 + |\nabla \phi|^2 + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right. \\
\left. - |\nabla A_x|^2 - |\nabla A_y|^2 - |\nabla A_z|^2 \right\}.
\]

\[ \varphi, \quad P_0 = -\frac{1}{4\pi c^2} \dot{\varphi}, \]
\[ A_x, \quad P_x = \frac{1}{4\pi c^2} \dot{A}_x, \]
\[ A_y, \quad P_y = \frac{1}{4\pi c^2} \dot{A}_y, \]
\[ A_z, \quad P_z = \frac{1}{4\pi c^2} \dot{A}_z, \]

\[ \Box \varphi = 0, \]
\[ \Box A = 0. \]
\[ E = -\nabla \varphi - \frac{1}{c} \dot{A}. \]

\[ H = \nabla \times A. \]

\[ H = P_0 \dot{\varphi} + P_x \dot{A}_x + P_y \dot{A}_y + P_z \dot{A}_z - L \]

\[ = \frac{1}{8\pi} \left\{ \frac{1}{c^2} \varphi^2 - |\nabla \varphi|^2 + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) + |\nabla A_x|^2 \right. \]

\[ + |\nabla A_y|^2 + |\nabla A_z|^2 \right\} \]

\[ = 2\pi c^2 (-P_0^2 + P_x^2 + P_y^2 + P_z^2) \]

\[ + \frac{1}{8\pi} \left( -|\nabla \varphi|^2 + |\nabla A_x|^2 + |\nabla A_y|^2 + |\nabla A_z|^2 \right), \]

\[ 4\pi c P_0 = \nabla \cdot A, \]

\[ \frac{1}{c} \dot{\varphi} + \nabla \cdot A = 0, \]

\[ \nabla^2 \varphi + \frac{1}{c} \nabla \cdot \dot{A} = 0. \]

\[ \int H \, ds = \frac{1}{8\pi} \int \left\{ -(\nabla \cdot A)^2 - |\nabla \varphi|^2 + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right. \]

\[ + |\nabla A_x|^2 + |\nabla A_y|^2 + |\nabla A_z|^2 \right\} \, ds \]

\[ = \frac{1}{8\pi} \int \left\{ -(\nabla \cdot A)^2 + \varphi \nabla^2 \varphi + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right. \]

\[ - A \cdot \nabla^2 A \right\} \, ds. \]

\[ \mathcal{E} = -\nabla \varphi - \frac{1}{c} \dot{A}, \]

\[ \int \mathcal{E}^2 \, ds = \int \left\{ |\nabla \varphi|^2 + \frac{2}{c} (\nabla \varphi) \cdot \dot{A} + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right\} \, ds \]

\[ = \int \left\{ -\varphi \nabla^2 \varphi - \frac{2}{c} \varphi \nabla \cdot \dot{A} + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right\} \, ds \]

\[ = \int \left\{ \varphi \nabla^2 \varphi + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right\} \, ds, \]
\[
\mathcal{H} = \nabla \times A,
\]
\[
\int \mathcal{H}^2 ds = \int |\nabla \times A|^2 ds = \int A \cdot \nabla \times \nabla \times A \, ds
\]
\[
= \int \{ A \cdot \nabla (\nabla \cdot A) - A \cdot \nabla^2 A \} \, ds
\]
\[
= \int \{ -(\nabla \cdot A)^2 - A \cdot \nabla^2 A \} \, ds,
\]

[1]

\[
\int H \, ds = \frac{1}{8\pi} \int (E^2 + \mathcal{H}^2) \, ds.
\]

### 2.2. ANALOGY BETWEEN THE ELECTROMAGNETIC FIELD AND THE DIRAC FIELD

In the following pages, the author explored the possibility of describing the electromagnetic field in full analogy with what usually done for a Dirac field. In a three-dimensional formalism, he then introduced a wavefunction \( \psi \) in terms of the electric and magnetic fields \( E, H \) (and, more specifically, in terms of quantities \( E \pm iH \)), and its dynamics (for free fields) was developed in close analogy with the Dirac procedure for spin-1/2 fields. Commutation (rather than anticommutation) rules for Dirac-like matrices were adopted, and energy eigenvalues and eigenvectors were calculated.


---

\(^{1@}\) In the original manuscript, the author pointed out that, from:

\[
\frac{1}{c} \dot{\varphi} + \nabla \cdot A = 0, \quad \Box \varphi = 0,
\]

it follows that:

\[
\nabla^2 \varphi + \frac{1}{c} \nabla \cdot A = 0.
\]
\[ 4\pi \rho - \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0, \]

\[ 4\pi \mathbf{I} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \quad -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = \nabla \times \mathbf{E}. \]

\begin{align*}
\psi_1 &= E_1 - iH_1 = E_x - iH_x, \\
\psi_2 &= E_2 - iH_2 = E_y - iH_y, \\
\psi_3 &= E_3 - iH_3 = E_z - iH_z.
\end{align*}

\[ \nabla \cdot \psi = \nabla \cdot \mathbf{E} - i \nabla \cdot \mathbf{H} = 4\pi \rho. \]  \hspace{1cm} (1)

\[ \nabla \times \psi = \nabla \times \mathbf{E} - i \nabla \times \mathbf{H} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - 4\pi i \mathbf{I} \]

\[ = -\frac{i}{c} \left( \frac{\partial \mathbf{E}}{\partial t} - i \frac{\partial \mathbf{H}}{\partial t} \right) - 4\pi i \mathbf{I}, \]

\[ 4\pi \mathbf{I} + \frac{1}{c} \frac{\partial \psi}{\partial t} = +i \nabla \times \psi. \]  \hspace{1cm} (2)

The Maxwell equations are given by:

\[ \frac{1}{c} \frac{\partial \psi}{\partial t} - i \nabla \times \psi + 4\pi \mathbf{I} = 0, \]

\[ \nabla \cdot \psi - 4\pi \rho = 0. \]
Without charge:

\[
\begin{align*}
\frac{W}{c} \psi_1 + ip_y \psi_3 - ip_z \psi_2 &= 0, \\
\frac{W}{c} \psi_2 + ip_z \psi_1 - ip_x \psi_3 &= 0, \\
\frac{W}{c} \psi_3 + ip_x \psi_2 - ip_y \psi_1 &= 0, \\
p_x \psi_1 + p_y \psi_2 + p_z \psi_3 &= 0.
\end{align*}
\]

\[2\]

\[
\left( \frac{W}{c} + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z \right) \psi = 0. \tag{3}
\]

\[
\alpha_x = \begin{vmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & +i & 0 \\
\end{vmatrix}, \quad \alpha_y = \begin{vmatrix}
0 & 0 & +i \\
0 & 0 & 0 \\
-i & 0 & 0 \\
\end{vmatrix},
\]

\[
\alpha_z = \begin{vmatrix}
0 & -i & 0 \\
+i & 0 & 0 \\
0 & 0 & 0 \\
\end{vmatrix}, \quad \beta_x = \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{vmatrix}.
\]

\[3\]

\[
\alpha_x \alpha_y - \alpha_y \alpha_x = -i \alpha_z,
\]

\[
[\alpha_x, \alpha_z] = +i \alpha_y,
\]

\[
[\alpha_y, \alpha_z] = i \alpha_x.
\]

\[
\beta_x = |1 \ 0 \ 0|, \quad \beta_y = |0 \ 1 \ 0|, \quad \beta_z = |0 \ 0 \ 1|.
\]

\[
(\beta_x p_x + \beta_y p_y + \beta_z p_z) \psi = 0. \tag{4}
\]

Following the Dirac method, the eigenvalues of the Maxwell equation are obtained from:

\[2\] The line before the fourth equation means that it is deduced from the previous three equations.

\[3\] Note that the signs on the RHS of the following two equations were wrong: correctly, we have \(\alpha_x \alpha_y - \alpha_y \alpha_x = i \alpha_z\) and \([\alpha_x, \alpha_z] = -i \alpha_y\).
\[
\begin{bmatrix}
W/c & -ip_z & ip_y \\
-ip_z & W/c & -ip_x \\
-ip_y & ip_x & W/c
\end{bmatrix} = 0,
\]
\[
\left(\frac{W}{c}\right)^3 - p^2 \frac{W}{c} = 0,
\]
\[
\frac{W}{c} = \begin{cases}
p, \\
-p, \\
0,
\end{cases}
\]
\[
p = \sqrt{p_x^2 + p_y^2 + p_z^2}.
\]

For \( t = 0 \):
\[
\psi_1 = a \delta(x - x_0) \delta'(y - y_0) \delta'(r - r_0),
\]
\[
\psi_2 = b \delta'(x - x_0) \delta(y - y_0) \delta'(z - z_0),
\]
\[
\psi_3 = -(a + b) \delta'(x - x_0) \delta'(y - y_0) \delta(z - z_0).
\]
\[
\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} = 0.
\]
\[
\psi_1(x, y, z) = \int A(x_0, y_0, z_0) \delta(x - x_0) \delta'(y - y_0) \delta'(z - z_0) \, dx_0 dy_0 dz_0,
\]
\[
\psi_2(x, y, z) = \int B(x_0, y_0, z_0) \delta'(x - x_0) \delta(y - y_0) \delta'(z - z_0) \, dx_0 dy_0 dz_0,
\]
\[
\psi_3(x, y, z) = \int -(A + B) \delta'(x - x_0) \delta'(y - y_0) \delta(z - z_0) \, dx_0 dy_0 dz_0.
\]
\[
\psi_1 = \frac{\partial^2 A}{\partial y \partial z}, \quad \psi_2 = \frac{\partial^2 B}{\partial z \partial x}, \quad \psi_3 = -\frac{\partial^2 (A + B)}{\partial x \partial y};
\]
\[
\frac{\partial \psi_1}{\partial x} = \frac{\partial^3 A}{\partial x \partial y \partial z}, \quad \frac{\partial \psi_2}{\partial y} = \frac{\partial^2 B}{\partial x \partial y \partial z}, \quad \frac{\partial \psi_3}{\partial z} = -\frac{\partial^2 (A + B)}{\partial x \partial y \partial z}.
\]

\[
\frac{\partial' A}{\partial y \partial z} = \psi_1, \\
\frac{\partial A}{\partial y} = \int \psi_1 dz + f_y, \\
A = A_0 + F_1(x, y) + F_2(x, z); \\
\frac{\partial^2 B}{\partial z \partial x} = \psi_2, \\
B = B_0 + F_3(x, y) + F_4(y, z).
\]

\[
\psi_3 = -\frac{\partial^2 (A + B)}{\partial x \partial y} = -\frac{\partial^2 (A_0 + B_0)}{\partial x \partial y} + F(x, y).
\]

By substituting the expressions:

\[
\psi_1 = \frac{\partial^2 A}{\partial y \partial z}, \quad \psi_2 = \frac{\partial^2 B}{\partial z \partial x}, \quad \psi_3 = \frac{\partial^2 C}{\partial x \partial y},
\]

into the Maxwell equations, we get:

\[
\frac{1}{c} \frac{\partial^3 A}{\partial y \partial z \partial t} - \frac{i}{c} \frac{\partial^3 C}{\partial x^2 y} + \frac{i}{c} \frac{\partial^2 B}{\partial x^2 z} = 0,
\]

\[
\frac{1}{c} \frac{\partial^3 B}{\partial z \partial x \partial t} - \frac{i}{c} \frac{\partial^3 A}{\partial y^2 z} + \frac{i}{c} \frac{\partial^2 C}{\partial y^2 x} = 0,
\]

\[
\frac{1}{c} \frac{\partial^3 C}{\partial x \partial y \partial t} - \frac{i}{c} \frac{\partial^3 B}{\partial z^2 x} + \frac{i}{c} \frac{\partial^2 A}{\partial z^2 y} = 0;
\]

\[
\frac{\partial^3 (A + B + C)}{\partial x \partial y \partial z} = 0.
\]

\[
A + B + C = 0.
\]
\[
\left[ \frac{\partial}{\partial y} \left( \frac{1}{c} \frac{\partial^2}{\partial x \partial t} - i \frac{\partial^2}{\partial y \partial z} \right) A - \frac{\partial}{\partial x} \left( \frac{1}{c} \frac{\partial^2}{\partial y \partial t} + i \frac{\partial^2}{\partial x \partial z} \right) B \right] = 0.
\]

\[
A = -a e^{i(\gamma_1 x + \gamma_2 y + \gamma_3 z)},
\]

\[
B = -b e^{i(\gamma_1 x + \gamma_2 y + \gamma_3 z)},
\]

\[
C = -c e^{i(\gamma_1 x + \gamma_2 y + \gamma_3 z)};
\]

\[
\psi_1 = a \gamma_2 \gamma_3 e^{i(\gamma_1 x + \gamma_2 y + \gamma_3 z)},
\]

\[
\psi_2 = b \gamma_3 \gamma_1 e^{i(\gamma_1 x + \gamma_2 y + \gamma_3 z)},
\]

\[
\psi_3 = c \gamma_1 \gamma_2 e^{i(\gamma_1 x + \gamma_2 y + \gamma_3 z)}.
\]

### 2.3. ELECTROMAGNETIC FIELD: PLANE WAVE OPERATORS

Plane wave expansion of the electromagnetic field was considered in a way similar to what is usually done for a Dirac or a Klein-Gordon field. In the second part, the author again introduced a sort of photon wave field \(\Psi\), in close analogy to the Dirac field for a spin-1/2 particle and in a full Lorentz-invariant formalism. The properties of this field are deduced from general group-theoretic arguments.

\[
\varphi, \quad P_0 = -\frac{1}{4\pi c^2} \dot{\varphi}, \quad \dot{\varphi} = 4\pi c^2 P_0;
\]

\[
A_x, \quad P_x = \frac{1}{4\pi c^2} \dot{A}_x, \quad \dot{A}_x = 4\pi c^2 P_x;
\]

\[
A_y, \quad P_y = \frac{1}{4\pi c^2} \dot{A}_y, \quad \dot{A}_y = 4\pi c^2 P_y;
\]

\[
A_z, \quad P_z = -\frac{1}{4\pi c^2} \dot{A}_z, \quad \dot{A}_z = 4\pi c^2 P_z;
\]

\[
P_0, \quad -\varphi, \quad \dot{P}_0 = -\frac{1}{4\pi} \nabla^2 \varphi;
\]

\[
P_x, \quad -A_x, \quad \dot{P}_x = \frac{1}{4\pi} \nabla^2 A_x;
\]

\[
P_y, \quad -A_y, \quad \dot{P}_y = \frac{1}{4\pi} \nabla^2 A_y;
\]

\[
P_z, \quad -A_z \quad \dot{P}_z = \frac{1}{4\pi} \nabla^2 A_z.
\]
\[ U_0(\gamma) = \int e^{-2\pi i (\gamma_1 x + \gamma_2 y + \gamma_3 z)} \varphi(x, y, z) \, dx \, dy \, dz, \]

\[ U_x(\gamma) = \int e^{-2\pi i \gamma \cdot q} A_x(q) \, dq, \]

\[ U_y(\gamma) = \int e^{-2\pi i \gamma \cdot q} A_y(q) \, dq, \]

\[ U_z(\gamma) = \int e^{-2\pi i \gamma \cdot q} A_z(q) \, dq. \]

\[ \int L(q) \, dq = \int M(\gamma) \, d\gamma, \]

\[ M = \frac{1}{8 \pi} \left\{ -\frac{1}{c^2} \overline{U}_0 \dot{U}_0 + 4\pi^2 \gamma^2 \overline{U}_0 U_0 + \frac{1}{c^2} (\overline{U}_x \dot{U}_x + \overline{U}_y \dot{U}_y + \overline{U}_z \dot{U}_z) \left. \right. - 4\pi^2 \gamma^2 (\overline{U}_x U_x + \overline{U}_y U_y + \overline{U}_z U_z) \right\}. \]

\[ U_0, \quad V_0 = -\frac{1}{4\pi c^2} \overline{U}_0, \]

\[ U_x, \quad V_x = \frac{1}{4\pi c^2} \overline{U}_x, \]

\[ U_y, \quad V_y = \frac{1}{4\pi c^2} \overline{U}_y, \]

\[ U_z, \quad V_z = \frac{1}{4\pi c^2} \overline{U}_z. \]

\[ U = (U_x, U_y, U_z), \quad V = (V_x, V_y, V_z), \]

\[ \dot{U} = (\dot{U}_x, \dot{U}_y, \dot{U}_z), \quad \dot{V} = (\dot{V}_x, \dot{V}_y, \dot{V}_z), \]

\[ \text{In the original manuscript, the author considered in what follows the role of the operators } \nabla^2 = L^2 \text{ and } L = \sqrt{\nabla^2}. \text{ He denoted with } \mathbf{q} \text{ the vector } (x, y, z). \]

\[ A \text{ bar over a quantity denotes complex conjugation.} \]
\[ U_0(\gamma) = U_0(-\gamma), \]
\[ \bar{U}_0(\gamma) = \bar{U}_0(-\gamma), \]
\[ U(\gamma) = U(-\gamma), \]
\[ \bar{U}(\gamma) = \bar{U}(-\gamma), \]
\[ V(\gamma) = V(-\gamma), \]
\[ \bar{V}(\gamma) = \bar{V}(-\gamma), \]
\[ V_0(\gamma) = V_0(-\gamma), \]
\[ \bar{V}_0(\gamma) = \bar{V}_0(-\gamma). \]

\[ \frac{1}{c^2} \ddot{U}_0 + 4\pi^2 \gamma^2 U_0 = 0, \]
\[ \frac{1}{c^2} \ddot{U} + 4\pi^2 \gamma^2 U = 0, \]
\[ \frac{1}{c} \dot{U}_0 + 2\pi i (\gamma_1 U_x + \gamma_2 U_y + \gamma_3 U_z) = 0, \]
\[ 2\pi i \gamma^2 U_0 + \frac{1}{c} (\gamma_1 \dot{U}_x + \gamma_2 \dot{U}_y + \gamma_3 \dot{U}_z) = 0. \]

\[ \psi_0(\gamma) = \int e^{-2\pi i \gamma \cdot q} \cdot \frac{1}{2c\sqrt{\hbar}} \left( \sqrt{2\pi \gamma c} \varphi(q) + \frac{i}{\sqrt{2\pi \gamma c}} \dot{\varphi}(q) \right) dq, \]
\[ \psi_x(\gamma) = \int e^{-2\pi i (\gamma \cdot q)} \cdot \frac{1}{2c\sqrt{\hbar}} \left( \sqrt{2\pi \gamma c} A_x(q) + \frac{i}{\sqrt{2\pi \gamma c}} \dot{A}_x(q) \right) dq, \]
\[ \psi_y(\gamma) = \int e^{-2\pi i \gamma \cdot q} \cdot \frac{1}{2c\sqrt{\hbar}} \left( \sqrt{2\pi \gamma c} A_y(q) + \frac{i}{\sqrt{2\pi \gamma c}} \dot{A}_y(q) \right) dq, \]
\[ \psi_z(\gamma) = \int e^{-2\pi i \gamma \cdot q} \cdot \frac{1}{2c\sqrt{\hbar}} \left( \sqrt{2\pi \gamma c} A_z(q) + \frac{i}{\sqrt{2\pi \gamma c}} \dot{A}_z(q) \right) dq. \]

\[ \varphi(q) = c\sqrt{\hbar} \int \frac{1}{\sqrt{2\pi \gamma c}} [\psi_0(\gamma) + \bar{\psi}_0(-\gamma)] e^{2\pi i \gamma \cdot q} dq, \]
\[ \dot{\varphi}(q) = \frac{c\sqrt{\hbar}}{i} \int \sqrt{2\pi \gamma c} [\psi_0(\gamma) - \bar{\psi}_0(-\gamma)] e^{2\pi i \gamma \cdot q} dq. \]

\[ ^{[6]} \text{Probably, the author proceeded in analogy with the Dirac field.} \]
\[ A_x(q) = c\sqrt{\hbar} \int \frac{1}{\sqrt{2\pi\gamma c}} [\psi_x(\gamma) + \bar{\psi}_x(-\gamma)] e^{2\pi i \gamma q} d\gamma, \]
\[ \dot{A}_x(q) = \frac{c\sqrt{\hbar}}{i} \int \sqrt{2\pi\gamma c} [\psi_x(\gamma) - \bar{\psi}_x(-\gamma)] e^{2\pi i \gamma q} d\gamma, \]
\[ \square \varphi = \frac{1}{c^2} \ddot{\varphi} - \nabla^2 \varphi \]
\[ = \frac{\sqrt{\hbar}}{ci} \int \sqrt{2\pi\gamma c} \left\{ \dot{\psi}_0(\gamma) - \bar{\psi}_0(-\gamma) + 2\pi\gamma c i \dot{\psi}_0(\gamma) + 2\pi\gamma c i \bar{\psi}_0(-\gamma) \right\} e^{2\pi i \gamma q} d\gamma. \]
\[ \psi_0(\gamma) = -2\pi\gamma c i \psi_0(\gamma), \quad \dot{\psi}_0(\gamma) = 2\pi\gamma c i \bar{\psi}_0(\gamma), \]
\[ \dot{\psi}(\gamma) = -2\pi\gamma c i \psi_x(\gamma), \quad \dot{\psi}_x(\gamma) = 2\pi\gamma c i \bar{\psi}_x(\gamma), \]
\[ \frac{1}{8\pi} \int \left\{ -\frac{1}{c^2} \ddot{\varphi}^2 - |\nabla \varphi|^2 + \frac{1}{c^2} (\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) + |\nabla A_x|^2 + |\nabla A_y|^2 + |\nabla A_z|^2 \right\} dq \]
\[ = \int h\gamma c \left\{ -\frac{\psi_0(\gamma) \bar{\psi}_0(\gamma) + \bar{\psi}_x(\gamma) \psi_x(\gamma)}{2} + \frac{\psi_x(\gamma) \bar{\psi}_x(\gamma) + \bar{\psi}_0(\gamma) \psi_0(\gamma)}{2} \right. \]
\[ \left. + \psi_y(\gamma) \bar{\psi}_y(\gamma) + \bar{\psi}_y(\gamma) \psi_y(\gamma) \right\} d\gamma, \]
\[ W = \int h\gamma c \left\{ -\psi_0(\gamma) \bar{\psi}_0(\gamma) + \bar{\psi}_x(\gamma) \psi_x(\gamma) + \bar{\psi}_y(\gamma) \psi_y(\gamma) + \bar{\psi}_z(\gamma) \psi_z(\gamma) \right\} d\gamma. \]

\footnote{In the original manuscript, the author also cited the following (seeming) identity, whose meaning in this general framework is not clear:
\[ 0 = \phi(q) - \phi(q) \]
\[ = c\sqrt{\hbar} \int \frac{1}{\sqrt{2\pi\gamma c}} \left\{ \psi_0(\gamma) + \bar{\psi}_0(-\gamma) + 2\pi\gamma c i \psi_0(q) - 2\pi\gamma c i \bar{\psi}_0(-\gamma) \right\} e^{2\pi i \gamma q} d\gamma. \]
\[ \psi_0(\gamma)\overline{\psi}_0(\gamma') - \overline{\psi}_0(\gamma')\psi_0(\gamma) = -\delta(\gamma - \gamma'), \]
\[ \psi_x(\gamma)\overline{\psi}_x(\gamma') - \overline{\psi}_x(\gamma')\psi_x(\gamma) = +\delta(\gamma - \gamma'), \]
\[ \ldots \]

\[ \nabla^2 \varphi + \frac{1}{c} \nabla \cdot \dot{A} = \]
\[ = c\sqrt{\hbar} \int 2\pi \sqrt{\frac{2\pi \gamma}{c}} \left\{ -\gamma [\psi_0(\gamma) + \overline{\psi}_0(-\gamma)] + \gamma_x [\psi_x(\gamma) - \overline{\psi}_x(-\gamma)] \right. \]
\[ + \gamma_y [\psi_y(\gamma) - \overline{\psi}_y(-\gamma)] + \gamma_z [\psi_z(\gamma) - \overline{\psi}_z(-\gamma)] \} \right\} e^{2\pi i \gamma \cdot q} d\gamma, \]
\[ \frac{1}{c} \dot{\varphi} + \nabla \cdot \dot{A} \]
\[ = \frac{\hbar}{\sqrt{i}} \int \sqrt{\frac{2\pi}{\gamma c}} \left\{ \gamma [\psi_0(\gamma) - \overline{\psi}_0(-\gamma)] - \gamma_x [\psi_x(\gamma) - \overline{\psi}_x(-\gamma)] \right. \]
\[ - \gamma_y [\psi_y(\gamma) - \overline{\psi}_y(-\gamma)] - \gamma_z [\psi_z(\gamma) - \overline{\psi}_z(-\gamma)] \} \right\} e^{2\pi i \gamma \cdot q} d\gamma, \]
\[ \gamma \psi_0 - \gamma_x \psi_x - \gamma_y \psi_y - \gamma_z \psi_z = 0, \]
\[ \gamma \overline{\psi}_0 + \gamma_x \overline{\psi}_x - \gamma_y \overline{\psi}_y - \gamma_z \overline{\psi}_z = 0, \]
\[ \psi_0 = \psi_0(\gamma), \psi_x = \psi_x(\gamma), \ldots, \overline{\psi}_0 = \overline{\psi}_0(\gamma), \overline{\psi}_x = \overline{\psi}_x(\gamma), \ldots \]

2.3.1 Dirac Formalism

\[ \Psi = (\psi_0, \psi_x, \psi_y, \psi_z), \]
\[ H = -\frac{\hbar}{2\pi i} \frac{\partial}{\partial t}, \quad p_x = \frac{\hbar}{2\pi i} \frac{\partial}{\partial x}, \quad p_y = \frac{\hbar}{2\pi i} \frac{\partial}{\partial y}, \quad p_z = \frac{\hbar}{2\pi i} \frac{\partial}{\partial z}; \]
\[ S_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \]
\[ T_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

1) \( \Psi' = H \Psi = h \gamma_x \Psi \)

2) \( \Psi' = p_x \Psi = h \gamma_x \Psi \)

3) \( \Psi' = p_y \Psi = h \gamma_y \Psi \)

4) \( \Psi' = p_z \Psi = h \gamma_z \Psi \)

5) \( \Psi' = S_x \Psi = \left\{ \begin{array}{c} -\gamma_y \frac{\partial}{\partial \gamma_z} + \gamma_z \frac{\partial}{\partial \gamma_y} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \right\} \Psi \)

6) \( \Psi' = S_y \Psi = \left\{ \begin{array}{c} -\gamma_z \frac{\partial}{\partial \gamma_z} + \gamma_x \frac{\partial}{\partial \gamma_z} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \right\} \Psi \)

7) \( \Psi' = S_z \Psi = \left\{ \begin{array}{c} -\gamma_x \frac{\partial}{\partial \gamma_y} + \gamma_y \frac{\partial}{\partial \gamma_x} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \right\} \Psi \)

8) \( \Psi' = T_x \Psi = \left\{ \begin{array}{c} -\gamma \frac{\partial}{\partial \gamma_x} - \frac{\gamma_x}{2\gamma} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -\gamma_x/\gamma & 0 & 0 \\ 0 & -\gamma_y/\gamma & 0 & 0 \\ 0 & -\gamma_z/\gamma & 0 & 0 \end{pmatrix} \end{array} \right\} \Psi \)
9) \[ \Psi' = T_y \Psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_x/\gamma & 0 \\ 1 & 0 & -\gamma_y/\gamma & 0 \\ 0 & 0 & -\gamma_z/\gamma & 0 \end{pmatrix} + \frac{1}{2\gamma} \begin{pmatrix} -\gamma \partial_{\gamma_y} & \gamma_y \\ \gamma_y & -\gamma \partial_{\gamma_y} \end{pmatrix} \Psi = 0 \]

10) \[ \Psi' = T_z \Psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_x/\gamma & 0 \\ 1 & 0 & -\gamma_y/\gamma & 0 \\ 0 & 0 & -\gamma_z/\gamma & 0 \end{pmatrix} + \frac{1}{2\gamma} \begin{pmatrix} -\gamma \partial_{\gamma_z} & \gamma_z \\ \gamma_z & -\gamma \partial_{\gamma_z} \end{pmatrix} \Psi = 0 \]

\[ \psi_0 = 0, \quad \Psi = (\psi_x, \psi_y, \psi_z). \]

\[ \gamma = (\gamma_x, \gamma_y, \gamma_z), \quad \gamma = \sqrt{\gamma_x^2 + \gamma_y^2 + \gamma_z^2}. \]

\[ (\gamma', \gamma'_x, \gamma'_y, \gamma'_z) = C(\gamma, \gamma_x, \gamma_y, \gamma_z), \]

\[ C = \|c_{ik}\| \quad (i, k = 0, 1, 2, 3) \]

\[ c_{00}^2 - \sum_{i=1}^{3} c_{0i}^2 = 1, \]

\[ c_{00} c_{0i} - \sum_{k=1}^{3} c_{0k} c_{ik} = 0, \quad (i = 1, 2, 3), \]

\[ c_{i0} c_{k0} - \sum_{k=10}^{3} c_{ik} c_{ki} = -\partial_{ik}, \quad (i, k = 1, 2, 3). \]

\[ \Psi' (\gamma') = e^{-2\pi i c (\gamma' - \gamma)t} \sqrt{\frac{\gamma}{\gamma'}} D \Psi (\gamma), \]

\[ D = \|d_{ik}\| \quad (i, k = 1, 2, 3) \]

\[ d_{11} = c_{11} - \frac{\gamma'_x}{\gamma'} c_{01}, \quad d_{21} = c_{21} - \frac{\gamma'_y}{\gamma'} c_{01}, \quad d_{31} = c_{31} - \frac{\gamma'_z}{\gamma'} c_{01}, \]

\[ d_{12} = c_{12} - \frac{\gamma'_x}{\gamma'} c_{02}, \quad d_{22} = c_{22} - \frac{\gamma'_y}{\gamma'} c_{02}, \quad d_{32} = c_{32} - \frac{\gamma'_z}{\gamma'} c_{02}, \]

\[ d_{13} = c_{13} - \frac{\gamma'_x}{\gamma'} c_{03}, \quad d_{23} = c_{23} - \frac{\gamma'_y}{\gamma'} c_{03}, \quad d_{33} = c_{33} - \frac{\gamma'_z}{\gamma'} c_{03}. \]
\[ \gamma_x' \Psi_x' + \gamma_y' \Psi_y' + \gamma_z' \Psi_z' = \sqrt{\frac{\gamma}{\gamma'}} e^{-2\pi c(\gamma' - \gamma)t} (\gamma_x \Psi_x + \gamma_y \Psi_y + \gamma_z \Psi_z). \]

\[ S_x = -\gamma_y \frac{\partial}{\partial \gamma_z} + \gamma_z \frac{\partial}{\partial \gamma_y} + \frac{0}{0} \frac{0}{0} \frac{0}{1} \frac{0}{0}, \]

\[ S_y = -\gamma_z \frac{\partial}{\partial \gamma_x} + \gamma_x \frac{\partial}{\partial \gamma_z} + \frac{0}{0} \frac{0}{1} \frac{1}{0} \frac{0}{0}, \]

\[ S_z = -\gamma_x \frac{\partial}{\partial \gamma_y} + \gamma_y \frac{\partial}{\partial \gamma_x} + \frac{0}{0} \frac{-1}{1} \frac{0}{0}, \]

\[ T_x = -\gamma \frac{\partial}{\partial \gamma_x} - \frac{\gamma_x}{2\gamma} - 2\pi e c \gamma_x t - \frac{\gamma_x/\gamma}{0} \frac{0}{0} \frac{0}{0}, \]

\[ T_y = -\gamma \frac{\partial}{\partial \gamma_y} - \frac{\gamma_y}{2\gamma} - 2\pi e c \gamma_y t - \frac{0}{0} \frac{\gamma_x/\gamma}{0} \frac{0}{0}, \]

\[ T_z = -\gamma \frac{\partial}{\partial \gamma_z} - \frac{\gamma_z}{2\gamma} - 2\pi e c \gamma_z t - \frac{0}{0} \frac{\gamma_x/\gamma}{0} \frac{0}{0}, \]

\[ \gamma_x \psi_x + \gamma_y \psi_y + \gamma_z \psi_z = 0. \]

2.4. QUANTIZATION OF THE ELECTROMAGNETIC FIELD

In what follows,\(^8\) the author considered the quantization of the electromagnetic field inside a box, obtaining the usual equations in terms of

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\(^8\) In the original manuscript, the title of this section is “Dispersion”.\(\)
oscillators. Particular care was devoted to distinguish the role of the right-handed polarized states from that of the left-handed ones.

$$\nabla \cdot E = \nabla \cdot C = 0.$$

dS = dx\,dy\,dz:

$$\frac{1}{8\pi} \int (E^2 - H^2)\,dS\,dt = \text{minimum},$$

$$\varphi = 0.$$

$$E = -\frac{1}{c}\frac{\partial C}{\partial t}, \quad H = \nabla \times C;$$

$$\delta E = -\frac{1}{c}\frac{\partial}{\partial t}\delta C, \quad \delta H = \nabla \times \delta C.$$

$$\frac{1}{c} \frac{\partial H}{\partial t} + \nabla \times E = 0, \quad \frac{1}{c} \frac{\partial E}{\partial t} = \nabla \times H = \nabla \times \nabla \times C = \nabla (\nabla \cdot C) - \nabla^2 C$$

$$= -\nabla^2 C.$$

Conjugate variables:

$$C_x, \quad C_y, \quad C_z;$$

$$-\frac{1}{4\pi c} E_x, \quad -\frac{1}{4\pi c} E_y, \quad -\frac{1}{4\pi c} E_z.$$

$$H = \frac{1}{8\pi} \int (E^2 + H^2)\,dS.$$

Let us consider the electromagnetic field confined inside a cube with side \(k\), its volume being \(S = k^3\):

$$\gamma_1 = \frac{n_1}{k}, \quad \gamma_2 = \frac{n_2}{k}, \quad \gamma_3 = \frac{n_3}{k},$$

$$dN = 2k^3\,d\gamma_1\,d\gamma_2\,d\gamma_3.$$

$$v = c\gamma.$$
\[ \gamma = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2} = \frac{v}{c}. \]

\[
\begin{align*}
A_1^s &= k_1 \cos 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z) + k_2 \sin 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z), \\
A_2^s &= -k_1 \sin 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z) + k_2 \cos 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z), \\
A_3^s &= k_1 \cos 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z) - k_2 \sin 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z), \\
A_4^s &= k_1 \sin 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z) + k_2 \cos 2\pi (\gamma_1 x + \gamma_2 y + \gamma_3 z);
\end{align*}
\]

\(A_1^s\) and \(A_2^s\) correspond to right-handed, circularly polarized waves, while \(A_3^s\) and \(A_4^s\) correspond to the left-handed ones.

The direction of \(s = (v_1, v_2, v_3)\) is defined by the right-handed direction of \(k_1, k_2\). Note that \(\gamma_1, \gamma_2, \gamma_3\) are given apart from a simultaneous change of sign!

\[
\begin{array}{c}
s \rightarrow -s, \\
k_1, k_2 \rightarrow k_2, k_1.
\end{array}
\]

\[
\begin{array}{c}
A_{-s}^1 = A_{s}^2, \\
A_{-s}^2 = A_{s}^1, \\
A_{-s}^3 = A_{s}^4, \\
A_{-s}^4 = A_{s}^3.
\end{array}
\]

\(|k_1| = 1, |k_2| = 1; S = k^3.\)

\[
C = \sum a_i A_i^s, \\
E = \sum b_i A_i^s.
\]

Notice that, in these sums, the terms corresponding to \(s\) and those corresponding to \(-s\) give the same contribution: \(s \equiv -s\). The terms with \(s\) and \(-s\) are counted only once; the sign of \(s\) is defined by the right-handed rotation of \(k_1, k_2\)!
\[ a^i_b^i - b^i_a^i = \frac{2hc}{iS}. \]

\[ b^i_s = -\frac{1}{c}\dot{a}^i_s, \quad a^i_s = \frac{1}{4\pi\gamma^2c}\dot{b}^i_s. \]

\[ \ddot{a}^i_s + 4\pi^2\gamma^2c^2a^i_s = 0, \quad \ddot{b}^i_s + 4\pi^2\gamma^2c^2b^i_s = 0, \]
\[ \gamma^2c^2 = \nu^2. \]

\[ \dot{a}^i_s = -cb^i_s, \quad \dot{b}^i_s = 4\pi^2\gamma^2ca^i_s. \]

\[ H = \sum_{s,i} \frac{4\pi^2\gamma^2a^i_s^2 + b^i_s^2}{8\pi}S. \]

\[ \dot{a}^i_s = -\frac{4\pi c}{S}\frac{\partial H}{\partial b^i_s}, \quad \dot{b}^i_s = \frac{4\pi c}{S}\frac{\partial H}{\partial a^i_s}. \]

\[ p^i_s = \sqrt{\frac{\nu S\pi}{hc}}a^i_s, \quad q^i_s = \sqrt{\frac{S}{4\pi\nu hc}}b^i_s, \]

\[ a^i_s = \sqrt{\frac{hc}{\nu S\pi}}p^i_s, \quad b^i_s = \sqrt{\frac{4\pi\nu hc}{S}}q^i_s. \]

\[ H = \sum_{\nu,i} \frac{1}{2}(p^i_s^2 + q^i_s^2)h\nu. \]

\[ p^i_s q^i_s - q^i_s p^i_s = \frac{1}{i}, \quad a^i_s b^i_s - b^i_s a^i_s = \frac{2hc}{iS}. \]

\[ \dot{p}^i_s = -2\pi\nu q^i_s = -\frac{2\pi}{h}\frac{\partial H}{\partial q^i_s}, \quad \dot{q}^i_s = 2\pi\nu p^i_s = \frac{2\pi}{h}\frac{\partial H}{\partial p^i_s}. \]
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\[ s \to p_s^R = \frac{p_s' - q_s^2}{\sqrt{2}}, \quad q_s^R = \frac{q_s' + p_s'}{\sqrt{2}}, \quad p_s^R q_s^R - q_s^R p_s^R = \frac{1}{i}; \]

\[ -s \to p_{-s}^R = \frac{p_{-s}^2 - q_{-s}^2}{\sqrt{2}}, \quad q_{-s}^R = \frac{q_{-s}^2 + p_{-s}'}{\sqrt{2}}, \quad p_{-s}^R q_{-s}^R - q_{-s}^R p_{-s}^R = \frac{1}{i}; \]

\[ s \to p_s^L = \frac{p_s^4 - q_s^3}{\sqrt{2}}, \quad q_s^L = \frac{q_s^4 + p_s^3}{\sqrt{2}}, \quad p_s^L q_s^L - q_s^L p_s^L = \frac{1}{i}; \]

\[ -s \to p_{-s}^L = \frac{p_{-s}^3 - q_{-s}^4}{\sqrt{2}}, \quad q_{-s}^L = \frac{q_{-s}^4 + p_{-s}^3}{\sqrt{2}}, \quad p_{-s}^L q_{-s}^L - q_{-s}^L p_{-s}^L = \frac{1}{i}. \]

From now on, the terms with \( s \) are distinct from those with \(-s\)!

\[ p_s^R q_s^R - q_s^R p_s^R = \frac{1}{i}, \quad p_s^L q_s^L - q_s^L p_s^L = \frac{1}{i}. \]

\[ a_s = \frac{p_s^R - i q_s^R}{\sqrt{2}} \quad b_s = \frac{p_s^L - i q_s^L}{\sqrt{2}} \]

\[ a_s^* = \frac{p_s^R + i q_s^R}{\sqrt{2}} \quad b_s^* = \frac{p_s^L + i q_s^L}{\sqrt{2}} \]

\[ a_s a_s^* - a_s^* a_s = 1 \quad b_s b_s^* - b_s^* b_s = 1 \]

\[ a_s^* a_s = \frac{1}{2} (p_s^2 + q_s^2) - \frac{1}{2} \quad b_s^* b_s = \frac{1}{2} (p_s^2 + q_s^2) - \frac{1}{2} \]

\[ a_s a_s = n_s, \quad (n_s = 0, 1, 2, \ldots) \quad b_s b_s = n_s' \]

\[ a_s(n_s, n_s + 1) = \sqrt{n_s + 1} \quad b_s(n_s', n_s' + 1) = \sqrt{n_s' + 1} \]

\[ a_s^*(n_s, n_s - 1) = \sqrt{n_s} \quad b_s^*(n_s', n_s' - 1) = \sqrt{n_s'} \]

\[ p_s^R = \frac{a_s + a_s^*}{\sqrt{2}} \quad p_s^L = \frac{b_s + b_s^*}{\sqrt{2}} ; \]

\[ q_s^R = i \frac{a_s - a_s^*}{\sqrt{2}} \quad q_s^L = i \frac{b_s + b_s^*}{\sqrt{2}}. \]
\[
W = \frac{1}{2} \sum_{s,i} \frac{1}{2} h \nu_s (p^2_s + q_s^2)
= \sum_s \frac{1}{2} h \nu_s (p^D_s + q^D_s) + \sum_s \frac{1}{2} h \nu_s (p^S_s + q^S_s)
= \sum_s h \nu_s (n_s + n'_s) (+ \text{an infinite constant}).
\]

\[
p^1_s = \frac{p^R_s + q^R_s}{\sqrt{2}}, \quad q^1_s = \frac{q^R_s - p^R_s}{\sqrt{2}},
\]
\[
p^2_s = \frac{p^R_{-s} + q^R_s}{\sqrt{2}}, \quad q^2_s = \frac{q^R_s - p^R_{-s}}{\sqrt{2}},
\]
\[
p^3_s = \frac{p^L_s + q^L_s}{\sqrt{2}}, \quad q^3_s = \frac{q^L_s - p^L_s}{\sqrt{2}},
\]
\[
p^4_s = \frac{p^L_{-s} + q^L_{-s}}{\sqrt{2}}, \quad q^4_s = \frac{q^L_{-s} - p^L_{-s}}{\sqrt{2}}.
\]

(in the LHS \(s\) and \(-s\) are gathered together, while on the RHS they are kept distinct).

\[
p^1_s = \frac{1}{2} [a_s + ia_{-s} + a^*_s - ia^*_{-s}], \quad q^1_s = \frac{1}{2} [ia_s - a_{-s} - ia^*_s - a^*_{-s}],
\]
\[
p^2_s = \frac{1}{2} [a_{-s} + ia_{s} + a^*_{-s} - ia^*_s], \quad q^2_s = \frac{1}{2} [ia_{-s} - a_{s} - ia^*_s - a^*_s],
\]
\[
p^3_s = \frac{1}{2} [b_{-s} + ib_{s} + b^*_{-s} - ib^*_s], \quad q^3_s = \frac{1}{2} [ib_{-s} - b_s - ib^*_s - b^*_s],
\]
\[
p^4_s = \frac{1}{2} [b_{s} + ib_{-s} + b^*_{s} - ib^*_{-s}], \quad q^4_s = \frac{1}{2} [ib_{s} - b_{-s} - ib^*_{s} - b^*_{-s}].
\]

\[\dot{a}_s = \ldots, \quad \dot{b}_s = \ldots, \quad \dot{a}^*_s = \ldots, \quad \dot{b}^*_s = \ldots.\]

In what follows, the orthogonal functions \(A^i_s\) are defined for all the values of \(s\) (see page 73); the indices of \(k_1, k_2\) are given in such a way that the vectors \(k_1, k_2, s\) form a right-handed trihedron. The vectors \(k_1\) and \(k_2\) transform one into the other by changing \(s\) into \(-s\). Each function \(A^i_s\) is counted twice, due to the relations:

\[
A^1_s = A^2_{-s}, \quad A^2_s = A^1_{-s}, \quad A^3_s = A^4_{-s}, \quad A^4_s = A^3_{-s}.
\]
\[ C = \frac{c}{2} \sqrt{\frac{\hbar}{\pi S}} \sum_s \frac{1}{\sqrt{\nu_s}} \left[ (a_s + a_s^*) A_s^1 + i(a_s - a_s^*) A_s^2 ight. \\
\left. \quad + i(b_s - b_s^*) A_s^3 + (b_s + b_s^*) A_s^4 \right], \]
\[ E = \sqrt{\frac{\pi \hbar}{S}} \sum_s \sqrt{\nu_s} \left[ i(a_s - a_s^*) A_s^1 - (a_s + a_s^*) A_s^2 \\
\quad - (b_s + b_s^*) A_s^3 + i(b_s - b_s^*) A_s^4 \right]. \]

\[ a_s(n_s, n_{s+1}) = \sqrt{n_s + 1}, \quad b_s(n'_s, n'_{s+1}) = \sqrt{n'_s + 1}, \]
\[ a_s^*(n_s, n_{s-1}) = \sqrt{n_s}, \quad b_s(n'_s, n'_{s-1}) = \sqrt{n'_s}, \]
\[ a_s a_s^* - a_s^* a_s = 1, \quad b_s b_s^* - b_s^* b_s = 1, \]
\[ a_s^* a_s = n_s, \quad b_s^* b_s = n'_s. \]

\[ W = \frac{1}{4} \sum_s h\nu_s \left[ \bar{\rho}_s^2 + \bar{\rho}_s^{*2} + a_s a_s^* + a_s^* a_s - \bar{\rho}_s^2 - \bar{\rho}_s^{*2} + a_s a_s^* + a_s^* a_s \\
- \bar{\rho}_s^2 - \bar{\rho}_s^{*2} + b_s b_s^* + b_s^* b_s + \bar{\rho}_s^2 + \bar{\rho}_s^{*2} + b_s b_s^* + b_s^* b_s \right] \]
\[ = \sum_s h\nu_s (n_s + n'_{s+1}) = \sum_s h\nu_s (n_s + N_s) + \text{an infinite constant}, \]

with:

\[ n_s = a_s^* a_s, \]
\[ N_s = b_s^* b_s. \]

By absorbing the \textit{infinite} constant \textit{into} \( W \), we have:

\[ W_R = \sum_s h\nu_s (n_s + N_s). \]

We have used \( N_s \) instead of \( n'_s \): \( n_s \) corresponds to right-handed polarized waves, while \( N_s \) to the left-handed ones.
The author continued\(^9\) to study the quantization of the electromagnetic field, obtaining explicit expressions for the matrix elements of the creation and the annihilation operators (in the number operator representation) and for the angular momentum of the field. Transformation properties of the \(n\)-photon states \(\psi\) were quickly outlined at the end of this Section.

\[
\begin{align*}
C &= \sum_k \sqrt{\frac{2hc}{k}} p_k f_k, \\
E &= \sum_k \sqrt{2hc} q_k f_k.
\end{align*}
\]

\[\dot{q}_k = kc p_k, \quad \dot{p}_k = -kc q_k.\]

\[
\begin{align*}
\frac{1}{c} \frac{\partial C}{\partial t} &= \sum_k \sqrt{\frac{2h}{ck}} \dot{p}_k f_k = -E = -\sum_k \sqrt{2hc} q_k f_k, \\
\frac{1}{c} \frac{\partial E}{\partial t} &= \sum_k \sqrt{\frac{2hk}{c}} \dot{q}_k f_k = -\nabla^2 C = \sum_k k\sqrt{2hc} p_k f_k.
\end{align*}
\]

\[
\begin{align*}
\dot{q}_k &= \frac{2\pi}{h} \frac{\partial W}{\partial p_k}, \\
\dot{p}_k &= -\frac{2\pi}{h} \frac{\partial W}{\partial q_k}.
\end{align*}
\]

\[
W = \sum h\nu_k \frac{1}{2}(p_k^2 + q_k^2) = \sum_k \frac{h}{2\pi c k} \frac{1}{2}(p_k^2 + q_k^2).
\]

\(^9\) In the original manuscript, the title of this section is “Irradiation”.
\[ \dot{q}_k = -\frac{2\pi i}{\hbar} (q_k W - W q_k), \]
\[ \dot{p}_k = -\frac{2\pi i}{\hbar} (p_k W - W q_k); \]
\[ i(q_k W - W q_k) = \frac{\partial W}{\partial p_k}, \]
\[ -i(p_k W - W p_k) = \frac{\partial W}{\partial q_k}; \]
\[ -i(q_k p_k - p_k q_k) = 1, \]
\[ +i(p_k q_k - q_k p_k) = 1. \]

\[ p_k q_k - q_k p_k = \frac{1}{i}. \]

\[ W = \sum_k h\nu_k \left( n_k + \frac{1}{2} \right) = \sum h\nu_k \frac{p_k^2 + q_k^2}{2}. \]

\[ \frac{1}{2} (p_k^2 + q_k^2) = \frac{p_k + iq_k}{\sqrt{2}} \frac{p_k - iq_k}{\sqrt{2}} + \frac{1}{2}, \]
\[ a_k = \frac{p_k - iq_k}{\sqrt{2}}, \quad a_k^* = \frac{p_k + iq_k}{\sqrt{2}}. \]
\[ a_k a_k^* - a_k^* a_k = \frac{i}{2} (p_k q_k - q_k p_k + p_k q_k - q_k p_k) = 1. \]

\[ a_k^* a_k = n_k, \]
\[ a_k a_k^* = n_k + 1. \]

\[ a_k = \frac{p_k - iq_k}{\sqrt{i}}, \quad p_k = \frac{a_k + a_k^*}{\sqrt{2}}, \]
\[ a_k^* = \frac{p_k + iq_k}{\sqrt{i}}, \quad q_k = \frac{a_k^* - a_k}{i\sqrt{2}}. \]
\[ a_k = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \sqrt{3} & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \sqrt{4} & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \sqrt{5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \]

\[ a_k^* = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & \ldots \\
0 & \sqrt{2} & 0 & 0 & 0 & \ldots \\
0 & 0 & \sqrt{3} & 0 & 0 & \ldots \\
0 & 0 & 0 & \sqrt{4} & 0 & \ldots \\
0 & 0 & 0 & 0 & \sqrt{5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} ; \]

\[ a_k^* a_k = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 2 & 0 & \ldots \\
0 & 0 & 0 & 3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \]

\[ a_k a_k^* = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 0 & 0 & \ldots \\
0 & 0 & 3 & 0 & \ldots \\
0 & 0 & 0 & 4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} ; \]

\[ p_k = \begin{bmatrix}
0 & 1/\sqrt{2} & 0 & \ldots \\
1/\sqrt{2} & 0 & 1 & \ldots \\
0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \]

\[ q_k = \begin{bmatrix}
0 & i/\sqrt{2} & 0 & \ldots \\
-i/\sqrt{2} & 0 & -i & \ldots \\
0 & i & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \]

\[ C' = C + \epsilon S C, \quad E' = E + \epsilon S E. \]

\[ p'_r = p_r + \epsilon \sum_s S_{rs} p_s, \quad q'_r = q_r + \epsilon \sum_s S_{rs} q_s. \]

\[ S_{rs} = -S_{sr}. \]
\[ \psi = \psi(n_1, n_2, \ldots), \]
\[ \psi' = \psi + \frac{T}{i} \varepsilon \psi; \]
\[ q' = q + \frac{\varepsilon}{i} (qT - Tq), \]
\[ p' = p + \frac{\varepsilon}{i} (pT - Tp). \]

\[ p_r T - T p_r = i \sum S_{rs} p_s, \]
\[ q_r T - T q_r = i \sum S_{rs} q_s. \]

\[
T = \sum_{rs} S_{rs} p_r q_s. \]

\( T \) is the angular momentum in units \( \hbar / 2\pi \).

\[
T = \sum S_{rs} p_r q_s = \sum_{r<s} S_{rs} (p_r q_s - p_s q_r). \]

\[
p_r q_s - p_s q_r = \frac{1}{2i} (a_r^* a_s^* - a_r a_s - a_r^* a_s + a_r a_s^*) - a_s^* a_r^* - a_s a_r + a_s^* a_r - a_s a_r^* \]
\[ = \frac{1}{i} (a_r a_s^* - a_s a_r^*). \]

\[
T = \sum_{r<s} \frac{1}{i} (a_r a_s^* - a_s a_r^*) S_{rs}. \]

For \( n \) photons:

\[
\psi = \psi(n_1, n_2, \ldots) \delta \left( \sum n_i - n \right). \]

For \( n = 1 \), \( \psi = \psi(n_1, n_2, \ldots) \) and all \( n_i \) but one vanish, and the non-zero number is equal to 1:

\[
\psi(1, 0, 0, 0, \ldots) = c_1, \]
\[
\psi(0, 1, 0, 0, \ldots) = c_2, \]
\[
\psi(0, 0, 1, 0, 0, \ldots) = c_3, \]
\[
\ldots \]
\[ \psi = (c_1, c_2, c_3, \ldots). \]

\[ \psi' = T\psi. \]

\[ \sum_{r<s} S_{rs}(a_r a_s^* - a_s a_r^*) = \sum_{r,s} S_{rs} a_r a_s^*, \]

\[ \frac{1}{i} \sum_{rs} S_{rs} a_r a_s^* \psi = (c_1', c_2', \ldots). \]

\[ c_s' = \frac{1}{i} \sum_{rs} S_{rs} c_r = i \sum_{sr} S_{rs} c_r. \]

\[ c_r' = i \sum_{rs} S_{rs} c_s. \]

2.6. CONTINUATION II: INCLUDING THE MATTER FIELDS

What had been studied in the Sect. 2.4 was tentatively generalized here to the case of an electromagnetic field interacting with a charged Dirac field \( \psi \). As above, the scalar potential is assumed to be zero, \( \varphi = 0 \), and again the box volume is \( S = k^3 \).

Dirac equations:

\[ \left[ \frac{W}{c} + \rho_3 \sigma \cdot \left( p + \frac{e}{c} C \right) + \rho_1 mc \right] \psi = 0. \]

\( p = (p_x, p_y, p_z) \). For plane waves, \( p_x, p_y, p_z \) are constant.

\[ \psi^r_p = (\psi_1, \psi_2, \psi_3, \psi_4) = e^{(2\pi i / h)(p_x x + p_y y + p_z z)} (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4), \]

\[ \frac{W}{c} = \begin{cases} +\sqrt{m^2 c^2 + p^2}, & \text{for } r = 1, 2, \\ -\sqrt{m^2 c^2 + p^2}, & \text{for } r = 3, 4. \end{cases} \]
The spinor factors are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\epsilon_1 \sqrt{2S(1 + \frac{W/c}{p_z} + \frac{p_z^2}{m^2 c^2})}$</th>
<th>$\epsilon_2 \sqrt{2S(\ldots)}$</th>
<th>$\epsilon_3 \sqrt{2S(\ldots)}$</th>
<th>$\epsilon_4 \sqrt{2S(\ldots)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$-\frac{W/c + p_z}{mc}$</td>
<td>$-\frac{p_x + ip_y}{mc}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{p_x - ip_y}{mc}$</td>
<td>$-\frac{W/c + p_z}{mc}$</td>
<td>0</td>
<td>$-\frac{p_x + ip_y}{mc}$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$-\frac{W/c + p_z}{mc}$</td>
<td>$-\frac{p_x + ip_y}{mc}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{p_x - ip_y}{mc}$</td>
<td>$-\frac{W/c + p_z}{mc}$</td>
<td>0</td>
<td>$-\frac{p_x + ip_y}{mc}$</td>
<td>1</td>
</tr>
</tbody>
</table>

$p_x = g_1 \frac{h}{k}$, $p_y = g_2 \frac{h}{k}$, $p_z = g_3 \frac{h}{k}$;
$g_1, g_2, g_3 = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$H = -\epsilon p_3 \sigma \cdot p - \rho_1 mc^2 + \sum_s h\nu_s(n_s + N_s) - \epsilon p_3 \sigma \cdot C$$

$$= H_0 - \epsilon p_3 \sigma \cdot C = H_0 + H_1.$$

$H_1 = -\epsilon p_3 \sigma \cdot C$. Quantities $n_s, N_s$ are the numbers of the right-handed and left-handed polarized waves, respectively.

$$\langle p, r, n_i, N_i | H_0 | p', r', n_i', N_i' \rangle = \delta(p - p') \delta(r - r_i) \delta(n - n') \delta(N - N') \times W^{p, r}_{\text{electr.}} + \sum_s h\nu_s(n_s + N_s).$$

Expression for $p_3 \sigma$ on the states $\psi^1_p, \psi^2_p, \psi^3_p, \psi^4_p$.

$$p_3 \sigma_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad p_3 \sigma_y = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}.$$
\[
\rho_{3\sigma_z} = \begin{vmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 1
\end{vmatrix}.
\]

\[
\psi_1^p = (1, 0, 0, 0) e^{(2\pi i/h)(p_x x + p_y y + p_z z)},
\]
\[
\psi_2^p = (0, 1, 0, 0) e^{(2\pi i/h)(p_x x + p_y y + p_z z)},
\]
\[
\psi_3^p = (0, 0, 1, 0) e^{(2\pi i/h)(p_x x + p_y y + p_z z)},
\]
\[
\psi_4^p = (0, 0, 0, 1) e^{(2\pi i/h)(p_x x + p_y y + p_z z)}.
\]

### 2.7. QUANTUM DYNAMICS OF ELECTRONS INTERACTING WITH AN ELECTROMAGNETIC FIELD

The dynamics of a system composed of interacting electrons and photons is considered in the realm of Quantum Field Theory (Klein-Gordon theory). The electrons are described by a field \( \psi \) (or \( P \), deduced from \( \psi \)), while the electromagnetic field is described in terms of the potential \( (\varphi, C) \). An expression for the quantized Hamiltonian is given, along with the commutation rules for creation/annihilation operators.

For a charge \(-e\) we have:

\[
\left[ \left( -\frac{h}{2\pi i c} \frac{\partial}{\partial t} + \frac{e}{c} \varphi \right)^2 - \sum_x \left( \frac{h}{2\pi i} \frac{\partial}{\partial x} + \frac{e}{c} C_x \right)^2 - m^2 c^2 \right] \psi = 0.
\]

\[
P = \frac{h^2}{8\pi^2 c^2 m} \left( \frac{\partial}{\partial t} + \frac{2\pi i}{h} e \varphi \right) \overline{\psi},
\]
\[
\overline{P} = \frac{h^2}{8\pi^2 c^2 m} \left( \frac{\partial}{\partial t} - \frac{2\pi i}{h} e \varphi \right) \psi.
\]

\[
\left[ \frac{1}{c^2} \left( \frac{\partial}{\partial t} - \frac{2\pi i}{h} e \varphi \right)^2 - \sum_x \left( \frac{\partial}{\partial x} + \frac{2\pi i}{hc} e C_x \right)^2 + \frac{4\pi^2}{h^2} m^2 c^2 \right] \psi = 0,
\]
\[
\left[ \frac{1}{c^2} \left( \frac{\partial}{\partial t} + \frac{2\pi i}{h} e \varphi \right)^2 - \sum_x \left( \frac{\partial}{\partial x} - \frac{2\pi i}{hc} e C_x \right)^2 + \frac{4\pi^2}{h^2} m^2 c^2 \right] \overline{\psi} = 0.
\]

\[
\nabla^2 C_x - \frac{\partial}{\partial x} \nabla \cdot C = \frac{\partial^2 C_x}{\partial y^2} + \frac{\partial^2 C_x}{\partial r^2} - \frac{\partial^2 C_y}{\partial x \partial y} - \frac{\partial^2 C_x}{\partial x \partial z}.
\]
\[
\begin{align*}
\left[ \frac{\hbar^2}{8\pi^2mc^2} \left( \frac{\partial}{\partial t} - \frac{2\pi i}{\hbar} e \varphi \right)^2 \\
- \frac{\hbar^2}{8\pi^2m} \sum_x \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e C_x \right)^2 + \frac{1}{2}mc^2 \right] \psi &= 0, \\
\left[ \frac{\hbar^2}{8\pi^2mc^2} \left( \frac{\partial}{\partial t} + \frac{2\pi i}{\hbar} e \varphi \right)^2 \\
- \frac{\hbar^2}{8\pi^2m} \sum_x \left( \frac{\partial}{\partial x} - \frac{2\pi i}{\hbar c} e C_x \right)^2 + \frac{1}{2}mc^2 \right] \overline{\psi} &= 0.
\end{align*}
\] (1)

\[
\left( \frac{\partial}{\partial t} - \frac{2\pi i}{\hbar} e \varphi \right) P = -\frac{1}{2}mc^2\psi + \frac{\hbar^2}{8\pi^2m} \sum_x \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e C_x \right)^2 \psi,
\] (3)

\[
\left( \frac{\partial}{\partial t} + \frac{2\pi i}{\hbar} e \varphi \right) P = -\frac{1}{2}mc^2\overline{\psi} + \frac{\hbar^2}{8\pi^2m} \sum_x \left( \frac{\partial}{\partial x} - \frac{2\pi i}{\hbar c} e C_x \right) \overline{\psi},
\] (4)

\[
\left( \frac{\partial}{\partial t} + \frac{2\pi i}{\hbar} e \varphi \right) \overline{\psi} = \frac{8\pi^2mc^2}{\hbar^2} P,
\] (5)

\[
\left( \frac{\partial}{\partial t} - \frac{2\pi i}{\hbar} e \varphi \right) \psi = \frac{8\pi^2mc^2}{\hbar^2} \overline{P}.
\] (6)

\[
\rho = \frac{\hbar e}{4\pi mc^2} \left[ \overline{\psi} \left( \frac{\partial}{\partial t} - \frac{2\pi i}{\hbar c} e \varphi \right) \psi - \psi \left( \frac{\partial}{\partial t} + \frac{2\pi i}{\hbar c} e \varphi \right) \overline{\psi} \right],
\]

\[
i_x = -\frac{\hbar e}{4\pi mc} \left[ -\overline{\psi} \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e \varphi \right) \psi - \psi \left( \frac{\partial}{\partial x} - \frac{2\pi i}{\hbar c} \psi \psi \right) \overline{\psi} \right],
\]

\[
\cdots.
\]

\[
d\tau = dV dt.
\] [10]

---

\textsuperscript{10} Notice that, more appropriately, one should write } d^4\tau = d^3V dt, since } d\tau \text{ denotes the 4-dimensional volume element, while } d\tau dV \text{ is the 3-dimensional space volume element.}
\[
\delta \int \left\{ \frac{\hbar^2}{8\pi^2m} \left[ \frac{1}{c^2} \left( \frac{\partial}{\partial t} + \frac{2\pi i}{\hbar} e^\varphi \right) \psi \left( \frac{\partial}{\partial t} - \frac{2\pi i}{\hbar} e^\varphi \right) \psi \right] - \sum_x \left( \frac{\partial}{\partial x} - \frac{2\pi i x}{\hbar c} e^{C_x} \right) \psi \cdot \left( \frac{\partial}{\partial x} + \frac{2\pi i x}{\hbar c} e^{C_x} \right) \psi \right\} \right. \\
- \frac{1}{2} m c^2 \mbox{\text{Re}} \psi + \frac{1}{8\pi} \left( \frac{1}{c} \frac{\partial \bf{C} }{\partial t} + \nabla \varphi \right)^2 - \left| \nabla \times \bf{C} \right|^2 \left\} \right. \left. \right\} d\tau = 0. \tag{7} \]

From this, the variation with respect to \( \overline{\psi} \) or \( \psi \) gives Eq. (1) or (2), respectively. The variation with respect to \( \varphi \) yields:

\[
- \frac{1}{4\pi} \sum_x \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial x} + \frac{1}{c} \frac{\partial C}{\partial t} \right) \\
- \frac{\hbar e}{4\pi i mc^2} \left[ \psi \left( \frac{\partial}{\partial t} - \frac{2\pi i}{\hbar} e^\varphi \right) \psi - \psi \left( \frac{\partial}{\partial t} + \frac{2\pi i}{\hbar} e^\varphi \right) \overline{\psi} \right] = 0,
\]

\[
\frac{1}{4\pi} \nabla \cdot \bf{E} - \rho = 0. \tag{8} \]

The variation with respect to \( C_x \) instead gives:

\[
- \frac{1}{4\pi c} \frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial x} + \frac{1}{c} \frac{\partial C}{\partial t} \right) - \frac{1}{4\pi} \left[ \frac{\partial}{\partial y} \left( \frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y} \right) \right] \\
- \frac{\partial}{\partial z} \left( \frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x} \right) \right] - \frac{\hbar e}{4\pi i mc} \left[ \overline{\psi} \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e^{C_x} \right) \psi \right. \\
- \psi \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e^{C_x} \right) \overline{\psi} \right] = 0,
\]

\[
\frac{1}{4\pi c} \frac{\partial E_x}{\partial t} - \frac{1}{4\pi} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial r} \right) + i_x = 0, \tag{9} \]

and similarly for the other components.

\[
A = \frac{1}{8\pi} \sum_x \left( \frac{1}{c} \frac{\partial C_x}{\partial t} + \frac{\partial \varphi}{\partial x} \right)^2 \\
= \frac{1}{8\pi c^2} \sum_x \left( \frac{\partial C_x}{\partial t} \right)^2 + \frac{1}{8\pi} \left( \frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{4\pi c} \sum_x \frac{\partial C_x}{\partial t} \frac{\partial \varphi}{\partial x},
\]

\[
B = \frac{1}{4\pi c^2} \sum_x \left( \frac{\partial C_x}{\partial t} \right)^2 + \frac{1}{4\pi c} \sum_x \frac{\partial C_x}{\partial t} \frac{\partial \varphi}{\partial x},
\]
\[ B - A = \frac{1}{8\pi c^2} \sum_x \left( \frac{\partial C_x}{\partial t} \right)^2 - \frac{1}{8\pi} \sum_x \left( \frac{\partial \varphi}{\partial x} \right)^2. \]

Without matter fields, the conjugate Hamiltonian variables are:

\[ C_x, \quad -\frac{1}{4\pi c} E_x; \]
\[ C_y, \quad -\frac{1}{4\pi c} E_y; \]
\[ C_z, \quad -\frac{1}{4\pi c} E_z; \]
\[ \varphi, \quad 0 \]

\[ \mathcal{H} = \frac{1}{8\pi} |\nabla \times \mathbf{C}|^2 + \frac{1}{8\pi} E^2 + \frac{1}{4\pi} \sum_x \frac{\partial \varphi}{\partial x} E_x, \]

\[ \dot{E}_x = c \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right), \]
\[ \dot{C}_x = -cE_x - c \frac{\partial \varphi}{\partial x}; \quad E_x = -\frac{\partial \varphi}{\partial x} - \frac{1}{c} \frac{\partial C_x}{\partial t}, \]
\[ \dot{\varphi} = \ldots \]
\[ \dot{0} = 0 = -\frac{1}{4\pi} \nabla \cdot \mathbf{E}. \]

In the following we consider a particle with charge \(-e\) and assume \(\varphi = 0\).

\[ \delta \int L d\tau = 0, \quad \text{with } d\tau = dV dt. \]

\[ \delta \int \left\{ \frac{\hbar^2}{8\pi^2 m} \left[ \frac{1}{c^2} \frac{\partial}{\partial t} \psi \frac{\partial}{\partial t} \psi \right] \right. \]
\[ - \sum_x \left( \frac{\partial}{\partial x} - \frac{2\pi i}{\hbar c} e C_x \right) \bar{\psi} \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e C_x \right) \psi \]  
\[ - \frac{1}{2} mc^2 \bar{\psi} \psi + \frac{1}{8\pi} \left[ \frac{1}{c} \left| \frac{\partial C}{\partial t} \right|^2 - |\nabla \times \mathbf{C}|^2 \right] \]  
\[ \left. \right\} d\tau = 0. \tag{7'} \]

\[ ^{11} \text{In the following, the author looked for the variable conjugate to } \varphi. \]
\[ \psi, \quad P = \frac{\hbar^2}{8\pi^2 mc^2} \frac{\partial}{\partial t} \psi; \]
\[ \overline{\psi}, \quad \overline{P} = \frac{\hbar^2}{8\pi^2 mc} \frac{\partial}{\partial t} \psi; \]
\[ C_x, \quad -\frac{E_x}{4\pi c} = \frac{1}{4\pi c^2} \frac{\partial C_x}{\partial t}; \]
\[ C_y, \quad -\frac{E_y}{4\pi c} = \frac{1}{4\pi c^2} \frac{\partial C_y}{\partial t}; \]
\[ C_z, \quad -\frac{E_z}{4\pi c} = \frac{1}{4\pi c^2} \frac{\partial C_z}{\partial t}. \]

\[ H = \int \left[ \frac{8\pi^2 mc^2}{\hbar^2} PP + \frac{1}{2} mc^2 \overline{\psi} \psi + \frac{\hbar^2}{8\pi^2 m} \sum_x \left( \frac{\partial}{\partial x} - \frac{2\pi i}{\hbar c} e C_x \right) \overline{\psi} \times \left( \frac{\partial}{\partial x} + \frac{2\pi i}{\hbar c} e C_x \right) \psi + \frac{1}{8\pi} (E^2 + H^2) \right] dV, \]
\[ H = \int \left[ \frac{8\pi^2 mc^2}{\hbar^2} PP + \frac{1}{2} mc^2 \overline{\psi} \psi + \frac{\hbar^2}{8\pi^2 m} \nabla \overline{\psi} \cdot \nabla \psi + \frac{hc}{4\pi imc} C \cdot (\overline{\psi} \nabla \psi - \psi \nabla \overline{\psi}) + \frac{c^2}{2mc^2} |C|^2 \overline{\psi} \psi + \frac{1}{8\pi} (E^2 + H^2) \right] dV. \]

\[ \rho = \frac{2\pi i}{h} e (\psi P - \overline{\psi} P), \]
\[ i = -\frac{he}{4\pi imc} \left[ \overline{\psi} \left( \nabla + \frac{2\pi i}{\hbar c} e C \right) \psi - \psi \left( \nabla - \frac{2\pi C}{\hbar c} e C \right) \overline{\psi} \right] = -\frac{he}{4\pi imc} (\overline{\psi} \nabla \psi - \psi \nabla \overline{\psi}) - \frac{c^2}{mc^2} \overline{\psi} \psi C. \]

\[ \nabla \cdot f_k' = 0, \quad f_\lambda = \nabla \varphi_\lambda; \quad \nabla^2 \varphi_\lambda + \lambda^2 \varphi_\lambda = 0. \]

\[ \begin{cases} \nabla^2 f_\lambda + \lambda^2 f_\lambda = 0, \\ \nabla^2 f_k' + k^2 f_k' = 0. \end{cases} \]
\[ \int f_\lambda \cdot f_{\lambda'} \, dV = \delta_{\lambda \lambda'}, \]
\[ \int f'_k \cdot f'_{k'} \, dV = \delta_{kk'}, \]
\[ \int f_\lambda \cdot f_k \, dV = 0; \]
\[ \int \varphi_\lambda \varphi_{\lambda'} \, dV = \frac{1}{\lambda'^2} \int f_\lambda \cdot f_{\lambda'} \, dV = \frac{1}{\lambda'^2} \delta_{\lambda \lambda'}, \]
\[ \lambda \varphi_\lambda = u_\lambda; \quad \int u_\lambda u_{\lambda'} \, dV = \delta_{\lambda \lambda'}. \]

\[ \psi = \sum [A_\lambda (q_\lambda + Q_\lambda) + iB_\lambda (p_\lambda - P_\lambda)] \lambda \varphi_\lambda, \quad (A_\lambda = B_\lambda) \]
\[ P = \sum [C_\lambda (p_\lambda + P_\lambda) + iD_\lambda (q_\lambda - Q_\lambda)] \lambda \varphi_\lambda; \quad (C_\lambda = D_\lambda) \]

\[ \int \overline{\psi} P \, dV = \sum [C^2_\lambda (p_\lambda + P_\lambda)^2 + D^2_\lambda (q_\lambda - Q_\lambda)^2], \]
\[ \int \overline{\psi} \psi \, dV = \sum [A^2_\lambda (q_\lambda + Q_\lambda)^2 + B^2_\lambda (p_\lambda - P_\lambda)^2]. \]

\[ \frac{8\pi^2mc^2}{\hbar^2} \int \overline{\psi} P \, dV + \frac{1}{2m} \left( m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2} \right) \int \overline{\psi} \psi \, dV \]
\[ = \frac{8\pi^2mc^2}{\hbar^2} \sum_\lambda \left[ C^2_\lambda (p_\lambda + P_\lambda)^2 + D^2_\lambda (q_\lambda - Q_\lambda)^2 \right] + \frac{1}{2m} \left( m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2} \right) \sum_\lambda \left[ A^2_\lambda (q_\lambda + Q_\lambda)^2 + B^2_\lambda (p_\lambda - P_\lambda)^2 \right] \]
\[ = \sum_\lambda \left[ \frac{1}{2} p^2_\lambda + \frac{1}{2} q^2_\lambda + \frac{1}{2} P^2_\lambda + \frac{1}{2} Q^2_\lambda \right] c \sqrt{m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}}, \]

\[ \frac{8\pi^2mc^2}{\hbar^2} C^2_\lambda + \frac{1}{2m} \left( m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2} \right) B^2_\lambda = \frac{1}{2} c \sqrt{m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}}, \]
\[ \frac{8\pi^2mc^2}{\hbar^2} D^2_\lambda + \frac{1}{2m} \left( m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2} \right) A^2_\lambda = \frac{1}{2} c \sqrt{m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}}, \]
\[
\frac{8\pi^2mc^2}{\hbar^2}C_\lambda^2 = \frac{1}{2m} \left( m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2} \right) B_\lambda^2,
\]
\[
\frac{8\pi^2mc^2}{\hbar^2}D_\lambda^2 = \frac{1}{2m} \left( m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2} \right) A_\lambda^2,
\]
\[
A_\lambda^2 = B_\lambda^2 = \frac{mc}{2\sqrt{m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}}},
\]
\[
C_\lambda^2 = D_\lambda^2 = \frac{\hbar^2}{32\pi^2mc} \sqrt{m^2c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}}.
\]

\[
\psi = \frac{1}{\sqrt{2}} \sum_\lambda \sqrt{\frac{mc}{m^2c^2 + \lambda^2 \hbar^2/4\pi^2}} [q_\lambda + q'_\lambda + i(p_\lambda - p'_\lambda)] u_\lambda.
\]
\[
P = \frac{\hbar}{4\pi \sqrt{2}} \sum_\lambda \sqrt{\frac{m^2c^2 + \lambda^2 \hbar^2/4\pi^2}{mc}} [p_\lambda + p'_\lambda + i(q_\lambda - q'_\lambda)] u_\lambda.
\]

\[
4/i = 2(p_\lambda q_\lambda - q_\lambda p_\lambda) + 2(p'_\lambda q'_\lambda - q'_\lambda p'_\lambda) \pm 2i(q_\lambda q'_\lambda - q'_\lambda q_\lambda)
\]
\[
\mp 2i(p_\lambda p'_\lambda - p'_\lambda p_\lambda),
\]
\[
0 = (p_\lambda q_\lambda - q_\lambda p_\lambda) - (p'_\lambda q'_\lambda - q'_\lambda p'_\lambda) + (p_\lambda q'_\lambda - q_\lambda p'_\lambda) - (p'_\lambda q_\lambda - q'_\lambda p_\lambda);
\]
\[
0 = (p_\lambda q_\lambda - q_\lambda p_\lambda) - (p'_\lambda q'_\lambda - q'_\lambda p'_\lambda) - (p_\lambda q'_\lambda - q_\lambda p'_\lambda) - (p'_\lambda q_\lambda - q'_\lambda p_\lambda);
\]
\[
0 = (p_\lambda q'_\lambda - q_\lambda p'_\lambda) + (p'_\lambda q_\lambda - q_\lambda p_\lambda) \pm (p_\lambda p'_\lambda - p'_\lambda p_\lambda) \pm (q_\lambda q'_\lambda - q'_\lambda q_\lambda).
\]

\[
\begin{array}{ll}
p_\lambda q_\lambda - q_\lambda p_\lambda &= 1/i, \\
p'_\lambda q'_\lambda - q'_\lambda p'_\lambda &= 1/i, \\
p_\lambda q'_\lambda - q_\lambda p'_\lambda &= 0, \\
p'_\lambda q_\lambda - q_\lambda p_\lambda &= 0, \\
p_\lambda p'_\lambda - p'_\lambda p_\lambda &= 0, \\
q_\lambda q'_\lambda - q'_\lambda q_\lambda &= 0.
\end{array}
\]

\[
-Z\epsilon = \int \rho \, dV = \frac{2\pi i}{\hbar} e \int (\psi P - \overline{\psi P}) \, dV,
\]
\[ Z = -\frac{2\pi i}{\hbar} \int (\psi P - \overline{\psi} \bar{P}) \, dV \]
\[ = \sum_{\lambda} \left( \frac{1}{2} p_{\lambda}^2 + \frac{1}{2} q_{\lambda}^2 - \frac{1}{2} p_{\lambda}'^2 - \frac{1}{2} q_{\lambda}'^2 \right) \]
\[ = \sum_{\lambda} \left[ \left( \frac{1}{2} p_{\lambda}^2 + \frac{1}{2} q_{\lambda}^2 - \frac{1}{2} \right) - \left( \frac{1}{2} p_{\lambda}'^2 + \frac{1}{2} q_{\lambda}'^2 - \frac{1}{2} \right) \right] \]
\[ = \sum_{\lambda} (N_{\lambda} - N'_{\lambda}) = \sum_{\lambda} Z_{\lambda}. \]

\[
\begin{align*}
H &= H_M + H_R, \\
H_M &= H_M^0 + H_M^1,
\end{align*}
\]

where \( H_M \) and \( H_R \) account for the matter and radiation field contribution to the Hamiltonian, respectively. \( H_M^0 \) is the free particle Hamiltonian, while \( H_M^1 \) describes the particle interaction and that between particles and light quanta.

\[ N_{\lambda} = \frac{1}{2} p_{\lambda}^2 + \frac{1}{2} q_{\lambda}^2 - \frac{1}{2}, \]
\[ N'_{\lambda} = \frac{1}{2} p_{\lambda}'^2 + \frac{1}{2} q_{\lambda}'^2 - \frac{1}{2}, \]
\[ Z_{\lambda} = N_{\lambda} - N'_{\lambda}. \]

\[
H_M^0 = \sum_{\lambda} \left( \frac{1}{2} p_{\lambda}^2 + \frac{1}{2} q_{\lambda}^2 \frac{1}{2} p_{\lambda}'^2 + \frac{1}{2} q_{\lambda}'^2 \right) c \sqrt{m^2 c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}}
\[
= \sum_{\lambda} (N_{\lambda} + N'_{\lambda}) c \sqrt{m^2 c^2 + \lambda^2 \frac{\hbar^2}{4\pi^2}} + \text{zero point energy.}
\]

\[ ^{\[12\]} \]

\[ ^{12}@ \text{In the original manuscript, some expressions were written in terms of } \nu \text{ instead of } k, \]
\[ \text{but the warning “use } k \text{ instead of } \nu \text{” appears. We have therefore chosen to use the symbol } \]
\[ k \text{ throughout.} \]
\[ C = \sum_k A_k Q_k f_k + \sum_\lambda B_\lambda P_\lambda f_\lambda, \]
\[ -E = \sum_k C_k P_k f_k - \sum_\lambda D_\lambda Q_\lambda f_\lambda \]
\[ (\nabla \times f_\lambda = 0). \]
\[ \int E^2 dV = \sum_k C^2_k P^2_k + \sum_\lambda D^2_\lambda Q^2_\lambda. \]
\[ H_x = \frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}, \]
\[ H^2_x = \left( \frac{\partial C_z}{\partial y} \right)^2 + \left( \frac{\partial C_H}{\partial r} \right)^2 - 2 \frac{\partial C_x}{\partial y} \frac{\partial C_y}{\partial x}, \]
\[ H^2 = \sum_x |\nabla C_x|^2 - \sum_{xy} \frac{\partial C_x}{\partial y} \frac{\partial C_y}{\partial x} = \sum_k A^2_k N^2 Q^2_k. \]
\[ \int H^2 dV = \ldots. \]

\[
\begin{align*}
P_k Q_k - Q_k P_k &= 1/i, \\
P_\lambda Q_\lambda - Q_\lambda P_\lambda &= 1/i.
\end{align*}
\]
\[
\begin{align*}
C_k^2 &= \frac{1}{8\pi} \frac{hck}{2}, \\
A_k^2 &= \frac{1}{8\pi} \frac{hck}{2}, \\
D_\lambda^2 &= \frac{1}{8\pi} 2, \\
C_k &= \sqrt{2hck}, \\
A_k &= \sqrt{\frac{2hc}{k}}, \\
D_\lambda &= \sqrt{\frac{4\pi}{4\pi}} = 2\sqrt{\pi}, \\
B_\lambda &= \frac{hc}{\sqrt{\pi}}.
\end{align*}
\]
\[ N_k = \frac{1}{2} (P_k^2 + Q_k^2) - \frac{1}{2}. \]

\[
C = \sum_k \sqrt{\frac{2\hbar c}{k}} Q_k f_k' + \sum_{\lambda} \frac{\hbar c}{\sqrt{\pi}} P_\lambda f_\lambda,
\]

\[-E = \sum_k \sqrt{2\hbar c k} P_k f_k' - \sum_{\lambda} \sqrt{4\pi} Q_{\lambda} f_\lambda.\]

\[
\nu_k = c \frac{k}{2\pi}.
\]

\[
H_R = \frac{1}{2} \sum_k \hbar c k (Q_k^2 + P_k^2) + \frac{1}{2} \sum_{\lambda} Q_{\lambda}^2
\]

\[
= \sum_k \frac{1}{2} (P_k^2 + Q_k^2) h\nu_k + \frac{1}{2} \sum_{\lambda} Q_{\lambda}^2
\]

\[
= \sum_k N k h\nu_k + \sum_{\lambda} \frac{1}{2} Q_{\lambda}^2 + \text{rest energy}.
\]

---

\[13\]

\[
\nabla u_\lambda = \nabla \lambda \varphi_\lambda = \lambda f_\lambda,
\]

\[
\nabla \psi = \frac{1}{\sqrt{2}} \sum_{\lambda} \sqrt{\frac{mc}{\sqrt{m^2 c^2 + \lambda^2 h^2 / 4\pi^2}}} \left[ q_\lambda + q'_\lambda + i(p_\lambda - p'_\lambda) \right] \lambda f_\lambda,
\]

\[
\overline{\psi} \nabla \psi - \psi \nabla \overline{\psi} = -i mc \sum_{\lambda, \lambda'} \frac{1}{\sqrt{(m^2 c^2 + \lambda^2 h^2 / 4\pi^2)(m'^2 c^2 + \lambda'^2 h^2 / 4\pi^2)}} \times \left[ (p_\lambda - p'_\lambda)(q_{\lambda'} + q'_{\lambda'}) - (p_{\lambda'} - p'_{\lambda'})(q_\lambda + q'_\lambda) \right] \lambda' u_{\lambda'} f_{\lambda'}.
\]

\[
\nabla \cdot \varphi_\lambda f_{\lambda'} = f_\lambda \cdot f_{\lambda'} - \lambda' 2 \varphi_\lambda \varphi_{\lambda'}.
\]

\[\text{\textsuperscript{13}\textsuperscript{@}}\text{ In the original manuscript, the expression } \nabla u_\lambda = \nabla \lambda u_\lambda = \lambda f_\lambda \text{ was written down, which is evidently incorrect.}\]
\[ \psi = \frac{1}{\sqrt{2}} \sum_\lambda \sqrt{mc} \left[ q_\lambda + q'_\lambda + i(p_\lambda - p'_\lambda) \right] u_\lambda, \]
\[ P = \frac{h}{4\pi \sqrt{2}} \sum_\lambda \sqrt{mc} \left[ p_\lambda + p'_\lambda + i(q_\lambda - q'_\lambda) \right] u_\lambda. \]

14\@ In the original manuscript the simple formulas \((a - ib)(a + ib) = a^2 + b^2 + i(ab - ba)\) and \((a + ib)(a - ib) = a^2 + b^2 - i(ab - ba)\) are noted on the side.

\[ a_\lambda = \frac{1}{\sqrt{2}} (q_\lambda + ip_\lambda), \quad b_\lambda = \frac{1}{\sqrt{2}} (q'_\lambda + ip'_\lambda), \]
\[ \bar{a}_\lambda = \frac{1}{\sqrt{2}} (q_\lambda - ip_\lambda), \quad \bar{b}_\lambda = \frac{1}{\sqrt{2}} (q'_\lambda - ip'_\lambda). \]

\[ [a_\lambda, a_\mu] - [b_\lambda, b_\mu] - [a_\lambda, b_\mu] - [b_\lambda, a_\mu] = 2\delta_{\lambda\mu}, \]
\[ -[\bar{a}_\lambda, a_\mu] + [b_\lambda, b_\mu] + [\bar{a}_\lambda, b_\mu] - [b_\lambda, a_\mu] = 2\delta_{\lambda\mu}. \]

\[ [x, y] = xy - yx, \]
where the upper/lower sign refers to Einstein/Fermi particles.

\[ [a_\lambda, a_\mu] + [\bar{b}_\lambda, \bar{b}_\mu] + [a_\lambda, \bar{b}_\mu] + [\bar{b}_\lambda, a_\mu] = 0, \]
\[ [\bar{a}_\lambda, \bar{a}_\mu] + [b_\lambda, b_\mu] + [\bar{a}_\lambda, b_\mu] + [b_\lambda, \bar{a}_\mu] = 0, \]
\[ [a_\lambda, \bar{a}_\mu] + [\bar{b}_\lambda, b_\mu] + [a_\lambda, b_\mu] + [\bar{b}_\lambda, \bar{a}_\mu] = 0, \]
\[ [\bar{a}_\lambda, a_\mu] + [\bar{b}_\lambda, b_\mu] - [a_\lambda, b_\mu] - [\bar{b}_\lambda, a_\mu] = 0, \]
\[ [a_\lambda, \bar{a}_\mu] + [\bar{b}_\lambda, \bar{b}_\mu] - [a_\lambda, \bar{b}_\mu] - [\bar{b}_\lambda, a_\mu] = 0, \]
\[ [\bar{a}_\lambda, a_\mu] - [\bar{b}_\lambda, \bar{b}_\mu] + [\bar{a}_\lambda, b_\mu] - [\bar{b}_\lambda, a_\mu] = 0, \]
\[ [a_\lambda, a_\mu] - [\bar{b}_\lambda, b_\mu] + [\bar{a}_\lambda, a_\mu] - [\bar{b}_\lambda, b_\mu] = 0, \]
\[ [\bar{a}_\lambda, \bar{a}_\mu] - [b_\lambda, b_\mu] + [\bar{a}_\lambda, \bar{a}_\mu] - [b_\lambda, b_\mu] = 0. \]

2.8. CONTINUATION

\[ \psi = \sum_\lambda \sqrt{mc} \frac{mc}{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}} (a_\lambda + \bar{b}_\lambda) u_\lambda, \]
\[ P = \frac{hi}{4\pi} \sum_\lambda \sqrt{mc} \frac{mc}{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}} (\bar{a}_\lambda - b_\lambda) u_\lambda, \]
\[ \bar{\psi} = \sum_\lambda \sqrt{\frac{mc}{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}}} (\bar{a}_\lambda + b_\lambda) u_\lambda, \]
\[ \mathcal{P} = -\frac{hi}{4\pi} \sum_\lambda \sqrt{\frac{m^2c^2 + h^2\lambda^2/4\pi^2}{mc}} (a_\lambda - \bar{b}_\lambda) u_\lambda. \]

From the commutation relations reported at the end of the previous Section, we deduce that:

\[ [a_\lambda, a_\mu] + [\bar{b}_\lambda, b_\mu] = 0, \]
\[ [a_\lambda, b_\mu] + [\bar{b}_\lambda, a_\mu] = 0, \]
\[ [a_\lambda, \bar{a}_\mu] + [\bar{b}_\lambda, b_\mu] = 0, \]
\[ [a_\lambda, \bar{b}_\mu] + [\bar{b}_\lambda, \bar{a}_\mu] = 0, \]
\[ [\bar{a}_\lambda, \bar{a}_\mu] + [b_\lambda, b_\mu] = 0, \]
\[ [\bar{a}_\lambda, b_\mu] + [b_\lambda, \bar{a}_\mu] = 0, \]
\[ [a_\lambda, a_\mu] + [\bar{b}_\lambda, b_\mu] = 0; \]
\[ [a_\lambda, \bar{a}_\mu] - [\bar{b}_\lambda, b_\mu] = 2\delta_{\lambda\mu}, \]
\[ [\bar{a}_\lambda, a_\mu] - [b_\lambda, \bar{b}_\mu] = -2\delta_{\lambda\mu}. \]

0 = [a + ib, a + ib] = [a, a] - [b, b] + i[a, b] + i[b, a],
0 = [a - ib, a - ib] = [a, a] - [b, b] - i[a, b] - i[b, a],
0 = [a + ib, a - ib] = [a, a] + [b, b] - i[a, b] + i[b, a],
0 = [a - ib, a + ib] = [a, a] + [b, b] + i[a, b] - i[b, a];
\[ [a, a] = [b, b] = [a, b] = [b, a] = 0. \]

### 2.9. QUANTIZED RADIATION FIELD

The author again considered the quantization of the electromagnetic field, but using now another expansion in a basis different from that adopted in Sects. 2.4, 2.5. In the original manuscript, the present Section and the following four Sections are placed in the Quaderno 17 just after what has been here reported in Sect. 7.1.

\[ E = -\frac{1}{c} \frac{\partial C}{\partial t}, \quad \frac{1}{c^2} \frac{\partial^2 C^2}{\partial t^2} = \nabla^2 C = -\frac{1}{c} \frac{\partial E}{\partial t}. \]
\[ C_x, \quad C_y, \quad C_z; \]
\[ -\frac{E_x}{4\pi c}, \quad -\frac{E_y}{4\pi c}, \quad -\frac{E_z}{4\pi c}. \]
\[ \gamma_1, \gamma_2, \gamma_3 = 0, \pm 1, \pm 2, \ldots; \]
\[ \gamma = \frac{c}{k} \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}; \]
\[ p_x = \frac{h}{k} \gamma_1, \quad p_y = \frac{h}{k} \gamma_2, \quad p_z = \frac{h}{k} \gamma_3. \]
\[ |k_s| = 1, \quad k_s = k_{-s}. \]

\[ f_s = k_s e^{2\pi i (\gamma^s_1 x/k + \gamma^s_2 y/k + \gamma^s_3 z/k)} \frac{1}{\sqrt{k^3}}. \]

\[ C = \sum a_s f_s, \]
\[ E = \sum b_s f_s. \]
\[ a_s = \tilde{a}_{-s}, \]
\[ b_s = \tilde{b}_{-s}. \]
\[ a_s a_{s'} - a_{s'} a_s = 0, \]
\[ b_s b_{s'} - b_{s'} b_s = 0, \]
\[ a_s \tilde{b}_{s'} - \tilde{b}_{s'} a_s = \frac{2hc}{i} \delta_{s,s'}. \]

\[ ^{15} @ \text{In the original manuscript, the normalization factor } 1/\sqrt{k^3} \text{ is incorrectly treated as a denominator instead of a numerator.} \]
\[ \dot{C} = -c \mathbf{E} = \sum -c b_s f_s; \quad \dot{E} = -c \nabla^2 C = \sum \frac{4\pi^2 \nu_s^2}{c} a_s f_s. \]

\[ \dot{a}_s = -c b_s, \]
\[ \dot{b}_s = \frac{4\pi^2 \nu_s^2}{c} a_s. \]

\[ \frac{d}{dt} \left( a_s + \frac{c}{2\pi \nu_s i} b_s \right) = -c b_s - 2\pi \nu_s i a_s = -2\pi \nu_s i \left( a_s + \frac{c}{2\pi \nu_s i} b_s \right), \]
\[ \frac{d}{dt} \left( a_s - \frac{c}{2\pi \nu_s i} b_s \right) = -c b_s + 2\pi \nu_s i a_s = 2\pi \nu_s i \left( a_s - \frac{c}{2\pi \nu_s i} b_s \right). \]

\[ A_s = a_s + \frac{c}{2\pi \nu_s i} b_s, \]
\[ B_s = a_s - \frac{c}{2\pi \nu_s i} b_s; \]
\[ \dot{A}_s = -2\pi \nu_s i A_s, \]
\[ \dot{B}_s = 2\pi \nu_s i B_s; \]
\[ \tilde{A}_s = B_{-s}, \]
\[ \tilde{B}_s = A_{-s}. \]

\[ A_s B_s - B_s A_s = 0, \]
\[ A_s \tilde{B}_s - \tilde{B}_s A_s = 0, \]
\[ A_s B_s - \tilde{B}_s A_s = 0, \]

\[ A_s \tilde{A}_s - \tilde{A}_s A_s = \frac{2\hbar c^2}{\pi \nu_s}, \]

\[ B_s \tilde{B}_s - \tilde{B}_s B_s = -\frac{2\hbar c^2}{\pi \nu_s}. \]
\[ A_s A_t - A_t A_s = 0, \]
\[ B_s B_t - B_t B_s = 0, \]
\[ \tilde{A}_s \tilde{A}_t - \tilde{A}_t \tilde{A}_s = 0, \]
\[ \tilde{B}_s \tilde{B}_t - \tilde{B}_t \tilde{B}_s = 0, \]
\[ A_s \tilde{B}_t - \tilde{B}_t A_s = 0, \]
\[ \tilde{A}_s B_t - B_t \tilde{A}_s = 0, \]
\[ A_s \tilde{A}_t - \tilde{A}_t A_s = \frac{2hc^2}{\pi \nu_s} \delta_{st}, \]
\[ B_s \tilde{B}_t - \tilde{B}_t B_s = -\frac{2hc^2}{\pi \nu_s} \delta_{st}. \]

\[ Z_s = \frac{1}{c} \sqrt{\frac{\pi \nu_s}{2\hbar}} A_s. \]

\[ Z_s Z_t - Z_t Z_s = 0, \]
\[ \tilde{Z}_s \tilde{Z}_t - \tilde{Z}_t \tilde{Z}_s = 0, \]
\[ Z_s \tilde{Z}_t - \tilde{Z}_t Z_s = \delta_{st}. \]

\[ \tilde{Z}_s Z_s = n_s. \]

\[ < n_s | Z_s | n_{s+1} > = \sqrt{n_s + 1}, \]
\[ < n_s | \tilde{Z}_s | n_{s-1} > = \sqrt{n_s}. \]

\[ A_s = c \sqrt{\frac{2\hbar}{\pi \nu_s}} Z_s, \]

[16] @ In the original manuscript, the unidentified Ref. 5.45 is here alluded to.
\[ a_s = \frac{A_s + \tilde{A}_{-s}}{2} = c \sqrt{\frac{2\hbar}{\pi \nu_s} \frac{Z_s + \tilde{Z}_{-s}}{2}}, \]
\[ b_s = \frac{2\pi \nu_s i}{c} \frac{A_s - \tilde{A}_s}{2} = i \sqrt{2\hbar \pi \nu_s} \left( Z_s - \tilde{Z}_{-s} \right). \]

\[ W_s = \frac{1}{8\pi} \sum \left( \tilde{b}_s b_s + \frac{4\pi^2 \nu_s^2}{c^2} \tilde{a}_s a_s \right) \]
\[ = \frac{1}{8\pi} \sum 2h\pi \nu_s \left[ (\tilde{Z}_s - Z_{-s}) (Z_s - \tilde{Z}_{-s}) + (\tilde{Z}_s + Z_{-s}) (Z_s + \tilde{Z}_{-s}) \right] \]
\[ = \frac{1}{4} \sum h\nu_s \{ 2\tilde{Z}_s Z_s + 2Z_{-s} \tilde{Z}_{-s} \} \]
\[ = \sum h\nu_s \frac{\tilde{Z}_s Z_s + Z_{-s} \tilde{Z}_{-s}}{2} = \sum \left( n_s + \frac{1}{2} \right) h\nu_s. \]

\[ f_s = \frac{1}{k^{3/2}} e^{2\pi i \left( \gamma_1 x/k + \gamma_2 y/k + \gamma_3 z/k \right)} k_s, \]
\[ f_{-s} = \frac{1}{k^{3/2}} e^{-2\pi i \left( \gamma_1 x/k + \gamma_2 y/k + \gamma_3 z/k \right)} k_s = \overline{f}_s. \]

\[ f_s = \frac{1}{k^{3/2}} e^{2\pi i \gamma_s \cdot r/k} k_s, \]
with \( r = (x, y, z). \)

\[ C = \sum_s \frac{c}{2} \sqrt{\frac{2\hbar}{\pi \nu_s}} \left( Z_s f_s + \tilde{Z}_s \overline{f}_s \right), \]
\[ E = \sum_s i \sqrt{2\hbar \pi \nu_s} \left( Z_s f_s - \tilde{Z}_s \overline{f}_s \right). \]
\[ E^2(r) = - \frac{2\hbar \pi}{k^3} \sum_{s,t} \sqrt{\nu_s \nu_t} \mathbf{k}_s \cdot \mathbf{k}_t \left\{ Z_s Z_t e^{2\pi i(\gamma_s + \gamma_t) \mathbf{r}/k} + \tilde{Z}_s \tilde{Z}_t e^{-2\pi i(\gamma_s + \gamma_t) \mathbf{r}/k} - Z_s \tilde{Z}_t e^{2\pi i(\gamma_s - \gamma_t) \mathbf{r}/k} - \tilde{Z}_s Z_t e^{2\pi i(-\gamma_s + \gamma_t) \mathbf{r}/k} \right\}. \]

\[ H^2(r) = - \frac{2\hbar \pi}{k^3} \sum_{s,t} \sqrt{\nu_s \nu_t} \mathbf{k}'_s \cdot \mathbf{k}'_t \left\{ Z_s Z_t e^{2\pi i(\gamma_s + \gamma_t) \mathbf{r}/k} + \tilde{Z}_s \tilde{Z}_t e^{-2\pi i(\gamma_s + \gamma_t) \mathbf{r}/k} - Z_s \tilde{Z}_t e^{2\pi i(\gamma_s - \gamma_t) \mathbf{r}/k} - \tilde{Z}_s Z_t e^{2\pi i(-\gamma_s + \gamma_t) \mathbf{r}/k} \right\}. \]

### 2.10. WAVE EQUATION OF LIGHT QUANTA

Quantized fields of the electromagnetic interaction were again considered in these pages, with an emphasis (the name of this Section is the original one) on the definition of a wavefunction \( \psi \) for the photon. Matrix elements of the annihilation and creation operators \( Z, \tilde{Z} \) were reported in the subsequent Section, along with quantum expressions for the photon energy and angular momentum.

\[ C = \sum a_s f_s, \quad E = \sum b_s f_s; \]
\[ a_s = c \sqrt{\frac{2\hbar}{\pi \nu_s}} \frac{Z_s + \overline{Z}_{-s}}{2}, \quad b_s = i \sqrt{2\hbar \pi \nu_s} (Z_s - \overline{Z}_{-s}). \]

\[ f_s = \frac{1}{k^{3/2}} e^{2\pi i \gamma_s \mathbf{r}/\hbar} \mathbf{k}_s, \]
\[ \overline{f}_s = f_{-s}. \]

\[ ^{17}C \sim (e^{2\pi i \gamma r/k}, 0, 0), \quad H \sim (0, 2\pi i (\gamma/k) e^{2\pi i \gamma r/k}, 0). \]

\[ ^{18}@ \text{The original manuscript alludes here to the unidentified Ref. 11.20.} \]
\[ \gamma^s = (\gamma_1^s, \gamma_2^s, \gamma_3^s), \]
\[ \gamma_1, \gamma_2, \gamma_3 = 0, \pm 1, \pm 2, \pm 3, \ldots; \]

\[ \nu_s = \frac{c}{k} \gamma^s, \quad h\nu_s = \frac{hc}{k} \gamma^s. \]

\[ \psi = \sum Z_s f_s. \]

\[ C = \sum_s c \sqrt{\frac{2h}{\pi \nu_s}} \frac{Z_s + \bar{Z} - s}{2} f_s = \sum_s c \sqrt{\frac{2h}{\pi \nu_s}} \frac{Z_s f_s + \bar{Z} - s f_s}{2}, \]
\[ E = \sum_s i \sqrt{2h\pi \nu_s} (Z_s - \bar{Z} - s) f_s = \sum_s i \sqrt{2h\pi \nu_s} (Z_s f_s - \bar{Z} - s f_s). \]

2.11. CONTINUATION

\[ \nabla \cdot C = 0. \]

\[ \frac{1}{c} \frac{\partial E}{\partial t} = \nabla \times \nabla \times C = -\nabla^2 C, \]
\[ \frac{1}{c} \frac{\partial H}{\partial t} - \nabla \times E = \frac{1}{c} \frac{\partial}{\partial t} \nabla \times C. \]

\[ C = \sum c \sqrt{\frac{h}{2\pi \nu_s}} (Z_s f_s + \tilde{Z} _s \bar{f}_s), \]
\[ \frac{\partial C}{\partial t} = \sum c \sqrt{\frac{h}{2\pi \nu_s}} (\dot{Z} _s f_s + \tilde{Z} _s \bar{f}_s), \]
\[ \nabla^2 C = \sum \frac{2\pi \nu_s}{c} \sqrt{2h\pi \nu_s} (Z_s f_s + \tilde{Z} _s \bar{f}_s); \]
\[ E = \sum i \sqrt{2h\pi \nu_s} (Z_s f_s - \tilde{Z} _s \bar{f}_s), \]
\[ \frac{\partial E}{\partial t} = \sum i \sqrt{2h\pi \nu_s} (\dot{Z} _s f_s - \tilde{Z} _s \bar{f}_s). \]
\[ i\sqrt{2\hbar\nu_s} (\dot{Z}_s - \dot{\tilde{Z}}_{-s}) - 2\pi\nu_s\sqrt{2\hbar\nu_s} (Z_s + \tilde{Z}_{-s}) = 0, \]
\[ \sqrt{\frac{\hbar}{2\pi\nu_s}} (\dot{Z}_s + \dot{\tilde{Z}}_{-s}) + i\sqrt{2\hbar\nu_s} (Z_s - \tilde{Z}_{-s}) = 0. \]
\[ \dot{Z}_s - \dot{\tilde{Z}}_{-s} = -2\pi i\nu_s (Z_s + \tilde{Z}_{-s}), \]
\[ \dot{Z}_s + \dot{\tilde{Z}}_{-s} = -2\pi i\nu_s (Z_s - \tilde{Z}_{-s}). \]

\[ \dot{\tilde{Z}}_s = -2\pi i\nu_s Z_s, \quad \dot{\tilde{Z}}_{-s} = 2\pi i\nu_s \tilde{Z}_s, \quad \dot{Z}_{-s} = 2\pi i\nu_s \tilde{Z}_{-s}. \]

\[ \int \frac{E^2}{8\pi} \, d\tau = \sum \frac{\hbar\nu_s}{4} (Z_s - \tilde{Z}_{-s})(\tilde{Z}_s - Z_{-s}) \]
\[ = \sum \frac{\hbar\nu_s}{4} (Z_s\tilde{Z}_s + \tilde{Z}_{-s}Z_{-s} - Z_sZ_{-s} - \tilde{Z}_{-s}\tilde{Z}_s) \]
\[ = \sum \frac{\hbar\nu_s}{2} \left( \frac{Z_s\tilde{Z}_s + \tilde{Z}_{-s}Z_{-s}}{2} - \frac{Z_sZ_{-s} + \tilde{Z}_{-s}\tilde{Z}_s}{2} \right). \]

\[ \int \frac{H^2}{8\pi} \, d\tau = \sum \frac{\hbar\nu_s}{4} (Z_s + \tilde{Z}_{-s})(\tilde{Z}_s + Z_{-s}) \]
\[ = \sum \frac{\hbar\nu_s}{4} (Z_s\tilde{Z}_s + \tilde{Z}_{-s}Z_{-s} + Z_sZ_{-s} + \tilde{Z}_{-s}\tilde{Z}_s) \]
\[ = \sum \frac{\hbar\nu_s}{2} \left( \frac{Z_s\tilde{Z}_s + \tilde{Z}_{-s}Z_{-s}}{2} + \frac{Z_sZ_{-s} + \tilde{Z}_{-s}\tilde{Z}_s}{2} \right). \]

\[ \int \frac{E^2 + H^2}{8\pi} \, d\tau = \sum \hbar\nu_s \frac{Z_s\tilde{Z}_s + \tilde{Z}_{-s}Z_{-s}}{2}. \]

\[ e^{iLx}(0,0,1) = f_s, \]
\[ iL e^{iLx}(0,-1,0) = \nabla \times f_s \]

\[ f_{-s} \times \nabla \times f_s = iL(1,0,0). \]
Let us denote with $r_s$ a unitary vector along the propagation direction:

$$\int \frac{E \times H}{4\pi c} \, d\tau = \sum -\frac{\hbar \nu_s}{2c} (Z_s - \tilde{Z}_s)(Z_s + \tilde{Z}_s)r_s$$

$$= \sum \frac{\hbar \nu_s}{2c} r_s (\tilde{Z}_sZ_s - Z_s\tilde{Z}_s - Z_sZ_s - \tilde{Z}_s\tilde{Z}_s)$$

$$= \sum \frac{\hbar \nu_s}{c} r_s \frac{\tilde{Z}_sZ_s + Z_s\tilde{Z}_s}{2}.$$ 

$$Z_s\tilde{Z}_s - \tilde{Z}_sZ_s = 1.$$ 

$\tilde{Z}_sZ_s = X.$

$$Z_sX - XZ_s = (Z_s, X) = Z_s, \quad Z_{ik}(X_k - X_i) = 1,$$

$$\tilde{Z}_sX - X\tilde{Z}_s = (\tilde{Z}_s, X) = -\tilde{Z}_s, \quad \tilde{Z}_{ik}(X_k - X_i) = -1.$$ 

$$< X|Z|X + 1 > = f(X),$$

$$< X + 1|\tilde{Z}|X > = \bar{f}(X).$$

$$< X|\tilde{Z}Z|X > = < X|\tilde{Z}|X - 1 > < X - 1|Z|X > = |f(X - 1)|^2,$$

$$< X|Z\tilde{Z}|X > = < X|Z|X + 1 > < X + 1|\tilde{Z}|X > = |f(X)|^2;$$

$$|f(X)|^2 = X + 1,$$

$$|f(X_0)|^2 = 1, \quad X_0 = 0.$$ 

$$|f(X)|^2 = |f(X - 1)|^2 + 1,$$

$$|f(X_0)|^2 = 1.$$ 

$$< X_0|\tilde{Z}Z|X_0 > = 0,$$

$$< X_0|Z\tilde{Z}|X_0 > = |f(0)|^2.$$ 

$$\tilde{Z}_sZ_s = n_s, \quad (n_s = 0, 1, 2\ldots)$$

$$< n_s|Z_s|n_s + 1 > = \sqrt{n_s + 1},$$

$$< n_s + 1|\tilde{Z}_s|n_s > = \sqrt{n_s}.$$
\( \bar{f}_s = f_{-s} \).

\[
\int \frac{E^2 + H^2}{8\pi} \, d\tau = \sum \hbar \nu_s \left( n_s + \frac{1}{2} \right),
\]

\[
\int \frac{E \times H}{4\pi c} \, d\tau = \sum \frac{\hbar \nu_s}{c} \, r_s \left( n_s + \frac{1}{2} \right).
\]

### 2.12. FREE ELECTRON SCATTERING

The interaction between electrons and electromagnetic radiation was here studied in detail, and expressions for the matrix elements of the interaction energy (as well as for the transition probability) were explicitly obtained. Some care was also devoted to the kinematics of the process here considered. The material reported in this Section starts with that present in Quaderno 17 on the page following 151bis, but the complete study of the subject starts at page 133 of the same Quaderno.

\[
\left[ \frac{W}{c} + \rho_1 \, \sigma \cdot \left( p + \frac{e}{c} C \right) + \rho_3 \, mc \right] \psi = 0.
\]

Using Dirac coordinates:

\[
\psi_r = u_r \frac{1}{\sqrt{k^3}} \, e^{2\pi i (\Gamma^r_{1z}x/k + \Gamma^r_{2y}y/k + \Gamma^r_{3z}z/k)},
\]

\[
\psi_r = (u_{1r}, u_{2r}, u_{3r}, u_{r}), \quad \bar{u}_u u_r = 1, \quad \Gamma = \sqrt{\Gamma^r_1 + \Gamma^r_2 + \Gamma^r_3}.
\]

\[
E_r = \pm c \sqrt{m^2 c^2 + \frac{\hbar^2}{k^2} \Gamma^2}.
\]

\[
H = H_0 + \mathcal{I},
\]

\[
H_0 = -c \, \rho_1 \, \sigma \cdot p - \rho_3 \, mc^2 + \sum_s n_s \hbar \nu_s,
\]

\[
\mathcal{I} = -c \, \rho_1 \, \sigma \cdot \frac{e}{c} C = -e \, \rho_1 \, \sigma \cdot C.
\]
\[
<\ldots|H_0|\ldots> = E_r + \sum n_s h \nu_s,
\]
\[
<r; n_s \ldots |I| r'; n_s + 1 \ldots> = -\sqrt{n_s + 1} \frac{ec}{2} \frac{2h}{\pi \nu_s} \times \int \tilde{\psi}_r \rho_1 \sigma \cdot f_s \psi_{r'} \, d\tau,
\]
\[
<r; n_s \ldots |I| r'; n_s - 1 \ldots> = -\sqrt{n_s} \frac{ec}{2} \frac{2h}{\pi \nu_s} \times \int \tilde{\psi}_r \rho_1 \sigma \cdot f_{-s} \psi_{r'} \, d\tau.
\]
\[
\psi_r = u_r \frac{1}{k^{3/2}} e^{2\pi i \Gamma^r \cdot r/k},
\]
\[
\psi_{r'} = u_{r'} \frac{1}{k^{3/2}} e^{2\pi i \Gamma^{r'} \cdot r/k},
\]
\[
f_s = k_s \frac{1}{k^{3/2}} e^{2\pi i \gamma^s \cdot r/k},
\]
\[
f_{-s} = f_s \frac{1}{k^{3/2}} e^{-2\pi i \gamma^s \cdot r/k}.
\]
\[
k_s = k_{-s}.
\]
\[
\int \tilde{\psi}_r \rho_1 \sigma \cdot f_s \psi_{r'} \, d\tau = \frac{k^{-7/2}}{k^{7/2}} \tilde{u}_r \rho_1 \sigma \cdot k_s u_{r'} \delta \Gamma^r, \Gamma^{r'} + \Gamma^s,
\]
\[
\int \tilde{\psi}_r \rho_1 \sigma \cdot f_{-s} \psi_{r'} \, d\tau = \frac{\tilde{u}_r \rho_1 \sigma \cdot k_s u_{r'}}{k^{7/2}} \delta \Gamma^r, \Gamma^{r'} - \Gamma^s.
\]
\[
<r; n_s \ldots |I| r'; n_s + 1 \ldots> = -\frac{ec}{2k^{3/2}} \sqrt{n_s + 1} \frac{2h}{\pi \nu_s} \times \tilde{u}_r \rho_1 \sigma \cdot k_s u_{r'} \delta \Gamma^r, \Gamma^{r'} + \Gamma^s,
\]
\[
<r; n_s \ldots |I| r'; n_s - 1 \ldots> = -\frac{ec}{2k^{3/2}} \sqrt{n_s} \frac{2h}{\pi \nu_s} \times \tilde{u}_r \rho_1 \sigma \cdot k_s u_{r'} \delta \Gamma^r, \Gamma^{r'} - \Gamma^s.
\]
For $t = 0$: $a_1 = 1$, $a_2, \ldots = 0$.

For $t \to 0$:
\[
\dot{a}_i = -\frac{2\pi i}{\hbar} e^{2\pi i (E_i - E_1) t/\hbar} H_{i1};
\]
\[
a_i = -\frac{1}{E_i - E_1} \left( e^{2\pi i (E_i - E_1) t/\hbar} - 1 \right) H_{i1}.
\]

$H_{12} = 0$.

\[
\dot{a}_2 = -\frac{2\pi i}{\hbar} \sum_i \frac{-1}{E_i - E_1} e^{2\pi i (E_2 - E_i) t/\hbar} \left( e^{2\pi i (E_i - E_1) t/\hbar} - 1 \right) H_{2i} H_{i1}
= \frac{2\pi i}{\hbar} \sum_i \frac{1}{E_i - E_1} \left( e^{2\pi i (E_2 - E_1) t/\hbar} - e^{2\pi i (E_2 - E_i) t/\hbar} \right) H_{2i} H_{i1};
\]
\[
a_2 = \sum_i \left[ \frac{1}{(E_i - E_1)(E_2 - E_1)} \left( e^{2\pi i (E_2 - E_1) t/\hbar} - 1 \right) - \frac{1}{(E_2 - E_i)(E_i - E_1)} e^{2\pi i (E_2 - E_i) t} \right] H_{2i} H_{i1}.
\]

\[
electron \ radiation
\]

2 b $n_t = 1$
$i, i' r, r'$ \quad n_t = 1, n_s = 1$
1 a $n_s = 1$

\[
\Gamma^a + \gamma^s = \Gamma^b + \gamma^t = \Gamma^r = \Gamma^{r'} + \gamma^s + \gamma^t
\]

$s, t$ label the incident and the scattered quanta, respectively.
\begin{align*}
\langle b; 0, 1 \ldots |I|r; 0, 0 \rangle &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi \nu_t}} \tilde{u}_b \rho_1 \sigma \cdot k_t u_r, \\
\langle r'; 0, 0 \ldots |I|a; 1, 0 \rangle &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi \nu_s}} \tilde{u}_{r'} \rho_1 \sigma \cdot k_s u_a, \\
\langle b; 0, 1 \ldots |I|r; 1, 1 \rangle &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi \nu_s}} \tilde{u}_b \rho_1 \sigma \cdot k_s u_r, \\
\langle r'; 1, 1 \ldots |I|a; 1, 0 \rangle &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi \nu_t}} \tilde{u}_{r'} \rho_1 \sigma \cdot k_t u_a.
\end{align*}

The probability for a transition at a time $t$ to occur is (taking into account only the term with the resonance denominator equal to $E_1 - E_2$ in the expression for $a_2$):

\begin{equation}
P_{12} = \frac{\sin^2[\pi(E_2 - E_1)t/h]}{(E_2 - E_1)^2} \cdot \frac{4}{4} \sum_i \frac{H_{2i}H_{i1}}{|E_i - E_1|^2}.
\end{equation}

\begin{align*}
p_a &= \frac{h}{k} \Gamma^a, \quad p_r = \frac{h}{k} (\Gamma^a + \gamma^s), \\
p_b &= \frac{h}{k} \Gamma^b, \quad p_{r'} = \frac{h}{k} (\Gamma^b - \gamma^t).
\end{align*}

\begin{align*}
\Gamma &= \Gamma^a + \gamma^s = \Gamma^b + \gamma^t, \\
\Gamma^b &= \Gamma^a + \gamma^s - \gamma^t.
\end{align*}

\begin{align*}
E_a &= c \sqrt{m^2 c^2 + h^2 k^2 \Gamma^a}, \\
E_b &= c \sqrt{m^2 c^2 + h^2 k^2 \Gamma^b}, \\
E_r &= \pm c \sqrt{m^2 c^2 + h^2 k^2 (\Gamma^a + \gamma^s)^2}, \\
E_{r'} &= \pm c \sqrt{m^2 c^2 + h^2 k^2 (\Gamma^b - \gamma^t)^2}.
\end{align*}
\[ E_1 = c \sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a + \gamma_s)^2 + h\nu_s}, \]
\[ E_2 = c \sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a + \gamma_s - \gamma_t)^2 + h\nu_t}, \]
\[ E_i = \pm c \sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a + \gamma_s)^2}, \]
\[ E_i' = \pm c \sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a - \gamma_t)^2 + h\nu_s + h\nu_t}. \]

Let us denote by \( u \) the spin function for a plane wave with momentum \( p_x, p_y, p_z \) and by \( u^0 \) that for a wave of zero momentum.

\[
\begin{align*}
u^p &= \left[ f_1 \mp f_2 \frac{\alpha \cdot p}{p} \right] u^0,
\end{align*}
\]

where the upper/lower sign refers to positive/negative energy waves.

\[
\begin{align*}
f_1 &= \sqrt{\frac{1 + \sqrt{1 + p^2/m^2c^2}}{2\sqrt{1 + p^2/m^2c^2}}}, \quad f_2 = \sqrt{\frac{-1 + \sqrt{1 + p^2/m^2c^2}}{2\sqrt{1 + p^2/m^2c^2}}};
\end{align*}
\]

\[|f_1^2| + |f_2^2| = 1.\]

\[\alpha = \rho_1 \sigma.\]

\[
\begin{align*}
u_b &= \left[ f_1^b - f_2^b \frac{\alpha \cdot p_b}{p_b} \right] u_b^0, \quad \nu_r \left[ f_1^r + f_2^r \frac{\alpha \cdot p_r}{p_r} \right] u_r^0,
\end{align*}
\]

\[
\begin{align*}
u_a &= \left[ f_1^a - f_2^a \frac{\alpha \cdot p_a}{p_a} \right] u_a^0, \quad \nu_{r'} \left[ f_1^{r'} + f_2^{r'} \frac{\alpha \cdot p_{r'}}{p_{r'}} \right] u_{r'}^0.
\end{align*}
\]

We consider positive waves \( u_a, u_b. \)
1) Positive $u_r$:

\[
\tilde{u}_b \alpha \cdot k_t \ u_r \ 	ilde{u}_r \ \alpha \cdot k_s \ u_a \\
= \tilde{u}_b^0 \left( f_1^b - f_2^b \frac{\alpha \cdot p_b}{p_b} \right) \alpha \cdot k_t \left( f_1^r - f_2^r \frac{\alpha \cdot p_r}{p_r} \right) u_r^0 \\
\times \tilde{u}_r^0 \left( f_1^r f_1^a \alpha \cdot k_t f_2^a \frac{\alpha \cdot p_a}{p_a} + f_2^r \frac{\alpha \cdot p_r}{p_r} \alpha \cdot k_s f_1^a \right) u_a^0.
\]

2) Negative $u_r$:

\[
\tilde{u}_b \alpha \cdot k_t \ u_r \ 	ilde{u}_r \ \alpha \cdot k_s \ u_a \\
= \tilde{u}_b^0 \left[ f_1^b \alpha \cdot k_t f_2^r \frac{\alpha \cdot p_r}{p_r} + f_2^b \frac{\alpha \cdot p_b}{p_b} \alpha \cdot k_t f_1^r \right] u_r^0 \\
\times \tilde{u}_r^0 \left[ f_1^r f_1^a \alpha \cdot k_s f_2^a \frac{\alpha \cdot p_a}{p_a} + f_2^r \frac{\alpha \cdot p_r}{p_r} \alpha \cdot k_s f_1^a \right] u_a^0.
\]

3) Positive $u_r'$:

\[
\tilde{u}_b \alpha \cdot k_s \ u_r' \ 	ilde{u}_r' \ \alpha \cdot k_t \ u_a = \ldots
\]

[which is obtained from 1) with the replacements $r \rightarrow r'$, $k_s \rightarrow k_t$, $k_t \rightarrow k_s$.]

4) Negative $u_r'$:

\[
\tilde{u}_b \alpha \cdot k_s \ u_r' \ 	ilde{u}_r' \ \alpha \cdot k_t \ u_a = \ldots
\]

[which is obtained from 1) with the replacements $r \rightarrow r'$, $k_s \rightarrow k_t$, $k_t \rightarrow k_s$].
1) 
\[ \sum_{\text{positive } u_r} \tilde{u}_b \alpha \cdot k_l \ u_r \tilde{u}_r \alpha \cdot k_s \ u_a \]
\[ = \tilde{u}_b^0 \left[ f_1^b f_2^r \alpha \cdot k_l \frac{\alpha \cdot p_r}{p_r} + f_1^r f_2^b \frac{\alpha \cdot p_b}{p_b} \alpha \cdot k_l \right] \times \left[ f_1^r f_2^a \alpha \cdot k_s \frac{\alpha \cdot p_a}{p_a} + f_2^r f_1^a \frac{\alpha \cdot p_r}{p_r} \alpha \cdot k_s \right] u_a^0 \]
\[ = \tilde{u}_b^0 \left[ f_1^b f_2^r \sigma \cdot k_l \frac{\sigma \cdot p_r}{p_r} + f_1^r f_2^b \frac{\sigma \cdot p_b}{p_b} \sigma \cdot k_l \right] \times \left[ f_1^r f_2^a \sigma \cdot k_s \frac{\sigma \cdot p_a}{p_a} + f_2^r f_1^a \frac{\sigma \cdot p_r}{p_r} \sigma \cdot k_s \right] u_a^0. \]

\[ (\sigma \cdot k_l)(\sigma \cdot p_r) = k_l \cdot p_r + i \sigma \cdot k_l \times p_r, \]
\[ (\sigma \cdot k_l)(\sigma \cdot p_r)(\sigma \cdot k_s)(\sigma \cdot p_r) \]
\[ = (k_l \cdot p_r)(k_s \cdot p_a) + i(k_l \cdot p_r)(\sigma \cdot k_s \times p_a) \]
\[ + i(k_s \cdot p_a)(\sigma \cdot k_l \times p_r) - (\sigma \cdot k_l \times p_r)(\sigma \cdot k_s \times p_a) \]
\[ = (k_l \cdot p_r)(k_s \cdot p_a) + i(k_l \cdot p_r)(\sigma \cdot k_s \times p_a) \]
\[ + i(k_s \cdot p_a)(\sigma \cdot k_l \times p_r) - (k_l \cdot p_r)(k_s \times p_a). \]

For \( u_a = u_a^0, \ p_a = 0 \): \( f_1^a = 1, \ f_2^a = 0 \).

1) 
\[ \sum_{\text{positive } u_r} \tilde{u}_b \alpha \cdot k_l \ u_r \tilde{u}_r \alpha \cdot k_s \ u_a \]
\[ = \tilde{u}_b^0 \left[ f_1^b f_2^r \sigma \cdot k_l \frac{\sigma \cdot p_r}{p_r} + f_2^b f_1^r \frac{\sigma \cdot p_b}{p_b} \sigma \cdot k_l \right] f_2^r \frac{\sigma \cdot p_r}{p_r} \sigma \cdot k_s \ u_a^0. \]

For \( k_s \cdot p_r = 0 \):
\[ (\sigma \cdot k_l)(\sigma \cdot p_r)(\sigma \cdot p_r)(\sigma \cdot k_s) = p_r^2(\sigma \cdot k_l)(\sigma \cdot k_s) \]
\[ = p_r^2(k_l \cdot k_s) + ip_r^2(\sigma \cdot k_l \times k_s), \]
\[ (\sigma \cdot p_b)(\sigma \cdot k_l)(\sigma \cdot p_r)(\sigma \cdot k_s) = (p_b \cdot k_l + i\sigma \cdot p_b \times k_l) i\sigma \cdot p_r \times k_s \]
\[ = -(p_b \cdot k_l) \cdot (p_r \times k_s) + i(p_b \cdot k_l)(\sigma \cdot p_r \times k_s) \]
\[ = -i\sigma \cdot (p_b \times k_l) \times (p_r \times k_s). \]
2) \[
\sum_{\text{negative } u_r} \tilde{u}_b \alpha \cdot k_t \ u_r \tilde{u}_r \alpha \cdot k_s \ u_a
= \tilde{u}_b^0 \left[ f_{11}^b f_{11}' \sigma \cdot k_t - f_{21}^b f_{21}' \sigma \cdot p_b \sigma \cdot k_s \frac{\sigma \cdot p_r}{p_r} \right] f_1^r \sigma \cdot k_t u_a^0.
\]

3) \[
\sum_{\text{positive } u_r'} \tilde{u}_b \alpha \cdot k_s \ u_{r'} \tilde{u}_{r'} \alpha \cdot k_t \ u_a
= \tilde{u}_b^0 \left[ f_{12}^b f_{12}' \sigma \cdot k_s - f_{22}^b f_{22}' \sigma \cdot p_b \sigma \cdot k_s \frac{\sigma \cdot p_r}{p_r} \right] f_2^{r'} \sigma \cdot k_t u_a^0.
\]

4) \[
\sum_{\text{negative } u_{r'}} \tilde{u}_b \alpha \cdot k_s \ u_{r'} \tilde{u}_{r'} \alpha \cdot k_t \ u_a
= \tilde{u}_b^0 \left[ f_{12}^b f_{12}' \sigma \cdot k_s - f_{22}^b f_{22}' \sigma \cdot p_b \sigma \cdot k_s \frac{\sigma \cdot p_r}{p_r} \right] f_1^{r'} \sigma \cdot k_t u_a^0.
\]

Let us denote with \( \eta \) the average value with respect to \( u_b^0 \) and \( u_a^0 \):
\[
|\tilde{u}_b^0 A u_a^0|^2 = \tilde{u}_b^0 A u_a^0 \tilde{u}_b^0 A u_a^0 = \frac{1}{2} u_b^0 A \tilde{A} u_a^0 = \frac{1}{4} [(A \tilde{A})_{11} + (A \tilde{A})_{22}].
\]

\[
A = A_0 + i \sigma \cdot B,
\]
\[
A \tilde{A} = [A_0 + i \sigma \cdot B][\tilde{A}_0 - i \sigma \cdot \tilde{B}]
= A_0 \tilde{A}_0 + i A_0 \sigma \cdot B - i A_0 \sigma \cdot \tilde{B} + B \cdot \tilde{B} + i \sigma \cdot B \times \tilde{B}.
\]

\[
\gamma_s = (\gamma_s, 0, 0), \ k_s = (0, 0, 1), \ \gamma_t = (\gamma_t \sin \vartheta \cos \varphi, \gamma_t \sin \vartheta \sin \varphi, \gamma_t \cos \vartheta).
\]

Near the resonance we have:
\[
\nu_t = \frac{\nu_s}{1 + \frac{h \nu_s}{mc^2} (1 - \sin \vartheta \cos \varphi)},
\]
\[
p_r = \frac{h \nu_s}{c} (1, 0, 0), \ p_{r'} = -\frac{h \nu_t}{c} (\sin \vartheta \cos \varphi, \sin \vartheta \varphi, \cos \vartheta),
\]
\[ p_b = \frac{h\nu_t}{c} \left( 1 + \frac{h\nu_s}{mc^2} \right) (1 - \sin \vartheta \cos \varphi, - \sin \vartheta \sin \varphi, - \cos \vartheta) \].

\[
\begin{align*}
E_1 &= mc^2 + h\nu_s, \\
E'_1 &= \pm \sqrt{m^2c^4 + h^2\nu_t^2} + h\nu_s + h\nu_t, \\
E_i &= \pm \sqrt{m^2c^4 + h^2\nu_s^2}, \\
E_r &\sim E_1.
\end{align*}
\]

2.13. **BOUND ELECTRON SCATTERING**

Let us consider \( f \) bound electrons; the unperturbed energy of the system interacting with an electromagnetic field is \( E_n + \sum_s n_s h\nu_s \). Denoting with \( \psi_a(q_1, \ldots, q_f) \) the electron wavefunction corresponding to energy \( E_a \), the interaction with the electromagnetic field is described by:

\[
\begin{align*}
<a; n_s \ldots | I | b; n_s + 1 \ldots> &= -ec \sqrt{\frac{h(n_s + 1)}{2\pi \nu_s}} \\
&\quad \times \int \bar{\psi}_a \sum_{i=1}^{f} \alpha^i \cdot f_s(q_1) \psi_f \ d\tau, \\
<a; n_s \ldots | I | b; n_s - 1 \ldots> &= -ec \sqrt{\frac{hn_s}{2\pi \nu_s}} \\
&\quad \times \int \bar{\psi}_a \sum_{i=1}^{f} \alpha^i \cdot f_s(q_1) \psi_f \ d\tau.
\end{align*}
\]

\[ \alpha^i = \rho_1^i \sigma^i. \]

In first approximation, \( \lambda \gg |q_i| \):

\[ f_s(q_i) \sim f_s(0) = \frac{k_s}{k_{3/2}}. \]

For coherent scattering, by labelling with \( S, t \) the incident and scattered quantum, respectively, with wave-vectors \( k_s, k_t \), we have:
for resonant scattering, or otherwise

For $t = 0$: $a_1 = 1, a_2 = 0, n_i = 0; H_{12} = 0, H_{1i}, H_{2i} \neq 0$.

For $t \sim 0$:

$$\dot{a}_i = -\frac{2\pi i}{h} H_{i1} e^{2\pi i(E_i - E_1)t/h} - \frac{1}{2T} a_i.$$ 

$$a_i = -\frac{e^{-t/2T}}{E_i - E_1 + (h/4\pi i T)} \left( e^{2\pi i(E_i - E_1)t/h + t/2T} - 1 \right) H_{i1}.$$ 

$t \gg T$: 

$$a_i = \frac{-H_{i1}}{E_i - E_1 + (h/4\pi i T)} e^{2\pi i(E_i - E_1)t/h}.$$ 

$$\dot{a}_2 = \frac{2\pi i}{h} \sum_i \frac{H_{2i} H_{i1}}{E_i - E_1 + (h/4\pi i T)} e^{2\pi i(E_i - E_1)t/h}.$$ 

$$a_2 = \left( \sum_i \frac{H_{2i} H_{i1}}{E_i - E_1 + (h/4\pi i T)} e^{2\pi i(E_2 - E_1)t/h} - 1 \right) \frac{H_{21}}{E_2 - E_1}.$$ 


When a variable magnetic field $H = H(t)$ is included in the interaction, we have to consider also the diagonal magnetic moments $\mu_i$. For $H_x = H_y = 0, H_z = H(t)$:

$$\dot{a}_1 = \frac{2\pi i}{\hbar} H(t) \mu_1 a_1,$$

$$a_1 = e^{\left(\frac{2\pi i}{\hbar}\right) \int H dt}.$$

$$\dot{a}_i = -\frac{2\pi i}{\hbar} H_{i1} e^{2\pi i(E_i - E_1)t/\hbar} e^{\left(\frac{2\pi i}{\hbar}\right) \mu_1 \int H dt} - \frac{1}{2T} a_i + \frac{2\pi i}{\hbar} H \mu_i a_i.$$

$$a_i = e^{-t/2T} e^{\left(\frac{2\pi i}{\hbar}\right) \mu_i \int H dt} \left( -\frac{2\pi i}{\hbar} H_{i1} \right) \times \left[ \int e^{2\pi i(E_i - E_1)t/\hbar + t/2T + (2\pi i)/h (\mu_1 - \mu_i) \int H dt} dt + C \right].$$

$$\dot{a}_2 = -\frac{2\pi i}{\hbar} \sum_i H_{2i} e^{2\pi i (E_2 - E_1)t/\hbar} a_i + \frac{2\pi i}{\hbar} H \mu_2 a_2.$$

$$a_2 = \left( -\frac{2\pi i}{\hbar} \right) e^{\left(\frac{2\pi i}{\hbar}\right) \mu_2 \int H dt} \times \left[ \sum_i H_{2i} \int_0^t e^{2\pi i(E_2 - E_1)t/\hbar} - (2\pi i)/h \mu_2 \int H dt \right].$$

$$H = H_0 \cos 2\pi \nu t,$$

$$\int H dt = \frac{H_0}{2\pi \nu} \sin 2\pi \nu t,$$

$$\frac{2\pi}{\hbar} (\mu_1 - \mu_i) \int H dt = \frac{H_0 (\mu_1 - \mu_i)}{h \nu} \sin 2\pi \nu t,$$
\[ (2\pi i/\hbar)(\mu_1 - \mu_i) \int H dt = e^{i[H_0(\mu_1 - \mu_i)/\hbar]} \sin 2\pi \nu t \]
\[ = e^{iA_i \sin 2\pi \nu t}, \]

\[ A_i = \frac{H_0(\mu_1 - \mu_i)}{\hbar \nu}. \]

\[ e^{iA_i \sin 2\pi \nu t} = c_0 + c_1 e^{2\pi \nu it} + c_{-1} e^{-2\pi \nu it} + c_2 e^{4\pi \nu it} + c_{-2} e^{-4\pi \nu it} + \ldots. \]

\[ \omega = 2\pi \nu t: \]
\[ e^{iA_i \sin \omega} = c_0 + c_1 e^{i\omega} + c_{-1} e^{-i\omega} + c_2 e^{2i\omega} + c_{-2} e^{-2i\omega} + \ldots. \]

\[ c_0^i = \frac{1}{2\pi} \int_0^{2\pi} e^{iA_i \sin \omega} d\omega. \]

\[ \zeta = e^{i\omega}, \quad \sin \omega = \frac{\zeta - \zeta^{-1}}{2i}, \quad d\zeta = i\zeta d\omega, \quad d\omega = -\frac{i}{\zeta} d\zeta; \]
\[ e^{iA_i \sin \omega} d\omega = \frac{1}{i\zeta} e^{A_i(\zeta - \zeta^{-1})/2} d\zeta. \]
\[ c_0^i = \frac{1}{2\pi i} \oint \frac{1}{\zeta} e^{A_i(\zeta - \zeta^{-1})/2} d\zeta. \]

\[ e^{A_i(\zeta - \zeta^{-1})/2} = 1 + A_i \left( \frac{\zeta - \zeta^{-1}}{2} \right) + \frac{A_i^2}{2!} \left( \frac{\zeta - \zeta^{-1}}{2} \right)^2 + \frac{A_i^3}{3!} \left( \frac{\zeta - \zeta^{-1}}{2} \right)^3 + \ldots. \]

\[ (\zeta - \zeta^{-1})^n = \sum_{r=0}^{n} \zeta^{n-2r} \binom{n}{r} (-1)^r, \]

\[ \text{[20] The original manuscript alludes here to the unidentified Ref. 11.05.} \]
\[(\zeta - \zeta^{-1})^{2n} = \sum_{r=0}^{2n} (-1)^r \zeta^{2n-2r} \binom{2n}{r} \]
\[= \sum_{s=-n}^{n} (-1)^n \binom{2n}{n+s} \zeta^{-2s} (-1)^n \binom{2n}{n} \]
\[= \frac{(2n)!}{n!^2} (-1)^n. \]

\[c_i^0 = 1 - A_i^2 + \frac{A_i^4}{2!^2 \cdot 2^4} - \ldots = I_0(A_i). \]

### 2.14. RETARDED FIELDS

The possibility is considered, in the following pages, of introducing an intrinsic constant time delay \(\tau\) (or an intrinsic space constant \(\varepsilon = c\tau\)) in the expressions for the electromagnetic retarded fields, generically denoted with \(f(x, y, z, t)\).

\[f = f(x, y, z, t). \]

\[\phi(x, y, z, t) = f(x, y, z, t - \frac{r}{c}) = \tilde{f}(x, y, z, t). \]

\[\phi'_x(x, y, z, t) = f'_x(x, y, z, t - \frac{r}{c}) - \frac{x}{rc} f'_t(x, y, z, t - \frac{r}{c}) \]
\[= \tilde{f}'_x(x, y, z, t) - \frac{x}{rc} \tilde{f}'_t(x, y, z, t), \]

\[\phi''_x(x, y, z, t) = f''_{xx}(x, y, z, t - \frac{r}{c}) - \frac{2x}{rc} f''_{xt}(x, y, z, t - \frac{r}{c}) \]
\[+ \frac{x^2}{r^2 c^2} f''_{tt}(x, y, z, t - \frac{r}{c}) - \frac{r^2 - x^2}{r^3 c} f'_t(x, y, z, t - \frac{r}{c}) \]
\[= \tilde{f}''_{xx}(x, y, z, t) - \frac{2x}{rc} \tilde{f}''_{xt}(x, y, z, t) + \frac{x^2}{r^2 c^2} \tilde{f}''_{tt}(x, y, z, t) \]
\[ - \frac{r^2 - x^2}{r^3 c} \tilde{f}'_t(x, y, z, t). \]

\[\phi'_t(x, y, z, t) = f'_t(x, y, z, t - \frac{r}{c}) = \tilde{f}'_t(x, y, z, t), \]

\[\phi''_{tt}(x, y, z, t) = f''_{tt}(x, y, z, t - \frac{r}{c}) = \tilde{f}''_{tt}(x, y, z, t). \]
\[ \Box = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}; \]

\[ \Box \varphi(x, y, z, t) = \nabla^2 f \left( x, y, z, t - \frac{r}{c} \right) - \frac{2}{c} \frac{\partial}{\partial r \partial t} f_t(x, y, z, t) - \frac{2}{c} \frac{r}{c} f_t'(x, y, z, t), \]

\[ \frac{\partial}{\partial r} \varphi(x, y, z, t) = \sum_x \frac{x}{r} f_x' \left( x, y, z, t - \frac{r}{c} \right) - \frac{1}{c} f_t' \left( x, y, z, t - \frac{r}{c} \right), \]

\[ \frac{\partial^2}{\partial r \partial t} \varphi(x, y, z, t) = \frac{\partial^2}{\partial r \partial t} f \left( x, y, z, t \right) - \frac{1}{c} f_t'' \left( x, y, z, t \right). \]

\[ \Box \varphi + \frac{2}{c} \frac{\partial^2}{\partial z \partial t} \varphi = \nabla^2 f - \frac{2}{c^2} f_t'' - \frac{2}{rc} f_t', \]

\[ \Box f = \nabla^2 \varphi + \frac{2}{rc} \varphi' + \frac{2}{c} \frac{\partial^2}{\partial z \partial t} \varphi. \]

\[ \varphi(x, y, z, t) = f \left( x, y, z, t - \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) = \tilde{f}(x, y, z, t). \]

\[ f(x, y, z, t) = \varphi \left( x, y, z, t - \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right), \]

\[ f'_x(x, y, z, t) = \varphi'_x \left( x, y, z, t - \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) + \frac{x}{c \sqrt{r^2 + \varepsilon^2}} \varphi'_t \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c^2} \right), \]
\[ f''_{x x}(x, y, z, t) = \varphi''_{x x} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) \]
\[ + \frac{2x}{c\sqrt{r^2 + \varepsilon^2}} \varphi''_{x t} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c^2} \right) \]
\[ + \frac{r^2 + \varepsilon^2 - x^2}{c(r^2 + \varepsilon^2)^{3/2}} \varphi''_{t} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) \]
\[ + \frac{x^2}{c^2(r^2 + \varepsilon^2)} \varphi''_{t t} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) , \]

\[ f''_{t t}(x, y, z, t) = \varphi''_{t t} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right). \]

\[ \square f''_{t t}(x, y, z, t) = \nabla^2 \varphi \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) \]
\[ - \frac{\varepsilon^2}{c^2(r^2 + \varepsilon^2)} \varphi''_{t t} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) \]
\[ + \frac{2r^2 + 3\varepsilon^2}{c(\sqrt{r^2 + \varepsilon^2})^3} \varphi''_{t} \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right) \]
\[ + \frac{2r}{c\sqrt{r^2 + \varepsilon^2}} \frac{\partial^2}{\partial r \partial t} \varphi \left( x, y, z, t + \frac{\sqrt{r^2 + \varepsilon^2}}{c} \right). \]

\[ \hat{f} = \nabla^2 \varphi - \frac{\varepsilon^2}{c^2(r^2 + \varepsilon^2)} \varphi + \frac{2r^2 + 3\varepsilon^2}{c(r^2 + \varepsilon^2)^{3/2}} \varphi + \frac{2z}{c\sqrt{r^2 + \varepsilon^2}} \frac{\partial^2}{\partial r \partial t} \varphi. \]

2.14.1 Time Delay

With the introduction of a time delay \( \tau \), which is a universal constant (classically \( \tau = 0 \)), by setting

\[ \varepsilon = \tau c, \]
we get:

\[ \Phi = \int \frac{1}{\sqrt{r^2 + \varepsilon^2}} S \left( t - \frac{\sqrt{z_2 + \varepsilon^2}}{c}, x, y, z \right) dx \, dy \, dz, \]

and, for \( \varepsilon \rightarrow 0 \):

\[ \Phi = \int \frac{1}{r} S \left( t - \frac{r}{c}, x, y, z \right) dx \, dy \, dz \]
\[ -\varepsilon^2 \left\{ \int \frac{1}{2r^3} S \left( t - \frac{r}{c}, x, y, z \right) dx \, dy \, dz \right. \]
\[ + \int \frac{1}{2r^2c} \dot{S} \left( t - \frac{r}{c}, x, y, z \right) dx \, dy \, dz \right\} + \ldots. \]

### 2.15. MAGNETIC CHARGES

A modification of the classical Maxwell equations was considered in the following pages, in order to include also the effect of magnetic charges.

\[ A(q) = -\frac{1}{4\pi} \int \frac{\nabla \cdot g(q')}{r} dq'. \]

\[ g^0 = -\frac{1}{4\pi} \nabla \int \frac{\nabla \cdot g(q')}{r} dq', \]
\[ g^1 = g - g^0. \]

\[ g = (\delta(q - q_0); 0; 0), \]

\[ \nabla \cdot g = \delta'(x - x_0) \delta(y - y_0) \delta(z - z_0). \]

\[ r = |q' - q|: \]

\[ \int \frac{\delta'(x' - x_0) \delta(y - y_0) \delta(z - z_0)}{r} dq' \]
\[ = \int \frac{\delta'(x' - x_0)}{\sqrt{(y_0 - y)^2 + (z_0 - z)^2 + (x' - x)^2}} dx' \]
\[ = \int \frac{\delta(x' - x_0)}{\left[ (y_0 - y)^2 + (z_0 - z)^2 + (x' - x)^2 \right]^{3/2}} dx' \]
\[ = -\frac{1}{\left[ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \right]^{3/2}} = -\frac{x - x_0}{R^3}. \]
\[
g_1^0 = \frac{3(x-x_0)^2 - R^2}{R^5}, \quad g_2^0 = \frac{3(x-x_0)(y-y_0)}{R^5}, \quad g_3^0 = \frac{3(x-x_0)(z-z_0)}{R^5};
\]

\[
g_1^1 = \delta(q-q_0) - \frac{3(x-x_0)^2 - R^2}{R^5}, \quad g_2^1 = -\frac{3(x-x_0)(y-y_0)}{R^5}, \quad g_3^1 = -\frac{3(x-x_0)(z-z_0)}{R^5}.
\]

\[
E = \frac{E' + E''}{2}, \quad H = \frac{H' + H''}{2}.
\]

\[
4\pi I \int \frac{1}{c} \frac{\partial E'}{\partial t} = \nabla \times H', \quad 4\pi I \frac{1}{c} \frac{\partial E''}{\partial t} = \nabla \times H'',
\]

\[
-4\pi I - \frac{1}{c} \frac{\partial H'}{\partial t} = \nabla \times E', \quad 4\pi I - \frac{1}{c} \frac{\partial H''}{\partial t} = \nabla \times E'',
\]

\[
\nabla \cdot E' = 4\pi \rho, \quad \nabla \cdot E'' = 4\pi \rho,
\]

\[
\nabla \cdot H' = 4\pi \rho, \quad \nabla \cdot H'' = -4\pi \rho.
\]

\[
\begin{cases}
4\pi I \ (1 - i) + \frac{1}{c} \frac{\partial (E' - iH')}{\partial t} = i \nabla \times (E' - iH'), \\
\n\end{cases}
\]

\[
\nabla \cdot (E' - iH') = 4\pi \rho \ (1 - i),
\]

\[
\begin{cases}
4\pi I \ (1 + i) + \frac{1}{c} \frac{\partial (E'' - iH'')}{\partial t} = i \nabla \times (E'' - iH''), \\
\n\end{cases}
\]

\[
\nabla \cdot (E'' - iH'') = 4\pi \rho \ (1 + i),
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
4\pi i (1 + i) + \frac{1}{c} \frac{\partial (E' + iH')}{\partial t} = -i \nabla \times (E' + iH'), \\
\nabla \cdot (E' + iH') = 4\pi \rho (1 + i), \\
\end{array} \right.
\end{align*}
\]

\[
\left\{ \begin{array}{l}
4\pi i (1 - i) + \frac{1}{c} \frac{\partial (E'' + iH'')}{\partial t} = -i \nabla \times (E'' + iH''), \\
\nabla \cdot (E'' + iH'') = 4\pi \rho (1 - i), \\
\end{array} \right.
\]

For \( E' = -H'' \), \( H' = E'' \) we re-obtain the Maxwell equations:

\[
\begin{align*}
E &= \frac{E' + H'}{2}, \\
H &= \frac{H' - E'}{2}.
\end{align*}
\]

[21]

Appendix:
Potential experienced by an electric charge: a particular case

For a charge-1 particle:

\[
\frac{dV}{dt} = -\frac{1}{2(a^2 + t)(a^2 + t)(c^2 + t)} = -\frac{1}{2(a^2 + t)\sqrt{c^2 + t}}.
\]

\[21\] The page ended with an attempt to generalize the above results to arbitrary linear combinations of the \( E \) and \( H \) fields (with space-time dependent coefficients), in the case of Maxwell equations without sources:

\[
\begin{align*}
E' &= \alpha E + \beta H, \\
H' &= -\beta E + \alpha H; \\
\alpha &= \alpha(q,t), \beta = \beta(q,t); \\
\frac{1}{c} \frac{\partial E}{\partial t} &= \nabla \times H, \\
\frac{1}{c} \frac{\partial H}{\partial t} &= \nabla \times E, \\
\nabla \cdot E &= 0, \quad \nabla \cdot H = 0; \\
\nabla \cdot E' &= \nabla \alpha \cdot E + \nabla \beta \cdot H, \\
\nabla \cdot H' &= -\nabla \beta \cdot E + \nabla \alpha \cdot H.
\end{align*}
\]
\begin{align*}
-\frac{1}{c} &= V = \int_{0}^{\infty} \frac{dt}{2(a^2 + t)\sqrt{(c^2 + t)}} \\
&= \int_{c}^{\infty} \frac{dz}{z^2 + (a^2 - c^2)} = \frac{1}{\sqrt{a^2 - c^2}} \left( \frac{\pi}{2} - \arctan \frac{c}{\sqrt{a^2 - c^2}} \right) \\
&= \frac{1}{\sqrt{a^2 - c^2}} \arctan \frac{\sqrt{a^2 - c^2}}{c}.
\end{align*}

\begin{align*}
z &= \sqrt{c^2 + t}, \\
z^2 &= c^2 + t, \\
dt &= 2z \, dz, \\
t &= z^2 - c^2, \\
a^2 + t &= z^2 + (a^2 - c^2).
\end{align*}

\begin{align*}
c &= a\sqrt{1 - \beta^2}, \quad a^2 - c^2 = a^2 \beta^2. \\
\frac{1}{c} &= V = \frac{1}{a \beta} \arctan \frac{\beta}{\sqrt{1 - \beta^2}} = \frac{1}{a \beta} \arcsin \beta.
\end{align*}

\begin{equation*}
\begin{bmatrix}
c &= a \frac{\beta}{\arcsin \beta}; \\
V &= \frac{1}{c} = \frac{1}{a} \frac{\arcsin \beta}{\beta}
\end{bmatrix}
\end{equation*}

\begin{align*}
\left( \frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial C_x} \right)^2 + \left( \frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y} \right)^2 + \left( \frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z} \right)^2 \\
= |\nabla C_x|^2 + |\nabla C_y|^2 + |\nabla C_z|^2 - \sum_{xy} \frac{\partial C_x}{\partial y} \frac{\partial C_y}{\partial x}.
\end{align*}
PART II
3

ATOMIC PHYSICS

3.1. GROUND STATE ENERGY OF A TWO-ELECTRON ATOM

Let us consider a nucleus of charge $Z$ with two electrons. In electronic units we have:

$$\nabla^2 \psi + 2(E - V)\psi = 0,$$

$$V = -\frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_3}.$$

In the same units, but denoting with $W$ the energy in rydberg, we have $W = 2E$ and thus:

$$W\psi = V\psi - \nabla^2 \psi,$$

that is:

$$W\psi = -2Z \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \psi + \frac{2}{r_3} \psi - \nabla^2 \psi = H\psi.$$

3.1.1 Perturbation Method

In first approximation, neglecting the interaction and up to a normalization constant, we have:
\[ \psi = e^{-Zr_1} e^{-Zr_2}, \]

and:

\[ H_0 \psi = W_0 \psi = -2Z^2 \psi, \]

where \( H_0 \) is the unperturbed Hamiltonian:

\[ H_0 = -2Z \left( \frac{1}{r_1} + \frac{1}{r_2} \right) - \nabla^2. \]

In fact:

\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial r_1^2} + \frac{\partial^2 \psi}{\partial r_2^2} + \frac{2}{r_1} \frac{\partial \psi}{\partial r_1} + \frac{2}{r_2} \frac{\partial \psi}{\partial r_2} = 2Z^2 \psi - 2Z \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \psi. \]

In first approximation, assuming a normalized \( \psi \), we have:

\[ dW = \int \frac{2}{r_3} \psi^2 d\tau, \]

and since, evidently,

\[ W_0 = \int \psi H_0 \psi d\tau, \]

more expressively we can write:

\[ W = W_0 + \Delta W = \int \psi \left( H_0 + \frac{2}{r} \right) \psi d\tau = \int \psi H \psi d\tau. \]

The correct value \( W \) appears, then, to be the mean value of the energy relative to the function \( \psi \) that, in first approximation, coincides with the energy eigenfunction. This will be useful in comparing the results obtained with the perturbation method with those of the variational method.\(^1\)

We thus have:

\[ dW = \frac{\int \frac{2}{r_3} e^{-2Z(r_1 + r_2)} d\tau}{\int e^{-2Z(r_1 + r_2)} d\tau}. \]

The integration with respect to the angular coordinates gives:

\(^1\) In the original manuscript, the variational method is appropriately called the “minimum method”.
\[
dW = \frac{2 \int \int \frac{r_1^2 r_2^2}{\rho} e^{-2Z(r_1+r_2)} \, dr_1 \, dr_2}{\int \int r_1^2 r_2^2 e^{-2Z(r_1+r_2)} \, dr_1 \, dr_2},
\]

where \( \rho \) is the greater value between \( r_1 \) and \( r_2 \). By restricting the double integration field to the region \( r_1 \leq r_2 \), the numerator and the denominator will be divided by a factor two, so that:

\[
dW = \frac{\int_0^\infty 2r_2 e^{-2Zr_2} \, dr_2 \int_0^{r_2} r_1^2 e^{-2Zr_1} \, dr_1}{\int_0^\infty r_2^2 e^{-2Zr_2} \, dr_2 \int_0^{r_2} r_1^2 e^{-2Zr_1} \, dr_1}.
\]

Now we have:

\[
\int r_1^2 e^{-2Zr_1} \, dr_1 = -\frac{r_1^2}{2Z} e^{-2Zr_1} + \frac{1}{Z} \int r_1 e^{-2Zr_1} \, dr_1
\]

\[
= -\frac{r_1^2}{2Z} e^{-2Zr_1} - \frac{r_1}{2Z^2} e^{-2Zr_1} + \frac{1}{2Z^2} \int e^{-2Zr_1} \, dr_1
\]

\[
= \left( -\frac{r_1^2}{2Z} - \frac{r_1}{2Z^2} - \frac{1}{4Z^3} \right) e^{-2Zr_1},
\]

so that:

\[
\int_0^{r_2} r_1^2 e^{-2Zr_1} \, dr_1 = \frac{1}{4Z^3} - \left( \frac{1}{4Z^3} + \frac{r_2}{2Z^2} + \frac{r_2^2}{2Z} \right) e^{-2Zr_2}.
\]

We thus have:

\[
dW = \frac{N}{D},
\]

\[
N = \int_0^\infty \frac{r_2^2}{2Z^3} e^{-2Zr_2} \, dr_2 - \int_0^\infty \frac{r_2^2}{2Z^3} e^{-4Zr_2} \, dr_2 - \int_0^\infty \frac{r_2^2}{Z} e^{-4Zr_2} \, dr_2
\]

\[
= \frac{1}{8Z^5} - \frac{1}{32Z^5} - \frac{1}{32Z^5} - \frac{3}{128Z^5} = \frac{5}{128Z^5},
\]

\[
D = \int_0^\infty \frac{r_2^4}{4Z^3} e^{-2Zr_2} \, dr_2 - \int_0^\infty \frac{r_2^4}{4Z^3} e^{-4Zr_2} \, dr_2 - \int_0^\infty \frac{r_2^2}{2Z^2} e^{-4Zr_2} \, dr_2
\]

\[
= \frac{1}{16Z^6} - \frac{1}{128Z^6} - \frac{3}{256Z^6} - \frac{3}{256Z^6} = \frac{1}{32Z^6},
\]
\[ dW = \frac{5}{4} Z, \]

and therefore:

\[ W = W_0 + \Delta W = -2Z^2 + \frac{5}{4} Z. \]

The ionization energy consequently is:\(^2\)

\[ W_j = -Z^2 - W = Z^2 - \frac{5}{4} Z \left[ = \left( -Z - \frac{5}{8} \right)^2 - \frac{25}{64} \right]. \]

For the helium atom we thus have:\(^3\)

\[ W_j = 4 - \frac{5}{4} \cdot 2 = \frac{3}{2} = 20.31 \text{ V}. \]

For the lithium atom, the second ionization potential is:

\[ W_j = 9 - \frac{5}{4} \cdot 3 = \frac{21}{4} = 71.08 \text{ V}. \]

### 3.1.2 Variational Method

The ground state energy is the minimum value of the expression

\[ \frac{\int \phi H \phi \, d\tau}{\int \phi^2 \, d\tau}, \]

i.e., the minimum value assumed by the mean value of the energy with respect to any wavefunction \( \phi \). If we consider only a given set of functions \( \phi \), the minimum will correspond to an approximate value. The given approximation improves when the set is enlarged. When this set reduces to the only unperturbed wavefunction considered in the perturbation method, we obtain the same result given by that method. If the set is composed also of further wavefunctions besides the unperturbed wavefunction, in general we will have a better approximation.

\(^2\) Note that, in the following, the author uses to write volt instead of eV for the energy unit.

\(^3\) Here and in the following pages, Majorana usually employed the electron-volt as energy unit. The symbol used by him was V (the same as for volt) rather than eV.
3.1.2.1 First case. To this end, we consider the functions
\[ \varphi = e^{-k(r_1 + r_2)} \]
with arbitrary \( k \). We have:
\[
H \varphi = -2k^2 \varphi + 2(k - Z) \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \varphi + \frac{2}{r_3} \varphi;
\]
\[
\int \frac{\varphi H \varphi \, d\tau}{\int \varphi^2 \, d\tau} = -2k^2 + 4(k - Z) \int_0^\infty \int_0^\infty r_1 r_2 e^{-2kr_1} \, dr_1 \, dr_2 + \frac{5}{4} k
\]
that is:
\[
W_{\text{mean}} = 2k^2 - 4kZ + \frac{5}{4} k.
\]
The minimum will be reached when:
\[
4k - 4Z + \frac{5}{4} = 0,
\]
that is:
\[
k = Z - \frac{5}{16}.
\]
In this case we have:
\[
W = 2 \left( Z - \frac{5}{16} \right)^2 - 4Z \left( Z - \frac{5}{16} \right) + \frac{5}{4} \left( Z - \frac{5}{16} \right),
\]
that is:
\[
W = -2Z^2 + \frac{5}{4} Z - \frac{25}{128} = -2 \left( Z - \frac{5}{16} \right)^2 = -2k^2.
\]
The ionization energy will be
\[
W_j = -Z^2 - W = Z^2 - \frac{5}{4} Z + \frac{25}{128}.
\]
For the helium atom:
\[
W_j = \frac{217}{128} = 22.95 \, \text{V}.
\]
3.1.2.2 Second case. Let $\varphi$ be an arbitrary function; the wavefunction of the ground state can be approximated by an expression of the form:

$$y = a\varphi + bH\varphi,$$

so that we have:

$$W_{\text{mean}} = \frac{\int yHy \, d\tau}{\int y^2 \, d\tau} = \frac{\int (a\varphi + bH\varphi)(aH\varphi + bH^2\varphi) \, d\tau}{\int (a\varphi + bH\varphi)^2 \, d\tau}$$

$$= \frac{a^2\int H\varphi \, d\tau + b^2\int H\varphi \cdot H^2\varphi \, d\tau + ab\int H\varphi \cdot H\varphi \, d\tau + ab\int H\varphi \cdot H\varphi \, d\tau}{a^2\int \varphi^2 \, d\tau + b^2\int H\varphi \cdot H\varphi \, d\tau + 2ab\int \varphi \cdot \varphi \, d\tau}$$

By noting that

$$\int (H\varphi \cdot H\varphi - \varphi H^2\varphi) \, d\tau = \int [(H\varphi)H\varphi - \varphi H(H\varphi)] \, d\tau = 0$$

or:

$$\int H\varphi \cdot H\varphi \, d\tau = \int \varphi \cdot H^2\varphi \, d\tau,$$

and, in general,

$$\int H^m\varphi \cdot H^n\varphi \, d\tau = \int \varphi H^{m+n}\varphi \, d\tau,$$

we get:

$$W_{\text{mean}} = \frac{a^2A + 2abB + b^2C}{a^2 + 2abA + b^2B},$$

where

$$A = \frac{\int \varphi \cdot H\varphi \, d\tau}{\int \varphi^2 \, d\tau}, \quad B = \frac{\int \varphi \cdot H^2\varphi \, d\tau}{\int \varphi^2 \, d\tau}, \quad C = \frac{\int \varphi \cdot H^3\varphi \, d\tau}{\int \varphi^2 \, d\tau}.$$

If we consider the generalized trial function

$$y = a_0\varphi + a_1H\varphi + a_2H^2\varphi + \ldots + a_nH^n\varphi,$$
we analogously get:

\[ W_{\text{media}} = \frac{\sum_{i,k=0}^{n} a_i a_k A_{i+k+1}}{\sum_{i,k=0}^{n} a_i a_k A_{i+k}}, \]

where:

\[ A_r = \frac{\int \varphi H^r \varphi \, d\tau}{\int \varphi^2 \, d\tau}, \]

and \( W \) will be the smallest root of the following equation:

\[
\begin{vmatrix}
A_1 - W & A_2 - A_1 W & \ldots & A_n - A_{n-1} W \\
A_2 - A_1 W & A_3 - A_2 W & \ldots & A_{n+1} - A_n W \\
A_3 - A_2 W & A_4 - A_3 W & \ldots & A_{n+2} - A_{n+1} W \\
\vdots & \vdots & \ddots & \vdots \\
A_n - A_{n-1} W & A_{n+1} - A_n W & \ldots & A_{2n-1} - A_{2n-2} W
\end{vmatrix} = 0.
\]

For \( n = 1 \), we simply have:

\[
\begin{vmatrix}
A_1 - W & A_2 - A_1 W \\
A_2 - A_1 W & A_3 - A_2 W
\end{vmatrix} = 0.
\]

Often, this procedure does not converge, because, starting from a given value of \( n \), quantity \( H^n \varphi \) exhibits too many singularities, which forces us to consider only combinations of the form

\[ y = a_0 \varphi + a_1 H \varphi + \ldots + a_{n-1} H^{n-1} \varphi. \]

The inclusion of additional terms is not useful, since the corresponding \( a \) coefficients would necessarily vanish.

**3.1.2.3 Third case.** In our efforts for the search of the minimum value, let us consider the set of functions of the form:

\[ \varphi = e^{-kr_1} e^{-kr_2} e^{\ell r_3}, \]

with arbitrary \( k \) and \( \ell \). A particular case of this set (\( \ell = 0 \)) has been considered in Sect. 3.1.2.1; then, we will certainly obtain a better approximation. We get:
\[ W_{\text{mean}} = \frac{\int e^{-k(r_1+r_2)+\ell r_3} H e^{-k(r_1+r_2)+\ell r_3} \, d\tau}{\int e^{-2k(r_1+r_2)+2\ell r_3} \, d\tau}. \]

Now we have:

\[ \nabla^2 \varphi = \nabla^2 e^{-k(r_1+r_2)+\ell r_3} = 2k^2 \varphi + 2\ell^2 \varphi - 2k\ell \varphi \cos r_1 \cdot r_3 - 2k\ell \varphi \cos r_2 \cdot r_3 - \frac{2k}{r_1} \varphi - \frac{2k}{r_2} \varphi + \frac{4\ell}{r_3} \varphi, \]

or, by setting:

\[ \alpha_{13} = \cos \hat{r}_1 \hat{r}_3, \quad \alpha_{23} = \cos \hat{r}_2 \hat{r}_3, \]

and, remembering the expression for \( H \), we obtain:

\[ H \varphi = -2k^2 \varphi - 2\ell^2 \varphi + 2k\ell \alpha_{13} + 2k\ell \varphi \alpha_{23} - 2 \frac{Z - k}{r_1} \varphi - 2 \frac{Z - k}{r_2} \varphi + \frac{2 - 4\ell}{r_3} \varphi. \]

It follows that

\[ W_{\text{mean}} = -2k^2 - 2\ell^2 + 2k\ell \frac{\int \varphi^2 \alpha_{13} \, d\tau}{\int \varphi^2 \, d\tau} + 2k\ell \frac{\int \varphi^2 \alpha_{23} \, d\tau}{\int \varphi^2 \, d\tau} - 2(Z - k) \frac{\int \frac{1}{r_1} \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} - 2(Z - k) \frac{\int \frac{1}{r_2} \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} + (2 - 4\ell) \frac{\int \frac{1}{r_3} \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau}. \]

Due to the symmetry of function \( \varphi \) for the two electrons, \textit{the third and fourth term above in the r.h.s are equal}, as well as the fifth and the sixth terms. Moreover, by observing that

\[ \alpha_{13} = \frac{r_1^2 + r_2^2 - r_3^2}{2r_1r_3}, \quad \alpha_{23} = \frac{r_2^2 + r_3^2 - r_1^2}{2r_2r_3}, \]
we have:

\[
W_{\text{mean}} = -2k^2 - 2\ell^2 + k\ell \frac{\int r_1 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} + k\ell \frac{\int r_2 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} + k\ell \frac{\int r_3 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} + k\ell \frac{\int r_2 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} + k\ell \frac{\int r_3 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} - k\ell \frac{\int r_1 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} - k\ell \frac{\int r_2 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} - k\ell \frac{\int r_3 \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} - 2(Z-k) \frac{\int \frac{1}{r_1} \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} - 2(Z-k) \frac{\int \frac{1}{r_2} \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau} + (2-4\ell) \frac{\int \frac{1}{r_3} \varphi^2 \, d\tau}{\int \varphi^2 \, d\tau}.
\]

\[\text{[4]}\]

3.2. WAVEFUNCTIONS OF A TWO-ELECTRON ATOM

The author again considered two-electron atoms, but now he focused on possible expressions for their wavefunctions. The notation is similar to that of the previous Section.

\[y = 1 - 2r_1 - 2r_2 + \frac{1}{2}r_3 + a(r_1^2 + r_2^2) + br_3^2 + cr_1r_2 + d(r_1 + r_2)r_3 + \ldots,\]

\[\frac{\partial y}{\partial r_1} = -2 + 2ar_1 + cr_2 + dr_3 + \ldots.\]

\[y_{r_1=0, r_2=0, r_3=R} = 1 - \frac{3}{2}R + \ldots,\]

\[\left( \frac{\partial y}{\partial r_1} \right)_{r_1=0, r_2=0, r_3=R} = -2 + (c + d)R + \ldots.\]

\[c + d = 3.\]

\[\text{[5]}\]

\[\text{[4]} \text{ This Section is left incomplete in the original manuscript, which continues as follows: “By performing a first integration on the 4-dimensional surface } r_1 = \text{ const.}, r_2 = \text{ const.}, \text{ apart from a common factor in the numerator and in the denominator of the fractional terms above, and observing that on the considered surface we have the mean values of the following expressions, we find that ….”.}\]

\[\text{[5]} \text{ The numerical values for the coefficients } c, d \text{ are deduced by requiring that } y \text{ and its first derivative have a node at the same position when the two-electron system collapses into a one-electron one } [r_1 = 0 \text{ (or } r_2 = 0) \text{ and } r_3 = 0].\]
\[
\frac{\partial y}{\partial r_3} = -\frac{1}{2} + 2br_3 + d(r_1 + r_2) + \ldots;
\]

\[y_{r_3=0,r_1=r_2=R} = 1 - 4R + \ldots,\]

\[
\left(\frac{\partial y}{\partial r_2}\right)_{r_3=0,r_1=r_2=R} = \frac{1}{2} + 2dR + \ldots.
\]

d = -1.

\[y = 1 - 2r_1 - 2r_2 + \frac{1}{2}r_3 + a(r_1^2 + r_2^2) + br_3^2 + 4r_1r_2 - (r_1 + r_2)r_3 + \ldots\]

\[\begin{align*}
2\cos\alpha_1 &= \frac{r_1^2 + r_3^2 - r_2^2}{r_1r_3}, \\
2\cos\alpha_2 &= \frac{r_2^2 + r_3^2 - r_1^2}{r_2r_3};
\end{align*}\]

\[2\cos\alpha_1 + 2\cos\alpha_2 = \frac{r_1 + r_2}{r_3} + \frac{r_3}{r_1} - \frac{r_2^2}{r_1r_3} - \frac{r_1^2}{r_2r_3}.\]

\[\lambda\psi = L\psi,\]

\[L = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + \frac{2\partial^2}{\partial r_3^2} + \frac{2}{r_1} \frac{\partial}{\partial r_1} + \frac{2}{r_2} \frac{\partial}{\partial r_2} + \frac{4}{r_3} \frac{\partial}{\partial r_3} + 2\cos\alpha_1 \frac{\partial^2}{\partial r_1\partial r_3} + 2\cos\alpha_2 \frac{\partial^2}{\partial r_2\partial r_3}.\]

\[
\psi = \left(1 + \frac{1}{2}r_3\right) \left[\frac{e^{-2r_1 - (2-2\alpha)r_2}}{1+2\alpha r_2} + \frac{e^{-(2-2\alpha)r_1 - 2r_2}}{1+2\alpha r_1}\right],
\]

\[
\frac{\partial \psi}{\partial r_1} = \left(1 + \frac{1}{2}r_3\right) \left[-\frac{2}{1+2\alpha r_2} e^{-2r_1 - (2-2\alpha)r_2} + \frac{2\alpha}{1+2\alpha r_1} e^{-(2-2\alpha)r_1 - 2r_2}\right],
\]

\[
\frac{\partial \psi}{\partial r_2} = \left(1 + \frac{1}{2}r_3\right) \left[\left\{\frac{-(2-2\alpha)}{1+2\alpha r_1} - \frac{2\alpha}{(1+2\alpha r_1)^2}\right\} e^{-(2-2\alpha)r_1 - 2r_2} + \frac{-2}{1+2\alpha r_1} e^{-(2-2\alpha)r_1 - 2r_2}\right],
\]

\[6^\text{th} \text{ With reference to the figure on page 125, } \alpha_1 [\alpha_2] \text{ is the angle between } r_1 [r_2] \text{ and } r_3.\]
\[\frac{\partial \psi}{\partial r_3} = \frac{1}{2} \left[ \frac{e^{-2r_1-(2-2\alpha)r_2}}{1+2ar_2} + \frac{e^{-(2-2\alpha)r_1-2r_2}}{1+2ar_1} \right],\]

\[\frac{\partial^2 \psi}{\partial r_1^2} = \left(1 + \frac{1}{2} r_3\right) \left[ \frac{4}{1+2ar_2} e^{-2r_1-(2-2\alpha)r_2} \right. \]

\[+ \left\{ \frac{(2-2\alpha)^2}{1+2ar_1} + \frac{4\alpha(2-2\alpha)}{(1+2ar_1)^2} + \frac{8\alpha^2}{(1+2ar_1)^3} \right\} e^{-(2-2\alpha)r_1-2r_2} \],\]

\[\frac{\partial^2 \psi}{\partial r_2^2} = \left(1 + \frac{1}{2} r_3\right) \left[ \left\{ \frac{(2-2\alpha)^2}{1+2ar_2} + \frac{4\alpha(2-2\alpha)}{(1+2ar_2)^2} \right\} e^{-2r_1-(2-2\alpha)r_2} \right. \]

\[+ \frac{8\alpha^2}{(1+2ar_2)^3} \right\} e^{-2r_1-(2-2\alpha)r_2} + \frac{4}{1+2ar_1} e^{-(2-2\alpha)r_1-2r_2} \],\]

\[\frac{\partial^2 \psi}{\partial r_3^2} = 0,\]

\[\frac{\partial^2 \psi}{\partial r_1 \partial r_3} = \frac{-1}{1+2ar_2} e^{-2r_1-(2-2\alpha)r_1} \]

\[+ \left\{ \left(\frac{1}{2} r_3 - \frac{\alpha}{1+2ar_1} \right) \right\} e^{-(2-2\alpha)r_1-2r_2},\]

\[\frac{\partial^2 \psi}{\partial r_2 \partial r_3} = \left\{ \frac{-1}{1+2ar_2} - \frac{\alpha}{(1+2ar_2)^2} \right\} e^{-2r_1-(2-2\alpha)r_2} \]

\[+ \frac{-1}{1+2ar_1} e^{-2r_1-(2-2\alpha)1-2r_2}.\]

\[L\psi = P(r_1, r_2, r_3)e^{-2r_1-(2-2\alpha)r_2} + P(r_2, r_1, r_3)e^{-2(2-2\alpha)r_1-2r_2},\]

\[P = \frac{1+r_3/2}{1+2ar_2} \left\{ \frac{4}{r_1} + \frac{4}{r_2} + \frac{2}{r_3} + 4 + (2-2\alpha)^2 + \frac{4\alpha(2-2\alpha)}{1+2ar_2} \right. \]

\[+ \frac{8\alpha^2}{(1+2ar_2)^2} - \frac{4}{r_1} - \frac{4-4\alpha}{r_2} - \frac{4\alpha}{r_2(1+2ar_2)} + \frac{2}{r_3(1+r_3/2)} \]

\[+ \frac{2\cos \alpha_1}{1+r_3/2} - \frac{\cos \alpha_2}{1+1/2r_3} \left(2-2\alpha + \frac{2\alpha}{1+2ar_1}\right) \right\} \]

\[= \frac{1+r_3/2}{1+2ar_2} \left\{ 4 + (2-2\alpha)^2 + \frac{8\alpha^2}{1+2ar_2} - \frac{1}{1+r_3/2} - \frac{\cos \alpha_1}{1+r_3/2} \right. \]

\[- \frac{\cos \alpha_2}{1+r_3/2} \left(2-2\alpha + \frac{2\alpha}{1+2ar_1}\right) \right\}.
3.3. CONTINUATION: WAVEFUNCTIONS FOR THE HELIUM ATOM

\[ \psi = e^{-p}, \]
\[ p = \frac{2r_1 + 2r_2 - \frac{1}{2}r_3 + a(r_1^2 + r_2^2) + br_1r_2 + cr_3^2 + d(r_1 + r_2)r_3}{1 + e(r_1 + r_2) + fr_3}. \]

\[ \lambda = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + \nabla^2 \]
\[ = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + 2 \frac{\partial^2}{\partial r_3^2} + \frac{2}{r_1} \frac{\partial}{\partial r_1} + \frac{2}{r_2} \frac{\partial}{\partial r_2} + \frac{4}{r_3} \frac{\partial}{\partial r_3} + 2 \cos \alpha_1 \cdot \frac{\partial^2}{\partial r_1 \partial r_3} + 2 \cos \alpha_2 \cdot \frac{\partial^2}{\partial r_2 \partial r_3}. \]

\[ \alpha_1 = OP_1 - \hat{P}_1 P_1; \quad \alpha_2 = OP_2 - \hat{P}_1 P_2. \]

\[ \psi_0 = e^{-2r_1 - 2r_2 + \frac{1}{2}r_3}; \]
\[ \frac{\partial}{\partial r_1} = -2, \quad \frac{\partial}{\partial r_2} = -2, \quad \frac{\partial}{\partial r_3} = \frac{1}{2}, \]
\[ \frac{\partial^2}{\partial r_1^2} = 4, \quad \frac{\partial^2}{\partial r_2^2} = 4, \quad \frac{\partial^2}{\partial r_3^2} = \frac{1}{4}, \]
\[ \frac{\partial^2}{\partial r_1 \partial r_2} = 4, \quad \frac{\partial^2}{\partial r_1 \partial r_3} = -1, \quad \frac{\partial^2}{\partial r_2 \partial r_3} = -1. \]

\[ \lambda_0 = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + 4 + 4 + \frac{1}{2} - \frac{4}{r_1} - \frac{4}{r_2} + \frac{2}{r_3} - 2 \cos \alpha_1 - 2 \cos \alpha_2 \]
\[ = \frac{17}{2} - 2 \cos \alpha_1 - 2 \cos \alpha_2, \]
\[ \lambda_0^{\text{max}} = 8.5, \quad \lambda_0^{\text{min}} = 4.5. \]
2 = \left( \frac{\partial p}{\partial r_1} \right)_{r_1=0, r_2=r_3=R} \\
= \frac{(2 + bR + dR)(1 + eR + fR) - e\left[2R - \frac{1}{2}R + aR^2 + cR^2 + dR^2\right]}{(1 + eR + fR)^2} \\
= \frac{2 + R\left(b + d + 2e + 2f - \frac{3}{2}e\right) + R^2(be + bf + df - ae - ce - fe)}{1 + R(2e + 2f) + R^2(e^2 + f^2 + 2ef)} \\
= \frac{2 + R\left(b + d + \frac{1}{2}e + 2f\right) + R^2(-ae + be + bf - ce + df)}{1 + R(2e + 2f) + R^2(e^2 + f^2 + 2ef)}.

b + d + \frac{1}{2}e + 2f = 4e + 4f,

-ae + be + bf - ce + df = 2e^2 + 2f^2 + 4ef;

\begin{align*}
&b + d - \frac{7}{2}e - 2f = 0, \\
&ac - be - bf + ce - df + 2e^2 + 4ef + 2f^2 = 0.
\end{align*}

-\frac{1}{2} = \left( \frac{\partial p}{\partial r_3} \right)_{r_3=0, r_1=r_2=R} \\
= \frac{(-\frac{1}{2} + 2dR)(1 + 2eR) - f(2R + 2R + aR^2 + aR^2 + bR^2)}{(1 + 2eR)^2} \\
= \frac{-\frac{1}{2} + R(2d - e - 4f) + R^2(4de - 2af - bf)}{1 + 4eR + 4e^2R^2}.

2d - e - 4f = -2e,

4de - 2af - bf = -2e^2.

\begin{align*}
2d + e - 4f &= 0, \\
2b + 2d - 7e - 4f &= 0, \\
2af + bf - df - 2e^2 &= 0, \\
ae - be - bf + ce - df + 2e^2 + 4ef + 2f^2 &= 0.
\end{align*}
\[ e = A, \quad f = B, \]
\[ d = -\frac{A}{2} + 2B, \]
\[ b = 4A, \]
\[ a = 0, \]
\[ c = 2A - \frac{1}{2}B. \]

\[ p_0 = \frac{2r_1 + 2r_2 - \frac{1}{2}r_3 + 4Ar_1r_2 + (2A - \frac{1}{2}B) r_3^2 + (2B - \frac{1}{2}A) (r_1 + r_2)r_3}{1 + A(r_1 + r_2) + Br_3}. \]

\[ \psi_0 = e^{-p}, \quad \nabla^2 \psi_0 = (-\nabla^2 p + (\nabla p)^2)\psi_0. \]

\[ \lambda = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} - \frac{\partial^2 p}{\partial r_1^2} - \frac{\partial^2 p}{\partial r_2^2} - 2 \frac{\partial^2 p}{\partial r_3^2} - 2 \frac{\partial p}{\partial r_1} \frac{\partial p}{\partial r_1} - \frac{2}{r_2} \frac{\partial p}{\partial r_2} - \frac{4}{r_3} \frac{\partial p}{\partial r_3}. \]

\[ -2 \cos \alpha_1 \frac{\partial^2 p}{\partial r_1 \partial r_3} - 2 \cos \alpha_2 \frac{\partial^2 p}{\partial r_2 \partial r_3} + \left( \frac{\partial p}{\partial r_1} \right)^2 + \left( \frac{\partial p}{\partial r_2} \right)^2 + 2 \left( \frac{\partial p}{\partial r_3} \right)^2 \]

\[ + 2 \cos \alpha_1 \frac{\partial p}{\partial r_1} \frac{\partial p}{\partial r_3} + 2 \cos \alpha_2 \frac{\partial p}{\partial r_2} \frac{\partial p}{\partial r_3}. \]

\[ p = \frac{R}{S}. \]

\[ \frac{\partial p}{\partial r_1} = \frac{1}{S^2} \left( \frac{\partial R}{\partial r_1} S - \frac{dS}{dr_1} R \right), \quad \frac{\partial p}{\partial r_2} = \frac{1}{S^2} \left( \frac{\partial R}{\partial r_2} S - \frac{\partial S}{\partial r_2} R \right), \]

\[ \frac{\partial p}{\partial r_3} = \frac{1}{S^2} \left( \frac{\partial R}{\partial r_3} S - \frac{\partial S}{\partial r_3} R \right). \]

\[ \frac{\partial^2 p}{\partial r_1^2} = \frac{1}{S^3} \left[ \left( \frac{\partial^2 p}{\partial r_1^2} S + \frac{\partial R}{\partial r_1} \frac{\partial S}{\partial r_1} - \frac{\partial S}{\partial r_1} \frac{\partial R}{\partial r_1} - \frac{\partial^2 S}{\partial r_1^2} \right) S \right. \]

\[ - 2 \left( \frac{\partial R}{\partial r_1} S - \frac{\partial S}{\partial r_1} R \right) \frac{\partial S}{\partial r_1} \] \]

\[ = \frac{1}{S^3} \left[ S \left( \frac{\partial^2 R}{\partial r_1^2} - R \frac{\partial^2 S}{\partial r_1^2} \right) - 2 \frac{\partial S}{\partial r_1} \left( S \frac{\partial R}{\partial r_1} - R \frac{\partial S}{\partial r_1} \right) \right]; \]
\[
\frac{\partial^2 p}{\partial r_1 \partial r_2} = \frac{1}{S^3} \left[ S \left( \frac{\partial^2 R}{\partial r_1 \partial r_2} + \frac{\partial R}{\partial r_1} \frac{\partial S}{\partial r_2} - \frac{\partial S}{\partial r_1} \frac{\partial R}{\partial r_2} - R \frac{\partial^2 S}{\partial r_1 \partial r_2} \right) \\
-2 \frac{\partial S}{\partial r_2} \left( \frac{\partial R}{\partial r_1} - R \frac{\partial S}{\partial r_1} \right) \right] \\
= \frac{1}{S^3} \left[ S \left( \frac{\partial^2 R}{\partial r_1 \partial r_2} - R \frac{\partial^2 S}{\partial r_1 \partial r_2} - \frac{\partial R}{\partial r_1} \frac{\partial S}{\partial r_2} - \frac{\partial R}{\partial r_2} \frac{\partial S}{\partial r_1} \right) \\
+ 2R \frac{\partial S}{\partial r_1} \frac{\partial S}{\partial r_2} \right].
\]

\[
R = 2r_1 + 2r_2 - \frac{1}{2} r_3 + 4Ar_1r_2 + \left( 2A - \frac{1}{2} B \right) r_3^2 \\
+ \left( 2B - \frac{1}{2} A \right) (r_1 + r_2)r_3,
\]

\[
S = 1 + A(r_1 + r_2) + Br_3.
\]

\[
\frac{\partial R}{\partial r_1} = 2 + 4Ar_2 + \left( 2B - \frac{1}{2} A \right) r_3, \quad \frac{\partial R}{\partial r_2} = 2 + 4Ar_1 + \left( 2B - \frac{1}{2} A \right) r_3,
\]

\[
\frac{\partial R}{\partial r_3} = -\frac{1}{2} + \left( 2B - \frac{1}{2} A \right) (r_1 + r_2) + (4A - B)r_3;
\]

\[
\frac{\partial^2 R}{\partial r_1^2} = 0, \quad \frac{\partial^2 R}{\partial r_2^2} = 4A - B; \quad \frac{\partial^2 R}{\partial r_3^2} = 2B - \frac{1}{2} A.
\]

\[
\frac{\partial S}{\partial r_1} = A, \quad \frac{\partial S}{\partial r_2} = A, \quad \frac{\partial S}{\partial r_3} = B;
\]

\[
\frac{\partial^2 S}{\partial r_1^2} = 0, \quad \frac{\partial^2 S}{\partial r_2^2} = 0, \quad \frac{\partial^2 S}{\partial r_3^2} = 0; \quad \frac{\partial^2 S}{\partial r_1 \partial r_2} = 0, \quad \frac{\partial^2 S}{\partial r_1 \partial r_3} = 0, \quad \frac{\partial^2 S}{\partial r_2 \partial r_3} = 0.
\]

\[ p = p_0. \]

\[ @ \] The original manuscript then continues with some calculations aimed at evaluating the derivatives of \( p \). In the following we report only the final results.
\[
\frac{\partial p}{\partial r_1} = 2 + \frac{-4A r_1 - 2A^2 r_1^2 + 2A^2 r_2^2 - 2A^2 r_3^2 - 4A^2 r_1 r_2 - 4A B r_1 r_3}{[1 + A (r_1 + r_2) + B r_3]^2},
\]
\[
\frac{\partial p}{\partial r_2} = 2 + \frac{-4A r_2 - 2A^2 r_2^2 + 2A^2 r_1^2 - 2A^2 r_3^2 - 4A^2 r_1 r_2 - 4A B r_2 r_3}{[1 + A (r_1 + r_2) + B r_3]^2},
\]
\[
\frac{\partial p}{\partial r_3} = \frac{1}{[1 + A (r_1 + r_2) + B r_3]^2} \left\{ \left[ 1 + a (R_1 + R_2) + b R_3 \right] \left[ -\frac{1}{2} \right.ight.
\]
\[
\left. \left. + \left( 2B - \frac{1}{2} A \right) (r_1 + r_2) + (4A - B) r_3 \right] \right.
\]
\[
\left. - B \left[ 2r_1 + 2r_2 - \frac{1}{2} r_3 + 4A r_1 r_2 + \left( 2A - \frac{1}{2} B \right) r_3^2 \right. \right.
\]
\[
\left. \left. + \left( 2B - \frac{1}{2} A \right) (r_1 + r_2) r_3 \right] \right\}.
\]

\[
\psi_0 = \left( 1 + \frac{1}{2} r_3 \right) e^{-2r_1 - 2r_2}.
\]
\[
\frac{\partial \psi_0}{\partial r_1} = -2\psi_0, \quad \frac{\partial \psi_0}{\partial r_2} = -2\psi_0, \quad \frac{\partial \psi_0}{\partial r_3} = \frac{1}{2} e^{-2r_1 - 2r_2};
\]
\[
\frac{\partial^2 \psi_0}{\partial r_1^2} = 4\psi_0, \quad \frac{\partial^2 \psi_0}{\partial r_2^2} = 4\psi_0, \quad \frac{\partial^2 \psi_0}{\partial r_3^2} = 0;
\]
\[
\frac{\partial^2 \psi_0}{\partial r_1 \partial r_3} = 4\psi_0, \quad \frac{\partial^2 \psi_0}{\partial r_1 \partial r_3} = e^{-2r_1 - 2r_2}, \quad \frac{\partial^2 \psi_0}{\partial r_3^2} = e^{-2r_1 - 2r_2}.
\]
\[
\lambda_0 = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + 4 + 4 - \frac{4}{r_1} - \frac{4}{r_2} + \frac{1}{1 + \frac{1}{2} r_3} \frac{2}{r_3}
\]
\[
- \frac{2}{1 + \frac{1}{2} r_3} \cos \alpha_1 - \frac{2}{1 + \frac{1}{2} r_3} \cos \alpha_2
\]
\[
= 8 - \frac{1}{1 + \frac{1}{2} r_3} - \frac{2}{1 + \frac{1}{2} r_3} (\cos \alpha_1 + \cos \alpha_2),
\]
\[
\lambda_0^{\text{max}} = 8, \quad \lambda_0^{\text{min}} = 3.
\]
\[ \lambda \psi = L \psi, \quad L = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + \nabla^2. \]

\[ \chi = \sqrt{r_1 r_2 r_3} \psi. \]

\[ \lambda \chi = L' \chi = \sqrt{r_1 r_2 r_3} L \psi, \]

\[ L' = \sqrt{r_1 r_2 r_3} L \frac{1}{\sqrt{r_1 r_2 r_3}}. \]

\[ L' = \frac{4}{r_1} + \frac{4}{r_2} - \frac{2}{r_3} + \left( \frac{\partial^2}{\partial r_1^2} - \frac{1}{r_1} \frac{\partial}{\partial r_1} + \frac{3}{4r_1^2} \right) + \left( \frac{\partial^2}{\partial r_2^2} - \frac{1}{r_2} \frac{\partial}{\partial r_2} + \frac{3}{4r_2^2} \right) \]
\[ + \left( \frac{2}{r_3} \frac{\partial}{\partial r_3} \right) + \left( \frac{4}{r_1} \frac{\partial}{\partial r_1} - \frac{2}{r_1^2} \right) + \left( \frac{4}{r_2} \frac{\partial}{\partial r_2} - \frac{2}{r_2^2} \right) \]
\[ + 2 \cos \alpha_1 \left( \frac{\partial^2}{\partial r_1 \partial r_3} - \frac{1}{2r_1} \frac{\partial}{\partial r_1} - \frac{1}{2r_3} \frac{\partial}{\partial r_3} + \frac{1}{4r_1 r_3} \right) \]
\[ + 2 \cos \alpha_2 \left( \frac{\partial^2}{\partial r_2 \partial r_3} - \frac{1}{2r_2} \frac{\partial}{\partial r_2} - \frac{1}{2r_3} \frac{\partial}{\partial r_3} + \frac{1}{4r_2 r_3} \right). \]

\[ \frac{\partial}{\partial r_1} \frac{1}{\sqrt{r_1}} = -\frac{1}{2r} \frac{1}{\sqrt{r_1}}, \quad \frac{\partial^2}{\partial r_1^2} \frac{1}{\sqrt{r_1}} = \frac{3}{4r_1^2} \frac{1}{\sqrt{r_1}}, \quad \frac{\partial^2}{\partial r_1 \partial r_3} \frac{1}{\sqrt{r_1 r_3}} = \frac{1}{4r_1 r_3}. \]

\[ \frac{\partial^2}{\partial r_1^2} \rightarrow \frac{\partial^2}{\partial r_1^2} - \frac{1}{r_1} \frac{\partial}{\partial r_1} + \frac{3}{4r_1^2}, \]
\[ \frac{\partial}{\partial r_1} \rightarrow \frac{\partial}{\partial r_1} - \frac{1}{2r_1}, \]
\[ \frac{\partial^2}{\partial r_1 \partial r_3} \rightarrow \frac{\partial^2}{\partial r_1 \partial r_3} - \frac{1}{2r_1} \frac{\partial}{\partial r_1} - \frac{1}{2r_3} \frac{\partial}{\partial r_3} + \frac{1}{4r_1 r_3}. \]

### 3.4. SELF-CONSISTENT FIELD IN TWO-ELECTRON ATOMS

A self-consistent field method is here applied to the problem of two-electron atoms with nuclear charge \( Z \). The quantities \( r_1 \) and \( r_2 \) are
the distance of the two electrons from the nucleus, while \( r_{12} \) denotes the inter-electron distance.

\[
E\varphi = H\varphi = \left( -2\frac{Z}{r_1} - 2\frac{Z}{r_2} + 2\frac{2}{r_{12}} \right) \psi - \nabla^2 \varphi.
\]

\[
W = \frac{\int \varphi \psi d\tau}{\int \varphi^2 d\tau}.
\]

\[
\delta \int \varphi(H - W')\varphi d\tau = 0.
\]

\[
\int \varphi(H - W)\varphi d\tau = 0,
\]

\[
\delta \varphi = \alpha \varphi:
\]

\[
\delta \int \varphi(H - W')\varphi d\tau = 2\alpha \int \varphi(H - W')\varphi d\tau = 0;
\]

\[
W' = W.
\]

\[
\delta \int \varphi(H - W)\varphi d\tau = 0. \tag{1}
\]

\[
\varphi(r_1, r_2, r_{12}) = y(r_1) y(r_2),
\]

\[
\delta \varphi = y(r_1) \delta y(r_2) + y(r_2) \delta y(r_1).
\]

\[
\delta \int \varphi(H - W)\varphi d\tau
= 2 \int [y(r_1) \delta y(r_2) + y(r_2) \delta y(r_1)] (H - W) y(r_1) y(r_2) d\tau
= 4 \int y(r_2) \delta y(r_1) (H - W) [y(r_1) y(r_2)] d\tau = 0
\tag{2}
= -\frac{2Z}{r_1} y(r_1) y(r_2) - y(r_1) \frac{2Z}{r_2} y(r_2) + y(r_1) \frac{Z}{r_{12}} y(r_2) - y(r_1) \nabla^2 y(r_2)
- \nabla^2 y(r_1) \cdot y(r_2) - W y(r_1) y(r_2),
\]
\[ \delta \int \varphi (H - W) \varphi \, d\tau = 4 \int \delta y(r_1) \left\{ \left[ -\frac{2Z}{r_1} - \int \frac{2Zy^2(r_1)}{r_2} \, dq_2 \right. \\
+ \left. \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 - \int y(r_2) \nabla^2 y(r_2) \, d\tau - W \right] - \nabla^2 y(r_1) \right\} \, dq_1. \]

\[ \left[ -\frac{2Z}{r_1} - \int \frac{2Zy^2(r_2)}{r_2} \, dq_2 + \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 \\
- \int y(r_2) \nabla^2 y(r_2) \, dq_2 - W \right] y(r_1) - \nabla^2 y(r_1) = 0, \]

\[ \left[ -\frac{2Z}{r_2} - \int \frac{2Zy^2(r_1)}{r_1} \, dq_1 + \int \frac{2y^2(r_1)}{r_{12}} \, dq_1 \\
- \int y(r_1) \nabla^2 y(r_1) \, dq_1 - W \right] y(r_2) - \nabla^2 y(r_2) = 0. \]

\[ \int \left[ -\frac{2Zy^2(r_2)}{r_2} + y(r_2) \nabla^2 y(r_2) \right] \, dq_2 = A \]

\[ = \int \left[ -\frac{2Zy^2(r_2)}{r_1} - y(r_1) \nabla^2 y(r_2) \right] \, dq_1. \]

\[ \left( -\frac{2Z}{r_1} + \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 - W + A \right) y(r_1) - \nabla^2 y(r_1) = 0, \]

\[ (W - A) y(r_1) = \left( -\frac{2z}{r_1} + \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 \right) y(r_1) - \nabla^2 y(r_1). \]

\[ W - A = B, \quad r_1 = r: \]

\[ B y(r) = \left( -\frac{2Z}{r_1} + \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 \right) y(r) - \frac{d^2y}{dr^2} - \frac{2 \, dy}{r \, dr}; \]

\[ B = -\frac{2Z}{r_1} + \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 - \frac{1}{y''} - \frac{2}{ry'}. \]

\[ P = r y:\]

\[ B = -\frac{2Z}{r} + \int \frac{2y^2(r_2)}{r_{12}} \, dq_2 - \frac{P''}{P}. \]
\[ 0 = -8\pi \frac{P^2}{r^2} - \frac{1}{r} \frac{d^2}{dr^2} \frac{rP''}{P}, \]

\[ 8\pi P^2 + r \frac{d^2}{dr^2} \frac{rP''}{P} = 0, \]

\[-8\pi P^2 = r \left[ \frac{d}{dr} \left( \frac{rP''' + P''}{P} - \frac{rP''P'}{P^2} \right) \right] = r \left[ \frac{rP'''}{P^2} - \frac{2P''P' - 2rP'P''' - rP''^2}{P^2} + \frac{2rP'^2P''}{P^2} \right]. \]

### 3.5. 2s TERMS FOR TWO-ELECTRON ATOMS

An approximate expression for the energy (in rydbergs) \( W \) (which is equal to half the mean value of the potential energy) of the 2s terms of two-electron atoms with charge \( Z \) is given. For further details, see Sect. 15 of Volumetto III.

\[-W = \frac{5}{4} Z^2 - \frac{34}{81} \pm \frac{32}{729} Z = Z^2 + \frac{1}{4} Z^2 + \frac{306 \pm 32}{729} Z \]

\[ = \begin{cases} 
Z^2 + \frac{1}{4} Z^2 - 0.3759Z = Z^2 + \frac{1}{4} (Z^2 - 1.5034Z), \\
\text{for ortho-states,} \\
Z^2 + \frac{1}{4} Z^2 - 0.4636Z = Z^2 + \frac{1}{4} (Z^2 - 1.8546Z), \\
\text{for para-states.} 
\end{cases} \]

### 3.6. ENERGY LEVELS FOR TWO-ELECTRON ATOMS

In the following pages, the author evaluates the energies for a number of terms in two-electron atoms, by using certain expressions for the corresponding wavefunctions. The numerical values are grouped in few tables.
\[
\begin{align*}
  r\psi_1 &= y_1\varphi_1^m \quad (m = 1, 0, -1), \\
  r\psi_2 &= y_2\varphi_1^{m'} \quad (m' = 1, 0, -1).
\end{align*}
\]

\[
d\tau = \frac{dxdydz}{4\pi}.
\]

\[
\begin{align*}
  \varphi_1^1\varphi_1^1 &= \varphi_1^1\varphi_1^{-1} = 1 - \frac{1}{\sqrt{5}}\varphi_0^0, \\
  \varphi_0^0\varphi_0^0 &= (\varphi_0^0)^2 = 1 + \frac{2}{\sqrt{5}}\varphi_0^0.
\end{align*}
\]

For \(y_1\varphi_1^1\):

\[
V(r_2) = \frac{1}{r_2} \int_0^{r_2} y_1^2 dr_1 + \int_{r_2}^{\infty} \frac{1}{r_1^2}y_1^2 dr_1 \\
- \left( \frac{1}{5\sqrt{5}r_1^3} \int_0^{r_2} r_1^2 y_1^2 dr_1 + \frac{r_2^2}{5\sqrt{5}} \int_{r_2}^{\infty} \frac{1}{r_1^3}y_1^2 dr_1 \right) \varphi_0^0.
\]

For \(y_1\varphi_0^0\):

\[
V(r_2) = \frac{1}{r_2} \int_0^{r_1} y_1^2 dr_1 + \int_{r_2}^{\infty} \frac{1}{r_1^2}y_1^2 dr_1 \\
+ \left( \frac{2}{5\sqrt{5}r_2^3} \int_0^{r_2} r_1^2 y_1^2 dr_1 + \frac{2r_2^2}{5\sqrt{5}} \int_{r_2}^{\infty} \frac{1}{r_1^3}y_1^2 dr_1 \right) \varphi_0^0.
\]

\[
A = \int \frac{y_1^2 y_2^2}{r_1} dr_1 dr_2,
\]

\[
B = \int \frac{r_2^2 y_1^2 y_2^2}{r_1^3} dr_1 dr_2,
\]

with \(r_i \geq r_k\).
Electrostatic energy

\[
\begin{array}{|c|c|c|c|}
\hline
 & y_1 \varphi_1^1 & y_1 \varphi_1^0 & y_1 \varphi_1^{-1} \\
\hline
y_2 \varphi_1^1 & A + \frac{1}{25}B & A - \frac{2}{25}B & A + \frac{1}{25}B \\
y_2 \varphi_2^0 & A - \frac{2}{25}B & A + \frac{4}{25}B & A - \frac{2}{25}B \\
y_2 \varphi_2^{-1} & A + \frac{1}{25}B & A - \frac{2}{25}B & A + \frac{1}{25}B \\
\hline
\end{array}
\]

<table>
<thead>
<tr>
<th>\ell</th>
<th>Electrostatic energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( A + \frac{1}{25}B )</td>
</tr>
<tr>
<td>1</td>
<td>( A - \frac{1}{5}B )</td>
</tr>
<tr>
<td>0</td>
<td>( A + \frac{2}{5}B )</td>
</tr>
</tbody>
</table>

\[ E_2 = S, \]
\[ E_1 + E_2 = 2T, \quad E_1 = 2T - S, \]
\[ E_0 + E_1 + E_2 = 2S + R, \quad E_0 = 2S + R - 2T. \]

2\textit{p}2\textit{p}

\begin{align*}
\text{\textcolor{red}{1D}} : & \quad A + \frac{1}{25}B \\
\text{\textcolor{red}{3P}} : & \quad A - \frac{1}{5}B \\
\text{\textcolor{red}{1S}} : & \quad A + \frac{3}{5}B \\
\end{align*}

\[
\begin{array}{|c|c|c|}
\hline
\text{\textcolor{red}{1D}} & \frac{1}{Z}A = \frac{93}{512} & = 0.181640625 \\
\text{\textcolor{red}{3P}} & \frac{1}{Z}B = \frac{45}{512} & = 0.08789005 \\
\text{\textcolor{red}{1S}} & \frac{4}{Z}B = \frac{45}{128} & = 0.3515625 \\
\hline
\end{array}
\]
\[ ^1D \ A + \frac{1}{25} \ B = \frac{237}{1280} = 0.18515625 \]

\[ Z = 1 \quad ^3P \quad A - \frac{1}{5} \ B = \frac{21}{128} = 0.1640625 \]

\[ ^1S \quad A + \frac{2}{5} \ B = \frac{111}{512} = 0.216796875 \]

For \( y = x^2 e^{-\frac{1}{2}x} \), \( y^2 = x^4 e^{-x} \), \( N = 24 \) we have in fact:

\[
A = \frac{1}{N^2} \int \frac{y_1^2 y_2^2}{x_1} \, dx_1 \, dx_2 = \frac{2}{N^2} \int_0^{\infty} y_1^2 \, dx_1 \int_{x_1}^{\infty} \frac{y_2^2}{x_2} \, dx_2,
\]

\[
\int \frac{y_2^2}{x_2} \, dx_2 = \int x_2^3 e^{-x_2} \, dx_2 = -(x_2^3 + 3x_2^2 + 6x_2 + 6)e^{-x_2},
\]

\[
\int_{x_1}^{\infty} \frac{y_2^2}{x_2} \, dx_2 = (x_1^3 + 3x_1^2 + 6x_1 + 6)e^{-x_1},
\]

\[
N^2 A = 2 \int_0^{\infty} (x_1^7 + 3x_1^6 + 6x_1^5 + 6x_1^4)e^{-2x_1} \, dx_1
\]

\[
= 2 \left( \frac{7!}{2^8} + 3 \frac{6!}{2^7} + 6 \frac{5!}{2^6} + 6 \frac{4!}{2^5} \right) = \frac{315}{8} + \frac{135}{4} + \frac{45}{2} + 9 = \frac{837}{8},
\]

\[
A = \frac{837}{8 \cdot 576} = \frac{93}{8 \cdot 64} = \frac{93}{512}.
\]

\[
B = \frac{1}{N^2} \int \frac{x_1^2 y_1^2 y_2^2}{x_1^3} \, dx_1 \, dx_2 = \frac{2}{N^2} \int_0^{\infty} x_1^2 y_1^2 \, dx_1 \int_{x_1}^{\infty} \frac{y_2^2}{x_2^3} \, dx_2,
\]

\[
\int \frac{y_2^2}{x_2^3} \, dx_2 = \int x_2 e^{-x_2} \, dx_2 = -(x_2 + 1)e^{-x_2},
\]

\[
\int_{x_1}^{\infty} \frac{y_2^2}{x_2^3} \, dx_2 = (x_1 + 1)e^{-x_1},
\]

\[
N^2 B = 2 \int_0^{\infty} (x_1^7 + x_1^6)e^{-2x_1} \, dx_1
\]

\[
= 2 \left( \frac{7!}{2^8} + \frac{6!}{2^7} \right) = \frac{315}{8} + \frac{45}{4} = \frac{405}{8} = 50.625,
\]
For \( y = (x^2 - 2x)e^{-\frac{1}{2}x} \), \( y^2 = (x^4 - 4x^3 + 4x^2)e^{-x} \), \( N = 24 - 24 + 8 \) we have in fact:

\[
A = \frac{1}{N^2} \int \frac{y_1^2 y_2^2}{x_i} dx_1 dx_2 = \frac{2}{N^2} \int_0^\infty y_1^2 dx_1 \int_1^\infty \frac{y_2^2}{x_2} dx_2,
\]

\[
\int \frac{y_2^2}{x_2} dx_2 = \int (x_2^3 - 4x_2 + 4x_2)e^{-x_2} dx_2 = -(x_2^3 - x_2^2 + 2x_2 + 2)e^{-x_2},
\]

\[
\int_1^\infty \frac{y_2^2}{x_2} dx_2 = (x_1^3 - x_1^2 + 2x_1 + 2)e^{-x_1},
\]

\[
N^2 A = 2 \int_0^\infty (x_1^4 - 4x_1^3 + 4x_1^2)(x_1^3 - x_1^2 + 2x_1 + 2)e^{-2x_1} dx_1
\]

\[
= 2 \int_0^\infty (x_1^7 - 5x_1^5 + 10x_1^4 - 10x_1^3 + 8x_1^2)e^{-2x_1} dx_1
\]

\[
= 2 \left( \frac{7!}{256} - \frac{6!}{128} + \frac{5!}{64} - \frac{4!}{32} + \frac{2!}{8} \right)
\]

\[
= 315 \frac{225}{8} + 75 \frac{4}{2} - 15 + 4 = 77 \frac{625}{8} = 9.625.
\]

### 3.6.1 Preliminaries For The \( X \) And \( Y \) Terms

\[
(2s)^2S + 2p^2p^1S = X + Y.
\]

\[
d\tau_1 = \frac{dx_1 dy_1 dz_1}{4\pi}, \quad d\tau = d\tau_1 d\tau_2.
\]
ATOMIC PHYSICS

\[ (2s)^2 \, ^1S : \quad u = (r_1 - 2)(r_2 - 2)e^{-\frac{1}{2}(r_1 + r_2)}; \]

\[ 2p\, ^2p^1S : \quad v = (x_1 x_2 + y_1 y_2 + z_1 z_2)e^{-\frac{1}{2}(z_1 + z_2)}; \]

\[ uv = (x_1 x_2 + y_1 y_2 + z_1 z_2)(r_1 r_2 - 2r_1 - 2r_2 + 4)e^{-r_1 - r_2}. \]

\[
\int \frac{u^2 d\tau}{r_{12}} = 2 \int d\tau_1 \int_{r_2 > r_1} \frac{uv}{r_{12}} d\tau_2, \]

\[
\int_{r_2 > r_1} \frac{uv}{r_{12}} d\tau_2 = \frac{1}{3} \frac{r_1^2}{r_2^2} e^{-r_1} \int_0^{\infty} r_2 (r_1 r_2 - 2r_1 - 2r_2 + 4)e^{-r_2} dr_2 \]

\[ = \frac{1}{3} \frac{r_1^2}{r_2^2} e^{-2r_1} [r_1(r_1^2 + 2r_1 + 2) - 2r_1(r_1 + 1) - 2(r_1^2 + 2r_1 + 2) + 4(r_1 + 1)], \]

\[
\int \frac{uv}{r_{12}} d\tau = 2 \int_0^{\infty} \left( \frac{1}{3} \frac{r_1^4 - 2}{3 r_1^4} \right) e^{-2r_1} dr_1 \]

\[ = 2 \left( \frac{1}{3} \frac{7!}{28} - \frac{2}{3} \frac{6!}{27} \right) = \frac{105}{8} - \frac{15}{2} = \frac{45}{8}. \]
\[
\frac{1}{\sqrt{N_uN_v}} \int \frac{uv}{r_{12}} \, d\tau = \frac{1}{64\sqrt{3}} \frac{45}{8} = \frac{15\sqrt{3}}{512} = 0.050743676003.
\]

\[
\begin{vmatrix}
77 & 15\sqrt{3} \\
512 & 512 \\
15\sqrt{3} & 111 \\
512 & 512
\end{vmatrix}
\]

\[
a - E & c \\
c & b - E
\]

\[
E^2 - (a + b)E + ab - c^2 = 0,
\]

\[
E = \frac{a + b}{2} \pm \sqrt{\left(\frac{a - b}{2}\right)^2 + c^2}.
\]

\[
\frac{a + b}{2} = \frac{47}{256}, \quad -\frac{a - b}{2} = \frac{17}{512},
\]

\[
\left(\frac{a - b}{2}\right)^2 = \frac{17^2}{512^2} = \frac{289}{512^2}, \quad c^2 = \frac{675}{512^2},
\]

\[
\left(\frac{a - b}{2}\right)^2 + c^2 = \frac{964}{(512)^2}, \quad \sqrt{\left(\frac{a - b}{2}\right)^2 + c^2} = \frac{\sqrt{964}}{(512)^2}.
\]

\[
E_1 = \frac{94 - \sqrt{964}}{512} = \frac{47 - \sqrt{241}}{256},
\]

\[
E_2 = \frac{94 + \sqrt{964}}{512} = \frac{47 + \sqrt{241}}{256}.
\]

\[
\begin{array}{|c|c|}
\hline
X & E_1 = 0.122952443 \\
Y & E_2 = 0.244235057 \\
\hline
\end{array}
\]

\[
E_1 + E_2 = 0.3671975 = \frac{47}{128}.
\]
\[\begin{array}{cc}
a - E_1 & c \\
c & b - E_1 \\
\end{array} = \begin{array}{cc}
-17 + \sqrt{964} & 15\sqrt{3} \\
\frac{512}{512} & \frac{512}{512} \\
15\sqrt{3} & 17 + \sqrt{964} \\
\frac{512}{512} & \frac{512}{512} \\
\end{array}\]

\[= \begin{array}{cc}
0.027438182 & 0.050743676 \\
0.050743676 & 0.093844432 \\
\end{array},\]

\[\begin{array}{cc}
a - E_2 & c \\
c & b - E_2 \\
\end{array} = \begin{array}{cc}
-17 - \sqrt{964} & 15\sqrt{3} \\
\frac{512}{512} & \frac{512}{512} \\
15\sqrt{3} & 17 - \sqrt{964} \\
\frac{512}{512} & \frac{512}{512} \\
\end{array}\]

\[= \begin{array}{cc}
0.093844432 & 0.050743676 \\
0.050743676 & 0.027438182 \\
\end{array}\].

\[X = \sqrt{p_1} (2s)^2 \frac{1}{2} S - \sqrt{p_2} 2p2p^1 S,\]

\[Y = \sqrt{p_2} (2s)^2 \frac{1}{2} S + \sqrt{p_1} 2p2p^1 S,\]

\[p_1 + p_2 = 1.\]

\[p_1 = \frac{675}{1928 - 34\sqrt{964}} = \frac{964 + 17\sqrt{964}}{1928} = 0.774,\]

\[p_2 = \frac{675}{1928 + 34\sqrt{964}} = \frac{964 - 17\sqrt{964}}{1928} = 0.226.\]

### 3.6.2 Simple Terms

\[
\begin{array}{cccc}
2s2s & 2s2p_1 & 2s2p_0 & 2s2p_{-1} \\
2p_12s & 2p_12p_1 & 2p_12p_0 & 2p_12p_{-1} \\
2p_02s & 2p_02p_1 & 2p_02p_0 & 2p_02p_{-1} \\
2p_{-1}2s & 2p_{-1}2p_1 & 2p_{-1}2p_0 & 2p_{-1}2p_{-1} \\
\end{array}
\]
\[
m = 2 \\
\text{singlets} \\
\begin{array}{c|c|c}
2p_1^2p_1 & 237 & 2p_2^1D = \frac{237}{1280} \\
\hline
2p_1^2p_1 & 1280 & \end{array}
\]

\[
m = 1 \\
\text{singlets} \\
\begin{array}{c|c|c}
2s_2p_1 & 0 & \\
\hline
2p_1^2p_0 & 0 & \frac{237}{1280} & 2p_2^1D = \frac{237}{1280} \\
\end{array}
\]

\[
m = 1 \\
\text{triplets} \\
\begin{array}{c|c|c}
2s_2p_1 & \frac{17}{128} & 0 & 2s_2p_3^3P = \frac{17}{128} \\
\hline
2p_1^2p_0 & 0 & \frac{21}{128} & 2p_2^3P = \frac{21}{128} \\
\end{array}
\]

\[
m = 0 \\
\text{singlets} \\
\begin{array}{c|c|c|c|c}
2s_2p_0 & 2s_2s & 2p_1^2p_{-1} & 2p_0^2p_0 & \\
\hline
2s_2p_0 & \frac{49}{256} & 0 & 0 & 0 \\
\hline
2s_2s & 0 & \frac{77}{512} & \frac{15\sqrt{2}}{512} & \frac{15}{512} \\
\hline
2p_1^2p_{-1} & 0 & \frac{15\sqrt{2}}{512} & 33 & \frac{27\sqrt{2}}{2560} \\
\hline
2p_0^2p_0 & 0 & \frac{15}{512} & \frac{27\sqrt{2}}{2560} & \frac{501}{2560} \\
\end{array}
\]

*With a suitable change of states:*
\[ 2p2p^1D = \sqrt{\frac{1}{3}} 2p_1 2p_{-1} - \sqrt{\frac{2}{3}} 2p_0 2p_0, \]
\[ 2p2p^1S = \sqrt{\frac{2}{3}} 2p_1 2p_{-1} + \sqrt{\frac{1}{3}} 2p_0 2p_0, \]

we have: \(^8\)

\[
\begin{array}{cccc}
2s2p_0 & 2s2s & 2p2p^1D & 2p2p^1S \\
2s2p_0 & \frac{49}{256} & 0 & 0 & 0 \\
2s2s & 0 & \frac{77}{512} & 0 & \frac{15\sqrt{3}}{512} \\
2p2p^1D & 0 & 0 & \frac{237}{1280} & 0 \\
2p2p^1S & 0 & \frac{15\sqrt{3}}{512} & 0 & \frac{111}{512} \\
\end{array}
\]

\[ 2p2p^1D = \frac{237}{1280} \]

\[ X = \frac{47 - \sqrt{241}}{256} \left[ (2s)^2 1S \right], \]
\[ Y = \frac{47 + \sqrt{241}}{256} \left[ 2p2p^1S \right], \]

\(^9\)

\[
\begin{array}{cccc}
2s2p_0 & 2p_1 2p_{-1} \\
m = 0 \text{ triplets} & 2s2p_0 & 0 \\
2p_1 2p_{-1} & 0 & \frac{21}{128} & 2p2p^3P = \frac{21}{128} \\
\end{array}
\]

\(^8\) In the table below we have preferred to denote with the shorthand notations \(2p2p^1D\) and \(2p2p^1S\) (used even elsewhere in the original manuscript) what the author reported in the full expressions given above.

\(^9\) With \(X\) and \(Y\) the author denotes the eigenvalues of the subsystem formed by \(2s2s\) and \(2p2p^1S\) = \(\sqrt{\frac{2}{3}} 2p_1 2p_{-1} + \sqrt{\frac{1}{3}} 2p_0 2p_0\).
\[
2p2p^1D : \quad \frac{237}{1280} = 0.185.156.250; \\
2p2p^3P : \quad \frac{21}{128} = 0.164.062.500; \\
2s2p^3P : \quad \frac{17}{128} = 0.132.812.500; \\
2s2p^1P : \quad \frac{49}{256} = 0.191.406.250; \\
\]

\[
-\sqrt{p_2} 2s2s^1S + \sqrt{p_1} 2p2p^1S = Y : \quad \frac{47 + \sqrt{241}}{256} = 0.244.235.057; \\
\sqrt{p_1} 2s2s^1S + \sqrt{p_2} 2p2p^1S = X : \quad \frac{47 - \sqrt{241}}{256} = 0.122.952.443.
\]
3.6.3 Electrostatic Energy Of The 2s2p Term

\[ y_s^2 = (r_1^2 - 2r_1)^2 e^{-r_1}, \quad N_s = 8; \]
\[ y_p^2 = r_2^4 e^{-r_2}, \quad N_p = 24. \]

\[ r_i \geq r_1, r_2; \]

\[ \int \frac{y_s^2 y_p^2}{r_i} dr_1 dr_2 = \int y_s^2 dr_1 \int \frac{y_p^2}{r_i} dr_2 = \int y_p^2 dr_2 \int \frac{y_s^2}{r_i} dr_1, \]
\[ = \int_0^\infty y_s^2 dr_1 \int_0^\infty \frac{y_p^2}{r_2} dr_2 + \int_0^\infty y_p^2 dr_2 \int_0^\infty \frac{y_s^2}{r_1} dr_1, \]
\[ \int \frac{y_p^2}{r_i} dr_2 = \frac{1}{r_1} \int_0^{r_1} y_p^2 dr_2 + \int_0^\infty \frac{y_p^2}{r_2} dr_2, \]
\[ \int y_p^2 dr_2 = \int r_2^4 e^{-r_2} dr_2 = -(r_2^4 + 4r_2^4 + 12r_2^2 + 24r_s + 24)e^{-r_2}, \]
\[ \int \frac{1}{r_2} y_p^2 dr_2 = \int r_2^3 e^{r_2} dr_2 = -(r_2^4 + 3r_2^2 + 6r_2 + 6)e^{-r_2}, \]
\[ \int_0^{r_1} y_p^2 dr_2 = 24 - (r_1^4 + 4r_1^3 - 12r_1^2 + 24r_1 + 24)e^{-r_1}, \]
\[ \int_{r_1}^\infty \frac{1}{r_2} y_p^2 dr_2 = (r_1^3 + 3r_1^2 + 6r_1 + 6)e^{-r_1}, \]

\[ \frac{1}{r_1} \int_0^{r_1} y_p^2 dr_2 + \int_{r_1}^\infty \frac{1}{r_2} y_p^2 dr_2 = \frac{24}{r_1} - \left( \frac{24}{r_1} + 18 + 6r_1 + r_1^2 \right) e^{-r_1} \]
\[ = \int \frac{y_p^2}{r_i} dr_2 = V_p. \]

\[ ^{10} \text{In the original manuscript it is noted that:} \]
\[ \int V_p y_s^2 dr_1^2 = \int y_s^2 dr_1 \int \frac{y_p^2}{r_i} dr_2 = \int y_p^2 dr_2 \int \frac{y_s^2}{r_i} dr_1 = \int V_s y_s^2 dr_2, \]

where \( V \) denotes the electrostatic potential energy of the \( p \) or \( s \) state.
\[
\int y_s^2 V_p \, dr_1 = \int_0^\infty \left[(24r_1^3 - 96r_1^2 + 96r_1)e^{r_1} \right. \\
-\left.(r_1^6 + 2r_1^5 - 2r_1^4 + 24r_1^3 - 24r_1^2 + 96r_1)e^{-2r_1}\right] \, dr_1 \\
= 144 - 192 + 96 - \frac{720}{128} + \frac{240}{64} + \frac{48}{32} + \frac{144}{16} + \frac{48}{8} - \frac{96}{4}.
\]

\[
\frac{1}{N_s N_p} \int \frac{y_s^2 y_p^2}{r_i} \, dr_1 \, dr_2 = \frac{3}{4} - 1 + \frac{1}{2} - \frac{15}{256} + \frac{5}{128} + \frac{3}{64} + \frac{1}{32} - \frac{1}{8} \\
= \frac{83}{512} = M.
\]

\[
\frac{3}{2}y_s^2 - \frac{1}{2}y_p^2 = 12 \left(\frac{y_s^2}{N_s^2} - \frac{y_p^2}{N_p^2}\right) = (r_1^4 - 6r_1^3 + 6r_1^2)e^{-r_1} = t_1.
\]

\[
\int \frac{t_1 t_2}{r_i} \, dr_1 \, dr_2 = 2 \int_0^\infty t_1 \, dr_1 \int_{r_1}^\infty \frac{t_2}{r_2} \, dr_2,
\]

\[
\int \frac{t_2}{r_2} \, dr_2 = \int (r_2^3 - 6r_2^2 + 6r_2)e^{-r_2} \, dr_2 = -(r_2^3 - 3r_2^2)e^{-r_2}.
\]

\[
E_s + A_p - 2M = \frac{1}{144} \int \frac{t_1 t_2}{r_i} \, dr_1 \, dr_2
\]

\[
= \frac{1}{72} \int_0^\infty (r_1^4 - 6r_1^3 + 6r_1^2)(r_1^3 - 3r_1^2)e^{-2r_1} \, dr_1
\]

\[
= \int_0^\infty (r_1^7 - 9r_1^6 + 24r_1^5 - 18r_1^4)e^{-2r_1} \, dr_1
\]

\[
= \frac{1}{72} \left(\frac{5040}{256} - \frac{9\cdot720}{128} + \frac{24\cdot120}{64} - \frac{18\cdot24}{32}\right)
\]

\[
= \frac{35}{128} \cdot \frac{45}{8} \cdot \frac{5}{16} \cdot \frac{3}{128} = \frac{1}{128}.
\]

\[
E_s = \frac{77}{512}; \quad A_p = \frac{93}{512};
\]

\[
M = \frac{E_s + A_p}{2} - \frac{1}{256} = \frac{83}{512} = 0.162109375.
\]
3.6.4 Perturbation Theory For $s$ Terms

$$\psi = e^{r_1 - r_2}.$$

$$H_0 = -\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{2} \nabla_1^2 - \frac{1}{2} \nabla_2^2.$$  

$$H_0 \psi_0 = \left[-\frac{1}{r_1} - \frac{1}{r_2} - \left(\frac{1}{2} - \frac{1}{r_1}\right) - \left(\frac{1}{2} - \frac{1}{r_2}\right)\right] \psi_0 = -\psi_0.$$

Then: $\boxed{E_0 = -1}$. For $\lambda \to 0$:

$$H = H_0 + \lambda H_1,$$

$$H_1 = \frac{1}{r_{12}}.$$

$$\psi = \psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \ldots,$$

$$E = E_0 + \lambda E_1 + \lambda^2 E_2 + \ldots.$$

$$0 = (H - E)\psi$$

$$= (H_0 + \lambda H_1 - E_0 - \lambda E_1 - \lambda^2 E_2 \ldots)(\psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \ldots)$$

$$= \left(H_0 + \lambda H_1 - \sum_{i=0}^{\infty} \lambda^i E_i \right) \sum_{k=0}^{\infty} \lambda^k \psi_k;$$

$$\boxed{(H_0 - E_0)\psi_n = (E_1 - H_1)\psi_{n-1} + E_2 \psi_{n-2} + E_3 \psi_{n-3} + \ldots + E_n \psi_0.}$$

$$\boxed{(H_0 - E_0)\psi_0 = 0,}$$

$$\boxed{(H_0 - E_0)\psi_1 = (E_1 - H_1)\psi_0,}$$

$$E_1 = \frac{5}{8}.$$  

By setting:

$$\psi_1 = ye^{-r_1 - r_2},$$
we have:

\[ \nabla^2 \psi_1 = \left(2 - \frac{2}{r_1} - \frac{2}{r_2}\right) y e^{-r_1-r_2} - 2 \left(\frac{\partial y}{\partial r_1} + \frac{\partial y}{\partial r_2}\right) e^{-r_1-r_2} + \nabla^2 y e^{-r_1-r_2}, \]

\[ (H_0 - E_0)\psi_1 = \left(1 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{2} \nabla^2\right) \psi_1 \]

\[ = \left(\frac{\partial y}{\partial r_1} + \frac{\partial y}{\partial r_2} + 1 - 2y\right) e^{-r_1-r_2} = \left(\frac{5}{8} - \frac{1}{r_{12}}\right) e^{r_1-r_2}, \]

\[ \frac{\partial y}{\partial r_1} + \frac{\partial y}{\partial r_2} - \frac{1}{2} \nabla^2 y = \frac{5}{8} - \frac{1}{r_{12}}. \]

\[ y = \sum_{\ell=0}^{\infty} P_\ell(\cos \theta) f_i(r_1, r_2). \]

### 3.6.5 2s2p^3P Term

Let us consider the functions:

\[ (r_1 - 2)e^{-\frac{1}{2}r_1}, \quad r_2 e^{-\frac{1}{2}r_2}. \]

\[ \psi = e^{-\frac{1}{2}(r_1+r_2)} \left[(r_1 - 2)r_2 \varphi_1^0(q_2) - (r_2 - 2)r_1 \varphi_2^0(q_1)\right], \]

\[ \psi^2 = e^{-(r_1+r_2)} \left\{ (r_1 - 2)^2 r_2^2 + (r_2 - 2)^2 r_1^2 \right. \\
-2r_1r_2(r_1 - 2)(r_2 - 2)\varphi_1^0(q_1)\varphi_2^0(q_1) \\
\left. + \frac{2}{\sqrt{5}}(r_1 - 2)^2 r_2^2 \varphi_2^0(q_2) + \frac{2}{\sqrt{5}}(r_2 - 2)^2 r_1^2 \varphi_1^0(q_1)\right\}, \]

where we have used: \( \varphi_1^0 = 1 + \frac{2}{\sqrt{5}} \varphi_2^0. \)

\[ N = 384 = 2 \cdot 8 \cdot 24. \]
\[
\frac{4}{3} \int_0^\infty (r_1^5 - 2r_1^4)e^{r_1} dr_1 \int_{r_1}^\infty r_2(r_2 - 2)e^{-r_2} dr_2 \\
= \frac{4}{3} \int_0^\infty (r_1^7 - 2r_1^6)e^{-2r_1} dr_1 = \frac{45}{4}.
\]

\[I_{sp} = \frac{45}{4} \cdot \frac{384}{3} = \frac{15}{512}.
\]

### 3.6.6 X Term

\(Z = 2\).

\[d\tau = \frac{dxdydz}{4\pi}.
\]

\[y_1 = r_1r_2e^{-r_1 - r_2},
\]

\[y_2 = (r_1 + r_2)e^{-r_1 - r_2},
\]

\[y_3 = e^{-r_1 - r_2},
\]

\[y_4 = (x_1x_2 + y_1y_2 + z_1z_2)e^{-r_1 - r_2}.
\]

\[\int y_1^2 d\tau = \left( \int_0^\infty r_1^4 e^{-2r_1} dr_1 \right)^2 = \left( \frac{24}{32} \right)^2 = \frac{9}{16},
\]

\[\int y_2^2 d\tau = 2 \int_0^\infty r_1^4 e^{-2r_1} dr_1 \int_0^\infty r_2^2 e^{-2r_2} dr_2 + 2 \left( \int_0^\infty r_1^3 e^{2r_1} dr_1 \right)^2,
\]

\[= 2 \cdot \frac{3}{4} \cdot \frac{1}{4} + 2 \cdot \left( \frac{3}{8} \right)^2 = \frac{21}{32},
\]

\[\int y_3^2 d\tau = \left( \int_0^\infty r_1^2 e^{-2r_1} dr_1 \right)^2 = \left( \frac{1}{4} \right)^2 = \frac{1}{16},
\]

\[\int y_4^2 d\tau = \frac{1}{3} \left( \int_0^\infty r_1^4 e^{-2r_1} dr_1 \right)^2 = \frac{1}{3} \left( \frac{3}{4} \right)^2 = \frac{3}{16},
\]

\[\int y_1y_2 d\tau = 2 \int_0^\infty r_1^4 e^{2r_1} dr_1 \int_0^\infty r_2^2 e^{-2r_2} dr_2 = 2 \cdot \frac{3}{4} \cdot \frac{3}{8} = \frac{9}{16}.
\]

---

\[\text{Remember that the X term is a superposition of the 2s}^1S \text{ and } 2p^1S \text{ ones.}\]
\[ \int y_1 y_3 \, d\tau = \left( \int_0^\infty r_1^3 e^{-r_1} \, dr_1 \right)^2 = \left( \frac{3}{8} \right)^2 = \frac{9}{64}, \]

\[ \int y_2 y_3 \, d\tau = 2 \int_0^\infty r_1^3 e^{-r_1} \, dr_1 \int_0^\infty r_2^2 e^{-r_2} \, dr_2 = 2 \cdot \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{16}. \]

Kinetic energy: \[ T = -\frac{1}{2} \nabla^2. \]

Potential energy: \[ U = -\frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_3}. \]

\[ \nabla^2 r_1 e^{-r_1} = \left( r_1 - 4 + \frac{2}{r_1} \right) e^{-r_1} = \left( 1 - \frac{4}{r_1} + \frac{2}{r_1^2} \right) r_1 e^{-r_1}, \]

\[ -\frac{1}{2} \nabla^2 r_1 e^{-r_1} \left( -\frac{r_1}{2} + 2 - \frac{1}{r_1} \right) e^{-r_1} = \left( -\frac{1}{2} + \frac{2}{r_1} - \frac{1}{r_1^2} \right) r_1 e^{-r_1}. \]

\[ \int y_1 T y_1 \, d\tau = 2 \int_0^\infty \left( -\frac{r_1^4}{2} + 2 r_1^3 - r_1^2 \right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^4 e^{-2r_2} \, dr_2 \]

\[ = 2 \cdot \left( -\frac{3}{8} + \frac{3}{4} - \frac{1}{4} \right) \cdot \frac{3}{4} = \frac{3}{16}, \]

\[ \int y_2 T y_1 \, d\tau = 2 \int_0^\infty \left( -\frac{r_1^4}{2} + 2 r_1^3 - r_1^2 \right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 \]

\[ + 2 \int_0^\infty \left( -\frac{r_1^3}{2} + 2 r_1^2 - r_1 \right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_1 \]

\[ = 2 \cdot \frac{1}{8} \cdot \frac{3}{8} + 2 \left( -\frac{3}{16} + \frac{1}{2} - \frac{1}{4} \right) \cdot \frac{3}{4} = \frac{3}{32} + \frac{3}{32} = \frac{3}{16}. \]

\[ -\frac{1}{2} \nabla^2 r_1 e^{-r_1} = \left( -\frac{r_1}{2} + 2 - \frac{1}{r_1} \right) e^{-r_1}, \]

\[ -\frac{1}{2} \nabla^2 r_1 e^{-r_1} = \left( -\frac{1}{2} + \frac{1}{r_1} \right) e^{-r_1}, \]

\[ T y_2 = \left( -\frac{r_1}{2} + 2 - \frac{1}{r_1} \right) e^{-r_1 - r_2} + \left( -\frac{1}{2} + \frac{1}{r_1} \right) r_2 e^{-r_1 - r_2}. \]

\[ \int y_1 T y_2 \, d\tau = 2 \int_0^\infty \left( -\frac{r_1^4}{2} + 2 r_1^3 - r_1^2 \right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_1} \, dr_2 \]

\[ + 2 \int_0^\infty \left( -\frac{1}{2} r_1^3 + r_1^2 \right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_1} \, dr_2 \]

\[ = 2 \cdot \frac{1}{16} \cdot \frac{3}{16} + 2 \cdot \frac{1}{16} \cdot \frac{3}{4} = \frac{3}{16}. \]
\[ \int y_3 T y_1 \, d\tau = 2 \int_0^\infty \left(-\frac{r_1^3}{2} + 2r_1^2 - r_1\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 = 2 \cdot \frac{1}{16} \cdot \frac{3}{8}, \]

\[ \int y_4 T y \, d\tau = 0, \]

\[ \int y_2 T y_2 \, d\tau = 2 \int_0^\infty \left(-\frac{r_1^4}{2} + 2r_1^2 - r_1^2\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2 + 2 \int_0^\infty \left(-\frac{r_1^2}{2} + r_1\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 + 2 \int_0^\infty \left(-\frac{r_1^2}{2} + 2r_1^2 - r_1\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^4 e^{-2r_2} \, dr_2 = 2 \cdot \frac{1}{8} \cdot \frac{1}{4} + 2 \cdot \frac{1}{16} \cdot \frac{3}{8} + 2 \cdot \frac{1}{16} \cdot \frac{3}{8} + 2 \cdot \frac{1}{8} \cdot \frac{3}{4} = \frac{11}{32}, \]

\[ \int y_2 T y_3 \, d\tau = 2 \int_0^\infty \left(-\frac{r_1^2}{2} + r_1^2\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-r_2} \, dr_2 + 2 \int_0^\infty \left(-\frac{r_1^2}{2} + r_1\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 = 2 \cdot \frac{1}{16} \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} \cdot \frac{3}{8} = \frac{1}{32} + \frac{3}{32} = \frac{1}{8}, \]

\[ \int y_3 T y_3 \, d\tau = 2 \int_0^\infty \left(-\frac{r_1^2}{2} + r_1\right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^4 e^{-2r_2} \, dr_2 = 2 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{16}. \]

\[ T_1 (x_1 x_2 + y_1 y_2 + z_1 z_2) e^{-r_1 - r_2} = (x_1 x_2 + y_1 y_2 + z_1 z_2) \left(-\frac{1}{2} + \frac{1}{r_1}\right) e^{-r_1 - r_2} + (x_1 x_2 + y_1 y_2 + z_1 z_2) \frac{1}{r_1} e^{-r_1 - r_2}. \]
\[
\int y_4 T y_4 \, d\tau = 2 \int (x_1 x_2 + y_1 y_2 + r_1 r_2)^2 \left( -\frac{1}{2} + \frac{2}{r_1^2} \right) e^{-2r_1 - 2r_2} \, d\tau \\
= \frac{2}{3} \int_0^\infty \left( -\frac{r_2^4}{2} + 2r_1^3 \right) e^{-2r_1} \, dr_1 \int_0^\infty r_2^4 e^{-r_2} \, dr \\
= \frac{2}{3} \cdot \frac{3}{8} \cdot \frac{3}{4} = \frac{3}{16}.
\]

\[
\int y_1 U y_1 \, d\tau = -4 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_0^\infty r_2^4 e^{-2r_1} \, dr_2 \\
+ 2 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_{r_1}^\infty r_2^3 e^{-2r_2} \, dr_2 \\
= -4 \cdot \frac{3}{8} \cdot \frac{3}{4} + 2 \cdot \frac{837}{8192} = -\frac{9}{8} + \frac{837}{4096} = -\frac{3771}{4096},
\]
given that:

\[
\int r_2^3 e^{-2r_2} \, dr_2 = -\left( \frac{1}{2} r_2^3 + \frac{3}{4} r_2^2 + \frac{3}{4} r_2 + \frac{3}{8} \right) e^{-2r_2},
\]

\[
\int_{r_1}^\infty r_2^3 e^{-2r_2} \, dr_2 = \left( \frac{1}{2} r_1^3 + \frac{3}{4} r_1^2 + \frac{3}{4} r_1 + \frac{3}{8} \right) e^{-2r_1},
\]

\[
\int_0^\infty \left( \frac{1}{2} r_1^7 + \frac{3}{4} r_1^6 + \frac{3}{4} r_1^5 + \frac{3}{8} r_1^4 \right) e^{4r_1} \, dr_1 \\
= \frac{1}{2} \frac{5040}{4096} + \frac{3}{4} \frac{720}{4096} + \frac{3}{4} \frac{120}{4096} + \frac{3}{8} \frac{24}{4096} \\
= \frac{1}{2} \frac{5040}{4096} + \frac{3}{4} \frac{720}{4096} + \frac{3}{4} \frac{120}{4096} + \frac{3}{8} \frac{24}{4096} \\
= \frac{1}{2} \frac{5040}{4096} + \frac{3}{4} \frac{720}{4096} + \frac{3}{4} \frac{120}{4096} + \frac{3}{8} \frac{24}{4096} \\
= \frac{5040}{8192} + \frac{720}{4096} + \frac{120}{2048} + \frac{24}{1024} = \frac{837}{8192}.
\]

\[
\int U y_1 y_2 \, d\tau = -4 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 \\
-4 \int_0^\infty r_1^4 e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2 \\
+ 2 \int_0^\infty r_1^4 e^{-2r_1} \, dr_1 \int_{r_1}^\infty r_2^2 e^{-2r_2} \, dr_2 \\
+ 2 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \\
= -4 \cdot \frac{3}{8} \cdot \frac{3}{8} - 4 \cdot \frac{3}{4} \cdot \frac{1}{4} + 2 \cdot \frac{87}{2048} + 2 \cdot \frac{9}{128} \\
= -\frac{9}{16} \cdot \frac{3}{4} + \frac{87}{1024} + \frac{9}{64} = -\frac{1113}{1024}.
\]
because:

\[
\int_{r_1}^{\infty} r_2^2 e^{-2r_2} \, dr_2 = \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1},
\]

\[
\int_{r_2}^{\infty} r_1^3 e^{-2r_1} \, dr_1 = \left( \frac{1}{2} r_2^3 + \frac{3}{4} r_2^2 + \frac{3}{4} r_2 + \frac{3}{8} \right) e^{-2r_2},
\]

\[
\int_0^\infty \left( \frac{1}{2} r_1^6 + \frac{1}{2} r_1^5 + \frac{1}{4} r_1^4 \right) e^{-4r_2} \, dr_1 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{32}.
\]

\[
\int_{r_2}^{\infty} \left( \frac{1}{2} r_2^6 + \frac{3}{4} r_2^5 + \frac{3}{4} r_2^4 + \frac{3}{8} r_2^3 \right) e^{-4r_2} \, dr_2 = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{125}.
\]

\[
\int y_3 y_1 U \, d\tau = -4 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2 + 2 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_{r_1}^\infty r_2^2 e^{-2r_2} \, dr_2 = -4 \cdot \frac{1}{2} \cdot \frac{3}{8} + 2 \cdot \frac{33}{1024} = -\frac{3}{8} + \frac{33}{512} = -\frac{159}{512},
\]

since:

\[
\int_{r_1}^{\infty} r_2^2 e^{-2r_2} \, dr_2 = \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1},
\]

\[
\int_0^\infty \left( \frac{1}{2} r_1^6 + \frac{1}{2} r_1^5 + \frac{1}{4} r_1^4 \right) e^{-4r_2} \, dr_1 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{32}.
\]

\[
\int U y_2 y_2 \, d\tau = -12 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2 - 4 \int_0^\infty r_1^2 e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 + 2 \int_0^\infty r_1^2 e^{-2r_1} \, dr_1 \int_{r_1}^\infty r_2^3 e^{-2r_2} \, dr_2 + 2 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2 \int_0^\infty r_1^3 e^{-2r_1} \, dr_2 + 4 \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_{r_1}^\infty r_2^2 e^{-2r_2} \, dr_2.
\]
\[ \int_0^\infty r_2 e^{-2r_2} dr_2 = \left( \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1}, \]
\[ \int_0^\infty \left( \frac{1}{2} r_1^5 + \frac{1}{4} r_1^4 \right) e^{-4r_2} dr_2 = \frac{1120}{24096} + \frac{1}{24} + \frac{3}{2} = \frac{8}{512} + \frac{3}{2} = \frac{1024}{512} = 2, \]
\[ \int_0^\infty r_1^3 e^{-2r_1} dr_1 = \left( \frac{1}{2} r_2 + \frac{3}{4} r_2 + \frac{3}{8} r_1 + \frac{3}{8} \right) e^{-2r_1}, \]
\[ \int_0^\infty \left( \frac{1}{2} r_2^3 + \frac{3}{4} r_2^2 + \frac{3}{4} r_1 + \frac{3}{8} \right) e^{-4r_2} dr_2 = \frac{1120}{24096} + \frac{3}{24} + \frac{3}{2} = \frac{1024}{512} = 2, \]
\[ \int_0^\infty r_1^2 e^{-2r_2} dr_2 = \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1}, \]
\[ \int_0^\infty \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-4r_1} dr_1 = \frac{1120}{24096} + \frac{1}{24} + \frac{1}{2} = \frac{1024}{512} = 2. \]

\[ \int U y_3 y_2 d\tau = -4 \int_0^\infty r_1^2 e^{-2r_1} dr_1 \int_0^\infty r_2^2 e^{-2r_2} dr_2 - 4 \int_0^\infty r_1^3 e^{-2r_1} dr_1 \int_0^\infty r_2 e^{-2r_2} dr_2 + 2 \int_0^\infty r_2^2 e^{-2r_2} dr_2 \int_0^\infty r_1 e^{-2r_1} dr_1 + 2 \int_0^\infty r_1^3 e^{-2r_1} dr_1 \int_0^\infty r_2 e^{-2r_2} dr_2. \]
\[ \begin{align*}
&=-4 \cdot \frac{1}{4} \cdot \frac{1}{4} - 4 \cdot \frac{3}{8} \cdot \frac{1}{4} + 2 \cdot \frac{1}{32} + 2 \cdot \frac{9}{512} \\
&=-\frac{5}{8} + \frac{25}{256} = -\frac{135}{256},
\end{align*} \]
given that:
\[
\begin{align*}
\int_{r_2}^\infty r_1^2 e^{-2r_1} \, dr_1 &= \left( \frac{1}{2} r_2^2 + \frac{1}{2} r_2 + \frac{1}{4} \right) e^{-2r_2}, \\
\int_0^\infty \left( \frac{1}{2} r_2^2 + \frac{1}{2} r_2 + \frac{1}{4} \right) e^{-4r_2} \, dr_2 \\
&= \frac{1}{2} \frac{24}{1024} + \frac{1}{2} \frac{6}{256} + \frac{1}{4} \frac{4}{64} = \frac{1}{32}, \\
\int_{r_1}^\infty r_2 e^{-2r_2} \, dr_2 &= \left( \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1}, \\
\int_0^\infty \left( \frac{1}{2} r_1^4 + \frac{1}{4} r_1^3 \right) e^{-2r_1} \, dr_1 &= \frac{1}{2} \frac{24}{1024} + \frac{1}{4} \frac{4}{256} = \frac{9}{512}.
\end{align*} \]

\[
\begin{align*}
\int y_3 U y_3 \, d\tau &= -4 \int_0^\infty r_1 e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2 \\
&\quad + 2 \int_0^\infty r_1^2 e^{-2r_1} \, dr_1 \int_0^\infty r_2 e^{-2r_2} \, dr_2 \\
&= -4 \cdot \frac{1}{4} \cdot \frac{1}{4} + 2 \cdot \frac{5}{256} = -\frac{1}{4} + \frac{5}{128} = -\frac{27}{128},
\end{align*} \]
because:
\[
\begin{align*}
\int_{r_1}^\infty r_2 e^{-2r_2} \, dr_2 &= \left( \frac{1}{2} r_1 + \frac{1}{2} \right) e^{2r_1}, \\
\int_0^\infty \left( \frac{1}{2} r_1^3 + \frac{1}{4} r_1^2 \right) e^{-4r_1} \, dr_1 &= \frac{1}{2} \frac{6}{256} + \frac{1}{4} \frac{4}{64} = \frac{5}{256}.
\end{align*} \]

\[
\begin{align*}
\int y_4 U y_1 \, d\tau &= \frac{2}{3} \int_0^\infty r_1^4 e^{-2r_1} \, dr_1 \int_{r_1}^\infty r_2^2 e^{-2r_2} \, dr_2 \\
&= \frac{2}{3} \frac{555}{8192} = \frac{185}{4096}.
\end{align*} \]
since:

\[
\int_{r_1}^{\infty} r_2^2 e^{-2r_2} dr_2 = \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1},
\]

\[
\int_{0}^{\infty} \left( \frac{1}{2} r_1^7 + \frac{1}{2} r_1^6 + \frac{1}{4} r_1^5 \right) e^{-4r_1}
= \frac{1}{2} \cdot \frac{5040}{216 \cdot 1024} + \frac{1}{4} \cdot \frac{720}{4 \cdot 1024} + \frac{1}{4} \cdot \frac{120}{1024}
= \frac{315}{8192} + \frac{45}{2048} + \frac{15}{2048} = \frac{555}{8192}.
\]

\[
\int U y_2 y_4 d\tau = \int \frac{1}{r_{12}} y_2 y_4 d\tau
= \frac{2}{3} \int_{0}^{\infty} r_1^5 e^{-2r_1} dr_1 \int_{r_1}^{\infty} r_2 e^{-2r_2} dr_2
+ \frac{2}{3} \int_{0}^{\infty} r_2^4 e^{-2r_2} dr_2 \int_{r_1}^{\infty} r_1^2 e^{-2r_1} dr_1
= \frac{2}{3} \cdot \frac{15}{512} + \frac{2}{3} \cdot \frac{87}{2048} = \frac{5}{256} + \frac{29}{1024} = \frac{49}{1024},
\]

given that:

\[
\int_{0}^{\infty} \left( \frac{1}{2} r_1^6 + \frac{1}{4} r_1^5 \right) e^{-4r_1} dr_1
= \frac{1}{2} \cdot \frac{720}{4 \cdot 4096} + \frac{1}{4} \cdot \frac{120}{4 \cdot 1024}
= \frac{45}{2048} + \frac{15}{2048} = \frac{552}{2048} = \frac{221}{1024};
\]

\[
\int_{0}^{\infty} \left( \frac{1}{2} r_1^6 + \frac{1}{2} r_1^5 + \frac{1}{4} r_1^4 \right) e^{-4r_1} dr_1
= \frac{1}{2} \cdot \frac{720}{4 \cdot 4096} + \frac{1}{2} \cdot \frac{120}{4 \cdot 1024} + \frac{1}{4} \cdot \frac{24}{1024}
= \frac{45}{2048} + \frac{15}{2048} + \frac{3}{512} = \frac{201}{1024}.
\]

\[
\int \frac{1}{r_{12}} y_3 y_4 d\tau = \frac{2}{3} \int_{0}^{\infty} r_1^4 e^{-2r_1} dr_1 \int_{r_1}^{\infty} r_2 e^{-2r_2} dr_2
= \frac{7}{512},
\]

because:

\[
\int_{0}^{\infty} \left( \frac{1}{2} r_1^5 + \frac{1}{4} r_1^4 \right) e^{-4r_1} dr_1
= \frac{1}{2} \cdot \frac{120}{4 \cdot 1024} + \frac{1}{4} \cdot \frac{24}{1024}
= \frac{15}{1024} + \frac{3}{1024} = \frac{21}{1024}.
\]
\[
\int U y_1^2 \, d\tau = -\frac{4}{3} \int_0^\infty r_1^3 e^{-2r_1} \, dr_1 \int_0^\infty r_2^4 e^{-2r_2} \, dr_2 \\
+ \frac{2}{3} \int_0^\infty r_1^4 e^{-2r_1} \, dr_1 \int_0^\infty r_2^3 e^{-2r_2} \, dr_2 \\
+ \frac{4}{15} \int_0^\infty r_1^6 e^{-2r_1} \, dr_1 \int_0^\infty r_2^2 e^{-2r_2} \, dr_2
\]

\[
= -\frac{4}{3} \cdot \frac{3}{8} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{837}{8192} + \frac{4}{15} \cdot \frac{8192}{8192} \\
= -\frac{3}{8} - \frac{879}{4096} + \frac{27}{2048} = -\frac{8}{8} + \frac{333}{4096} = -\frac{1203}{4096},
\]

since:

\[
\int_0^\infty \left( \frac{1}{2} r_1^7 + \frac{3}{4} r_1^6 + \frac{3}{4} r_1^5 + \frac{3}{8} r_1^4 \right) e^{-4r_1} \, dr_1 \\
= \frac{1}{2} \cdot \frac{5040}{264 \cdot 1024} + \frac{3}{4} \cdot \frac{720}{16 \cdot 1024} + \frac{120}{4 \cdot 1024} + \frac{3}{8} \cdot \frac{24}{1024} \\
= \frac{315}{8192} + \frac{45}{4096} + \frac{9}{1024} = \frac{837}{8192}.
\]

\[
\int_0^\infty \left( \frac{1}{2} r_1^7 + \frac{1}{4} r_1^6 \right) e^{-4r_1} \, dr_1 \\
= \frac{1}{2} \cdot \frac{5040}{264 \cdot 1024} + \frac{1}{4} \cdot \frac{720}{16 \cdot 1024} \\
= \frac{315}{8192} + \frac{45}{4096} = \frac{405}{8192}.
\]

Normalization matrix

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
</tr>
</thead>
<tbody>
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<td>( y_1 )</td>
<td>( \frac{9}{16} )</td>
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<td>( \frac{9}{64} )</td>
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<tr>
<td>( y_2 )</td>
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<td>( 0 )</td>
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</table>

Kinetic energy

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<tbody>
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<tr>
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<td>( y_4 )</td>
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### Potential energy

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<th>$y_4$</th>
</tr>
</thead>
<tbody>
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<td>$-\frac{1113}{1024}$</td>
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<td>$\frac{135}{256}$</td>
<td>$49$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$\frac{159}{512}$</td>
<td>$\frac{135}{256}$</td>
<td>$\frac{27}{128}$</td>
<td>$\frac{7}{512}$</td>
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<tr>
<td>$y_4$</td>
<td>$\frac{185}{4096}$</td>
<td>$\frac{49}{1024}$</td>
<td>$\frac{7}{512}$</td>
<td>$-\frac{1203}{4096}$</td>
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### $\frac{1}{r_{12}}$ matrix

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### Potential energy without interaction

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<td>$-\frac{1}{4}$</td>
<td>$0$</td>
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<tr>
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### Energy without interaction

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<tr>
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<td>$-\frac{3}{16}$</td>
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### Total energy

<table>
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</thead>
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<tr>
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<td>$\frac{-921}{1024}$</td>
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<td>$y_2$</td>
<td>$\frac{921}{1024}$</td>
<td>$\frac{-317}{256}$</td>
<td>$\frac{103}{256}$</td>
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<tr>
<td>$y_3$</td>
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<td>$\frac{-103}{256}$</td>
<td>$\frac{-19}{128}$</td>
<td>$\frac{7}{512}$</td>
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<tr>
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<td>$\frac{-49}{1024}$</td>
<td>$\frac{7}{512}$</td>
<td>$\frac{435}{4096}$</td>
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</tbody>
</table>

#### 3.6.7 2s2s$^1$S And 2p2p$^1$S Terms

$2s2s^1S$: $y_1 - y_2 + y_3 = q$,

$2p2p^1S$: $y_4$.

$$H = T + U = T + U_0 + \frac{1}{r_{12}}$$

$$\int (y_1 - y_2 + y_3)^2 \frac{L}{(y_1 - y_2 + y_3)} d\tau = L_{11} + L_{22} + L_{33} - 2L_{12} + 2L_{23}$$

$$= \int qLq d\tau$$

$$\int q^2 d\tau = \frac{9}{16} + \frac{21}{32} + \frac{1}{16} - \frac{9}{8} + \frac{9}{32} - \frac{3}{8} = \frac{1}{16};$$

$$\int qU_0 q d\tau = -\frac{9}{8} - \frac{15}{8} - \frac{1}{4} + \frac{21}{8} - \frac{3}{4} + \frac{5}{4} = -\frac{1}{8};$$

Remember that the 2s2s$^1$S and 2p2p$^1$S terms are superpositions of the terms called X and Y by the author. The notation used here is the same as in the previous subsection.
\[
\int qTq \, d\tau = \frac{3}{16} + \frac{11}{32} + \frac{1}{16} - \frac{3}{8} + \frac{3}{32} - \frac{1}{4} = \frac{1}{16};
\]
\[
\int q \frac{1}{r_{12}} q \, d\tau = \frac{837}{4096} + \frac{75}{256} + \frac{5}{128} - \frac{231}{512} + \frac{33}{256} - \frac{25}{128} = \frac{77}{4096};
\]
\[
\int qHq \, d\tau = -\frac{3003}{4096} - \frac{317}{256} - \frac{19}{128} + \frac{921}{512} - \frac{135}{256} + \frac{103}{128} = -\frac{179}{4096}.
\]
\[
\int y_4^2 \, d\tau = \frac{3}{16};
\]
\[
\int y_4 U_0 y_4 \, d\tau = -\frac{3}{8};
\]
\[
\int y_4 T y_4 \, d\tau = \frac{3}{16};
\]
\[
\int y_4 r_{12} y_4 \, d\tau = \frac{333}{4096};
\]
\[
\int y_4 H y_4 \, d\tau = -\frac{435}{4096}.
\]
\[
\int y_1 (y_1 - y_2 + y_3) \, d\tau = \frac{9}{16} - \frac{9}{16} + \frac{9}{64} = \frac{9}{64},
\]
\[
\int y_1 (y_1 - y_2 + y_3) \, d\tau = \frac{9}{16} - \frac{21}{32} + \frac{3}{16} = \frac{3}{32},
\]
\[
\int y_3 (y_1 - y_2 + y_3) \, d\tau = \frac{9}{64} - \frac{3}{16} + \frac{1}{16} = \frac{1}{64}.
\]

### 3.6.8 1s1s Term

\(\psi \sim e^{-r_1-r_2},\)

\[
\int r_1^2 e^{-2r_1} \, dr_1 \int r_2^2 e^{-2r_2} \, dr_2 = \frac{1}{16},
\]

\[\psi^2 = 16 e^{-2r_1-2r_2}.\]

\(R - r < \ell < R + r, \ dp = \frac{\ell}{2Rr} \, d\ell.\)
\[ \int \frac{1}{\ell} \, dp = \int_{R-r}^{R+r} \frac{dl}{2Rr} = \frac{1}{R}, \]
\[ \int \frac{1}{\ell^2} \, dp = \frac{1}{2Rr} \int_{2-r}^{R+r} \frac{dl}{\ell} = \frac{1}{2Rr} \log \frac{R + r}{R - r}, \]
\[ \int (p + r_1) e^{-2p} \, dp = - \left( \frac{1}{2} p + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2p} + \frac{1}{4} r_1 + \frac{1}{4}. \]
\[ \int e^{-2r_1} \left\{ \left[ - \left( \frac{1}{2} p + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2p} + \frac{1}{2} r_1 + \frac{1}{4} \right] \log \frac{p + 2r_1}{p} \right. \]
\[ + \left. \left[ - \left( \frac{1}{2} p + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2p} + \frac{1}{2} r_1 + \frac{1}{4} \right] \frac{1}{p(2r_1 + p)} \right\} \, dp. \]
\[ \frac{1}{2Rr} \log \frac{R + r}{R - r} = \frac{1}{R^2} \left( 1 + \frac{1}{3} \frac{r^2}{R^2} + \frac{1}{5} \frac{r^4}{R^4} + \ldots + \frac{1}{2n+1} \frac{r^{2n}}{R^{2n}} + \ldots \right). \]
\[ \int \frac{1}{r_{12}} \psi^2 \, dr = 32 \left\{ \int_0^\infty r_1^2 e^{-2r_1} \, dr_1 \int_0^\infty e^{-2r_2} \, dr_2 \right. \]
\[ + \frac{1}{3} \int_0^\infty r_1^4 e^{-2r_1} \, dr_1 \int_0^\infty \frac{1}{r_2^2} e^{-2r_2} \, dr_2 \]
\[ + \frac{1}{5} \int_0^\infty r_1^6 e^{-2r_1} \, dr_1 \int_0^\infty \frac{1}{r_2^4} e^{-2r_2} \, dr_2 + \ldots \}, \]
\[ r_2 = tr_1 \ (t > 1): \]
\[ 16 \int \frac{1}{r_{12}} e^{-2r_1-2r_2} \, d\tau = 32 \int_{r_2 > r_1} \frac{1}{r_{12}} e^{-2r_1-2r_2} \, d\tau \]
\[ = 32 \int_{t > 1} \frac{1}{r_{12}} e^{-(2+2t)r_1} \, d\tau. \]
\[ r_1^2 r_2^2 dr_1 dr_2 = t^2 r_1^2 dr_1 d\tau \]

\[ \frac{1}{2r_1 r_2} \log \frac{r_2 + r_1}{r_2 - r_1} = \frac{1}{2r_1^2 t} \log \frac{t + 1}{t - 1}. \]

\[
\int \frac{1}{r_{12}} \psi^2 d\tau = 32 \int_{t>1} e^{-(2+2t)r_1} dt \\
= 16 \int_{t>1} tr_1^3 \log \frac{t + 1}{t - 1} e^{-2(1+t)r_1} dr_1 dt \\
= 16 \int_1^\infty t \log \frac{t + 1}{t - 1} dt \int_0^\infty r_1 e^{-2(1+t)r_1} dr_1 \\
= 6 \int_1^\infty \frac{t}{(t+1)^4} \log \frac{t + 1}{t - 1} dt,
\]

\[
\frac{t + 1}{t - 1} = e^r, \quad dt = \frac{-2e^r}{(e^r - 1)^2}, \\
\frac{t}{t + 1} = \frac{e^r + 1}{e^r - 1}, \quad t + 1 = \frac{2e^r}{e^r - 1}, \\
\frac{1}{t + 1} = \frac{e^r - 1}{2e^r}, \quad t = \frac{(e^r - 1)^3(e^r + 1)}{16e^{4r}}.
\]

\[
\frac{t}{(t + 1)^4} \log \frac{t + 1}{t - 1} dt = -\frac{e^r + 1}{e^r - 1} \frac{(e^r - 1)^4}{16e^{4r}} r \frac{2e^r}{(e^r - 1)^2} dr \\
= -\frac{(e^r + 1)(e^r - 1)}{8e^{2r}} dr.
\]

\[
\int \frac{1}{r_{12}} \psi^2 d\tau = \frac{3}{4} \int_0^\infty (e^{-r} + e^{-3r}) dr = \frac{3}{4} \left( 1 - \frac{1}{9} \right) = \frac{2}{3}.
\]

The probability curve \( p(\ell) \) \((r_1 + r_2 > \ell, \ |r_1 - r_2| < \ell)\) for the mutual distance \( r_{12} \) is obtained as follows.

\[ \psi = 4 e^{-r_1 - r_2}, \quad \psi^2 = 16 e^{-2r_1 - 2r_2}. \]

\[
p(\ell) = 8\ell \int_0^\ell r_1 e^{-2r_1} dr_1 \int_{|r_1-\ell|}^{r_1+\ell} r_2 e^{-2r_2} dr_2 \\
= 8\ell \left\{ \int_0^\ell r_1 e^{-2r_1} dr_1 \int_{|r_1-\ell|}^{r_1+\ell} r_2 e^{-2r_2} dr_2 \\
+ \int_\ell^\infty r_1 e^{-2r_1} dr_1 \int_{r_1-\ell}^{r_1+\ell} r_2 e^{-2r_2} dr_2 \right\}.
\]
\[
\int r_2 e^{-2r_2} dr_2 = - \left( \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_2},
\]
\[
\int_{r_1}^{r_1+\ell} r_2 e^{-2r_2} dr_2 = \left( \frac{1}{2} \ell - \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2\ell+2r_1} - \left( \frac{1}{2} \ell + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2\ell-2r_1},
\]
\[
\int_{r_1-\ell}^{r_1+\ell} e^{-2r_2} dr_2 = \left( \frac{1}{2} r_1 - \frac{1}{2} \ell + \frac{1}{4} \right) e^{-2r_1+2\ell} - \left( \frac{1}{2} r_1 + \frac{1}{2} \ell + \frac{1}{4} \right) e^{-2r_1-2\ell}.
\]
\[
p(\ell) = 8\ell \left\{ e^{-2\ell} \int_0^{\ell} \left( -\frac{1}{2} r_1^2 + \frac{1}{2} \ell r_1 + \frac{1}{4} r_1 \right) dr_1 \right.
\]
\[
+ e^{2\ell} \int_{\ell}^{\infty} \left( \frac{1}{2} r_1^2 - \frac{1}{2} \ell r_1 + \frac{1}{4} r_1 \right) e^{-4r_1} dr_1
\]
\[
- e^{-2\ell} \int_0^{\infty} \left( \frac{1}{2} r_1^2 + \frac{1}{2} \ell r_1 + \frac{1}{4} r_1 \right) e^{-4r_1} dr_1 \right\}
\]
\[
= 8\ell e^{-2\ell} \left( \frac{1}{12} \ell^3 + \frac{1}{8} \ell^2 + \frac{1}{16} \ell \right).
\]
\[
p(\ell) = \left( \frac{1}{2} \ell^2 + \ell^3 + \frac{2}{5} \ell^4 \right) e^{-2\ell} = \left( \frac{1}{2} + \ell + \frac{2}{3} \ell^2 \right) \ell^2 e^{-2\ell}.
\]
\[
\int e^{-2x} dx = \frac{1}{2}, \quad \int xe^{-2x} dx = \frac{1}{4}, \quad \int x^2 e^{-2x} dx = \frac{1}{4}, \quad \int x^3 e^{-2x} dx = \frac{3}{8}, \quad \int x^4 e^{-2x} dx = \frac{3}{4}, \quad \int x^5 e^{-2x} dx = \frac{15}{8}.
\]
\[ r_{12}^2 = \int_0^\infty \ell^2 p(\ell) \, d\ell = \ldots, \]
\[ r_{12} = \int_0^\infty \ell p(\ell) \, d\ell = \frac{3}{16} + \frac{3}{4} + \frac{5}{4} = \frac{35}{16}, \]
\[ 1 = \int_0^\infty p(\ell) \, d\ell = \frac{1}{8} + \frac{3}{8} + \frac{1}{2} = 1, \]
\[ \frac{1}{r_{12}} = \int_0^\infty \frac{1}{\ell} p(\ell) \, d\ell = \frac{1}{8} + \frac{1}{4} + \frac{1}{4} = \frac{5}{8}, \]
\[ \frac{1}{r_{12}^2} = \int_0^\infty \frac{1}{\ell^2} p(\ell) \, d\ell = \frac{1}{4} + \frac{1}{4} + \frac{1}{6} = \frac{2}{3}. \]

\[ p'(\ell) = \left( \ell + 2\ell^2 + \frac{2}{3} \ell^3 - \frac{4}{3} \ell^4 \right) e^{-2\ell}. \]

### 3.6.9 1s2s Term

The states are now given by:

\[ e^{-r_1 - r_2}, \quad (r_2 - 2)e^{-r_1 - \frac{1}{2}r_2}, \]

where the normalization factors are:

\[ N_1 = 16, \quad N_2 = 2, \quad N_1N_2 = \frac{1}{8}, \quad \frac{1}{\sqrt{N_1N_2}} = 2\sqrt{2}, \]

so that:

\[ 4e^{-r_1 - r_2}, \quad \frac{1}{\sqrt{2}} (r_2 - 2)e^{-r_1 - \frac{1}{2}r_2}. \]

\[ \int r_1^2 e^{-2r_1} \, dr_1 = - \left( \frac{1}{2} r_1^2 + \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1}, \]
\[ \frac{1}{r_2} \int_0^{r_2} r_1^2 e^{-2r_1} \, dr_1 = \frac{1}{4r_2} - \left( \frac{1}{4r_2} + \frac{1}{2} + \frac{1}{2} r_2 \right) e^{-2r_2}, \]
\[ \int r_1 e^{-2r_1} \, dr_1 = - \left( \frac{1}{2} r_1 + \frac{1}{4} \right) e^{-2r_1}, \]
\[ \int_{r_2}^\infty r_1 e^{-2r_1} \, dr_1 = \left( \frac{1}{4} + \frac{1}{2} r_2 \right) e^{-2r_2}. \]
\[
\int_0^\infty r_1^2 e^{-r_1} dr_1 \int_0^\infty (r_2^2 - 2r_2) e^{-\frac{3}{2} r_2} dr_2 \\
+ \int_0^\infty (r_2^3 - 2r_2^2) e^{-\frac{3}{2} r_2} \int_0^\infty r_1 e^{-r_1} dr_1 \\
= \int_0^\infty \left( \frac{2}{3} r_1^4 - \frac{4}{9} r_1^3 - \frac{8}{27} r_1^2 \right) e^{-\frac{3}{2} r_1} dr_1 \\
+ \int_0^\infty \left( \frac{1}{2} r_2^4 - \frac{3}{4} r_2^3 - \frac{1}{2} r_2^2 \right) e^{-\frac{3}{2} r_2} dr_2 \\
= \frac{2}{3} \cdot \frac{24}{7^5} \cdot \frac{32}{9} \cdot \frac{6}{7^4} \cdot \frac{16}{27} \cdot \frac{2}{7^3} \cdot \frac{8}{7^3} \\
+ \frac{1}{2} \cdot \frac{24}{7^5} \cdot \frac{32}{4} \cdot \frac{3}{4} \cdot \frac{6}{7^4} \cdot \frac{1}{2} \cdot \frac{2}{7^3} \cdot \frac{8}{7^3} \\
= \frac{1}{7^3} \left( \frac{512}{49} \cdot \frac{128}{3} \cdot \frac{128}{27} + \frac{384}{49} \cdot \frac{72}{7} \cdot \frac{7}{8} \right).
\]

3.6.10 Continuation

\[
e^{-Z(r_1+r_2)}:
\]

\[
H\psi = \left( -Z^2 + \frac{1}{r_{12}^2} \right) \psi, \quad H\psi \cdot H\psi = \left( Z^4 - \frac{2r_2^2}{r_{12}} + \frac{1}{r_{12}^4} \right) \psi^2.
\]

\[
\bar{H} = Z^2 - \frac{5}{8} Z, \quad (\bar{H})^2 = Z^4 - \frac{5}{4} Z^3 + \frac{25}{64} Z^2;
\]

\[
\int H\psi \cdot H\psi d\tau = \int \psi H^2 \psi d\tau = \bar{H}^2 = Z^4 - \frac{5}{4} Z^3 + \frac{2}{3} Z^2 = (\bar{H})^2 + \frac{53}{192} Z^2.
\]

\[
e^{\left( 5Z \right) - \frac{5}{8} (r_1+r_2)}, \quad Z^* = Z - \frac{5}{16}:
\]

\[
H\psi = \left( -Z^*^2 - \frac{5}{16} \frac{1}{r_1} - \frac{5}{16} \frac{1}{r_2} + \frac{1}{r_{12}} \right) \psi,
\]

\[
H\psi \cdot H\psi = \left( Z^{*4} - \frac{5}{8} \frac{Z^*^2}{r_1} - \frac{5}{8} \frac{Z^*^3}{r_2} + \frac{2Z^*}{r_{12}} + \frac{25}{256} \frac{1}{r_1^2} \right.
\]

\[
+ \frac{25}{256} \frac{1}{r_2^2} + \frac{25}{128} \frac{1}{r_1 r_2} - \frac{5}{8r_{12}} - \frac{5}{8r_{12}} + \frac{1}{r_{12}^2} \right).
\]
$\frac{1}{\lambda}$ method:

$1s2s^1S : \quad \frac{5}{4} (Z^2 - 0.1855)^2 = \frac{5}{4} Z^2 - 0.4637Z + 0.0430,$

$1s2s^3S : \quad \frac{5}{4} (Z^2 - 0.1503)^2 = \frac{5}{4} Z^2 - 0.3758Z + 0.0282.$

$1s2s^1S - Z^2 : \quad \frac{1}{4} Z^2 - 0.4637Z + 0.0430,$

$1s2s^3S - Z^2 : \quad \frac{1}{4} Z^2 - 0.3758Z + 0.0282.$

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### 3.6.11 Other Terms

Normalization matrix

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$p_1 = y_1 - 3y_2 + 9y_3,$

$p_2 = y_1 - 2y_2 + 3y_3,$

$p_3 = y_1 - y_2 + y_3,$

$p_4 = y_4.$
### Kinetic energy

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### Energy without interaction

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### Interaction ($1/r_{12}$)

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### Total energy $\lambda = 1$

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E. MAJORANA: RESEARCH NOTES ON THEORETICAL PHYSICS

Total potential energy

<table>
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\[ \int p_1 L p_1 \, d\tau = L_{11} + 9L_{22} + 81L_{33} + 18L_{13} - 6L_{12} - 54L_{23}, \]
\[ \int p_2 L p_1 \, d\tau = L_{11} + 6L_{22} + 27L_{33} + 12L_{13} - 5L_{12} - 27L_{23}, \]
\[ \int p_3 L p_1 \, d\tau = L_{11} + 3L_{22} + 9L_{33} + 10L_{13} - 4L_{12} - 12L_{23}, \]
\[ \int p_4 L p_1 \, d\tau = L_{14} - 3L_{24} + 9L_{24}, \]
\[ \int p_2 L p_2 \, d\tau = L_{11} + 4L_{22} + 9L_{33} + 6L_{13} - 4L_{12} - 12L_{23}, \]
\[ \int p_3 L p_2 \, d\tau = L_{11} + 2L_{22} + 3L_{33} + 4L_{13} - 3L_{12} - 5L_{23}, \]
\[ \int p_4 L p_2 \, d\tau = L_{14} - 2L_{24} + 3L_{34}, \]
\[ \int p_3 L p_3 \, d\tau = L_{11} + L_{22} + L_{33} + 2L_{13} - 2L_{12} - 2L_{23}, \]
\[ \int p_4 L p_3 \, d\tau = L_{14} - L_{24} + L_{34}, \]

\[13@\text{The author evaluates the matrix elements of operators } L, \text{ between } p \text{ states, in terms of those between } y \text{ states, already considered on the previous pages. In the following, we do not report the mere arithmetic calculations aimed at obtaining the numbers given in the tables.}\]
\[ \int p_4 L_4 \, d\tau = L_{44}. \]

\[ q_1 = \frac{4}{3} p_1, \]
\[ q_2 = 4 \sqrt{\frac{2}{3}} p_2, \]
\[ X = 4 \sqrt{\frac{1}{2} + \frac{17}{4 \sqrt{241}}} p_3 - 4 \sqrt{\frac{1}{2} - \frac{17}{4 \sqrt{241}}} p_4, \]
\[ Y' = 4 \sqrt{\frac{1}{2} + \frac{17}{4 \sqrt{241}}} p_3 + 4 \sqrt{\frac{1}{2} + \frac{17}{4 \sqrt{241}}} p_4; \]

\[ p_1 = \frac{3}{4} q_1, \]
\[ p_2 = \frac{1}{4} \sqrt{\frac{3}{2}} q_2, \]
\[ p_3 = \frac{1}{4} \sqrt{\frac{1}{2} + \frac{17}{4 \sqrt{241}}} X + \frac{1}{4} \sqrt{\frac{1}{2} - \frac{17}{4 \sqrt{241}}} Y', \]
\[ p_4 = -\frac{\sqrt{3}}{4} \sqrt{\frac{1}{2} - \frac{17}{4 \sqrt{241}}} X + \frac{\sqrt{3}}{4} \sqrt{\frac{1}{2} + \frac{17}{4 \sqrt{241}}} Y'. \]

\[ \{14\} \]

\[ \int q_1 A q_1 \, d\tau = \frac{16}{9} A^{11}, \]
\[ \int q_2 A q_1 \, d\tau = \frac{16}{3} \sqrt{\frac{2}{3}} A^{12}, \]
\[ \int X A q_1 \, d\tau = \frac{16}{3} \sqrt{\frac{1}{2} + \frac{17}{4 \sqrt{241}}} A^{13} - \frac{16}{3} \sqrt{\frac{1}{2} - \frac{17}{4 \sqrt{241}}} A^{14}, \]

\[ ^{14}\text{@ For the new states considered by the author, see the previous footnote.} \]
\[ \int Y^1 Aq_1 \,d\tau = \frac{16}{9} \sqrt{\frac{1}{2} - \frac{17}{4\sqrt{241}}} \, A^{13} + \frac{16}{3\sqrt{3}} \sqrt{\frac{1}{2} + \frac{17}{4\sqrt{241}}} \, A^{14}, \]
\[ \int q_2 Aq_2 \,d\tau = \frac{32}{3} \, A^{22}, \]
\[ \int X Aq_2 \,d\tau = 16 \sqrt{\frac{2}{3}} \left( \frac{1}{2} + \frac{17}{4\sqrt{241}} \right) A^{23} - \frac{16\sqrt{2}}{3} \sqrt{\frac{1}{2} - \frac{17}{4\sqrt{241}}} \, A^{24}, \]
\[ \int Y' Aq_2 \,d\tau = 16 \sqrt{\frac{2}{3}} \left( \frac{1}{2} - \frac{17}{4\sqrt{241}} \right) A^{23} + \frac{16\sqrt{2}}{3} \sqrt{\frac{1}{2} + \frac{17}{4\sqrt{241}}} \, A^{24}, \]
\[ \int XAX \,d\tau = 16 \left( \frac{1}{2} + \frac{17}{4\sqrt{241}} \right) A^{33} + \frac{16}{3} \left( \frac{1}{2} - \frac{17}{4\sqrt{241}} \right) A^{44} \]
\[ - \frac{16}{\sqrt{3}} \sqrt{\frac{675}{964}} \, A^{34}, \]
\[ \int Y'AX \,d\tau = 668 \sqrt{\frac{675}{964}} \, A^{33} - \frac{8}{3} \sqrt{\frac{675}{964}} \, A^{44} + \frac{16}{\sqrt{3}} \frac{17}{2\sqrt{241}} \, A^{34}, \]
\[ \int Y'AY' \,d\tau = 16 \left( \frac{1}{2} - \frac{17}{4\sqrt{241}} \right) A^{33} + \frac{16}{3} \left( \frac{1}{2} + \frac{17}{4\sqrt{241}} \right) A^{44} \]
\[ + \frac{16}{\sqrt{3}} \sqrt{\frac{675}{965}} \, A^{34}. \]

\[ [15] \]

\[ XX : \quad 12.38026 \, A^{33} + 1.20658 \, A^{44} - 7.72988 \, A^{34}, \]
\[ Y'Y' : \quad 3.61974 \, A^{33} + 4.12675 \, A^{44} + 7.72988 \, A^{34}, \]
\[ XY' : \quad 6.69427 \, A^{33} - 2.23142 \, A^{44} + 5.05789 \, A^{34}, \]
\[ Xq_1 : \quad 4.691 \, A^{13} - 1.465 \, A^{14}, \]
\[ Xq_2 : \quad 11.492 \, A^{23} - 3.588 \, A^{24}, \]
\[ Y'q_1 : \quad 2.5368 \, A^{13} + 2.7086 \, A^{14}, \]
\[ Y'q_2 : \quad 6.214 \, A^{23} + 6.635 \, A^{34}, \]

\[ [15] \text{In the original manuscript some numerical (arithmetic) calculations are given (not reported here), leading to the following expressions for the matrix elements.} \]
### Normalization matrix

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### Total energy $\lambda = 1$

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### 3.7. GROUND STATE OF THREE-ELECTRON ATOMS

An approximate expression for the energy (in rydbergs) \( W \) (which is equal to half the mean value of the potential energy) of the ground state of three-electron atoms with charge \( Z \) is here obtained, starting from particular forms for the wavefunctions \( \psi \) (or radial wavefunctions \( \chi \)) of the three electrons. For further details, see Sect. 15 of Volumetto III, referring to the case of two-electron atoms.

For \( Z \to \infty \) (\( \rho = Zr \)):

\[
\psi_1 = \psi_2 = a e^{-\rho}, \quad \psi_3 = \frac{a}{4\sqrt{2}} (2 - \rho) e^{-\rho/2}.
\]

\[
\chi_1 = \chi_2 = a \rho e^{-\rho}, \quad \chi_3 = \frac{a}{4\sqrt{2}} \rho (2 - \rho) e^{-\rho/2}.
\]

\[
2 \int \frac{1}{r_{12}} \psi_1^2(q_1) \psi_2^2(q_2) \, dq_1 \, dq_2 = \frac{5}{4},
\]

\[
2 \int \frac{1}{r_{13}} \psi_1^2(q_1) \psi_3^2(q_3) \, dq_1 \, dq_3 = \frac{1}{2} - \frac{13}{162} = \frac{1}{2} - 0.0802 = 0.4198,
\]

\[
2 \int \frac{1}{r_{13}} \psi_1(q_1) \psi_3(q_1) \psi_1(q_3) \psi_3(q_3) \, dq_1 \, dq_3 = \frac{32}{729} = 0.0439.
\]
\[ -W = \frac{q}{4}Z^2 - \frac{5}{4}Z - 2\left(\frac{1}{2} - \frac{13}{162}\right)Z + \frac{32}{729}Z \]
\[ = \frac{9}{4}Z^2 - \left(\frac{5}{4} + \frac{580}{729}\right)Z \]
\[ = 2Z^2 - \frac{5}{4}Z + \frac{1}{4}Z^2 - \frac{580}{729}Z \]
\[ = 2Z^2 - \frac{5}{4}Z + \frac{1}{4}Z^2 - 0.7956Z \]
\[ = 2Z^2 - \frac{5}{4}Z + \frac{1}{4}(Z^2 - 3.1824Z) \]
\[ = 2Z^2 - \frac{5}{4}Z + \frac{1}{4}(Z^2 - 4Z + 0.8176Z). \]

[16]

3.8. GROUND STATE OF THE LITHIUM ATOM

3.8.1 Electrostatic Potential

An expression for the electrostatic potential energy \( V \) of the lithium atom is obtained as a function of the distance \( r \) from the nucleus, by means of a semiclassical approach (a Poisson equation for \( V \) with an effective charge density). A table with numerical values for this potential is given as well. See also Sect. 3.11.

\[ 2 \int \varphi^2(q_1, q_2) dq_2 = ke^{-43r_1/8}. \]
\[ -\left(\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr}\right) = ke^{-43r/8}, \]
\[ -\frac{d^2(rV)}{dr^2} = kr e^{-43r/8} = kr e^{-\alpha r}, \]
\[ -\frac{d(rV)}{dr} = -\frac{k}{\alpha} \left( r + \frac{1}{\alpha} \right) e^{-\alpha k}, \]
\[ -rV = \frac{k}{\alpha^2} \left( r + \frac{2}{\alpha} \right) e^{-\alpha r} + 1. \]

\[ @ \text{In the original manuscript, in the last line of the previous expression, the first two terms are missing.} \]
\[ k = \alpha^3; \]

\[
-V = \frac{1}{r} + \left( \frac{2}{r} + \frac{43}{8} \right) e^{-\frac{43}{8}r},
\]

\[
-2V = \frac{2}{r} + \left( \frac{4}{r} + \frac{43}{4} \right) e^{-\frac{43}{8}r}.
\]

\[ \chi'' + 2(E - V)X = 0; \]

### 3.8.2 Ground State

The electrostatic potential inside the lithium atom considered above is now used in order to determine (mainly, numerically) the Schrödinger radial wavefunction for the ground state of this atom.

\[ \chi'' + 2(E - V)X = 0; \]

\[17\] The following expression for the electrostatic potential holds for the 2s term of lithium.

\[18\] The numerical values reported in the following table are obtained from the expression of \( V \) given just above. In the original manuscript the value in the sixth column corresponding to \( r = 2 \) is erroneously written as 2.9997 (instead of 1.9997).
\[ W = -2E; \]
\[
\chi'' = (2V + W)\chi.
\]
\[
2V = \frac{6}{r} + 10.75 + \ldots,
\]
\[
\chi'' = \left(\frac{6}{r} + 10.75 + W + \ldots\right)\chi.
\]
\[
\chi = x + ax^2 + bx^3 + \ldots; \quad (x = r)
\]
\[
\chi'' = 2a + 6bx + \ldots;
\]
\[
2a + 6br + \ldots = \left(\frac{6}{r} + 10.75 + W + \ldots\right)(r + ar^2 + \ldots),
\]
\[
2a + 6br = -6 + (10.75 + W - 6a)r;
\]
\[
a = -3, \quad b = \frac{28.75 + W}{6}.
\]

\[ [20] \]

\[ [19] \text{The energy } W \text{ is measured in rydbergs.} \]
\[ [20] \text{In the following tables, Majorana gave the numerical values of the Schrödinger radial wavefunction } \chi \text{ (and its derivatives) for some values of } r. \text{ They should have been obtained by solving the differential equation reported above. However, it is interesting to note that the quoted numerical values do not come out neither by using the series expansion method outlined in the notes (method I), nor by solving numerically the equation with the approximate expression for the potential quoted just above (method II). Probably, the numerical values given by the author were obtained by considering the complete potential considered at the end of the previous subsection (method III). To give an idea of the departure from the Majorana tables, in the following we give some values of } \chi \text{ and its derivatives for } W = 0.32, \text{ obtained by using the mentioned three methods. Notice that the series solution can be applied only for } r \ll 1: \]

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**Numerical solution (method III)**

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3.9. ASYMPTOTIC BEHAVIOR FOR THE $s$ TERMS IN ALKALI

The author looked for a solution of the Schrödinger equation for alkali metals, at large distances from the nucleus. In such an asymptotic limit the potential energy experienced by the external electron is approximatively coulombian. Two different methods were considered: in the first one, the eigenfunction is written in the form of a polynomial times an exponential decreasing factor, while the second one is that typical of homogeneous differential equations (for lowering the order of the equation by one unit).
3.9.1 First Method

\[ E = -2W. \]  

\[ y'' = \left( -\frac{2}{r} + E \right) y. \]

\[ y = P \ e^{-\sqrt{Ex}}, \]
\[ y' = (P' - \sqrt{EP}) \ e^{-\sqrt{Ex}}, \]
\[ y'' = (P'' - 2\sqrt{EP'} + EP) \ e^{-\sqrt{Ex}}, \]

\[ P'' - 2\sqrt{EP'} + EP = \left( -\frac{2}{r} + E \right) P, \]
\[ P'' - 2\sqrt{EP'} + \frac{2}{r} P = 0. \]

\[ P = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots, \]
\[ P' = n\alpha_n x^{n-1} + (n - 1)\alpha_{n-1} x^{n-2} + \ldots, \]
\[ P'' = n(n-1)\alpha_n x^{n-2} + (n - 1)(n - 2)\alpha_{n-1} x^{n-3} + \ldots. \]

\[ (r + 1) r \alpha_{r+1} - 2r\sqrt{E} \alpha_r + 2\alpha_r = 0; \]
\[ \alpha_{r+1} = \frac{2(r\sqrt{E} - 1)}{r(r + 1)} \alpha_r, \quad \alpha_r = \frac{r(r + 1)}{2(r\sqrt{E} - 1)} \alpha_{r+1}. \]

\[ n = \frac{1}{\sqrt{E}}, \quad E = \frac{1}{n^2}. \]

For \( n \to \infty \), \( \alpha_n = 1 \) and
\[ \alpha_{n-1} = \frac{-(n - 1)n}{2(1 - (n - 1)\sqrt{E})} = -\frac{(n - 1)n^2}{2}. \]

\[ ^{21} \text{Observe that the author apparently uses } x \text{ or } r \text{ to denote the same quantity. However, below, it is } r = k + x, \text{ quantity } k \text{ being the distance from the last node of the eigenfunction.} \]
Denoting with \( D \) the distance from the last node: \(^{22} \)

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Denoting with \( k \) the distance from the last node:

\[
r = k + x,
\]

\[
P'' - 2\sqrt{E}P' + \frac{2}{r}P = 0,
\]

\[
P'' - 2\sqrt{E}P' + \frac{2}{k+x}P = 0.
\]

\[
\begin{align*}
P &= a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \\
P' &= a_1 + 2a_2 x + 3a_3 x^2 + \ldots, \\
P'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots.
\end{align*}
\]

\[
\frac{1}{k+x} = \frac{1}{k} - \frac{x}{k^2} + \frac{x^2}{k^3} - \frac{x^3}{k^4} + \ldots.
\]

\(^{22} @ \) In the following table the author puts for some approximated expressions for the polynomial \( P \) for some maximum values \( n \) of the index \( r \). For a given \( n \), the first one of the coefficient \( \alpha_n \) is equal to 1, while the other non-vanishing coefficients (with decreasing \( r \)) are obtained from the formula

\[
\alpha_r = \frac{r(r+1)}{2(r\sqrt{E} - 1)} \alpha_{r+1}
\]

on setting \( \sqrt{E} = 1/n \). In the last column of the table, Majorana reports the distance from \( x = 0 \) of the greatest root of the considered polynomial. In the following, such a distance will be indicated by \( k \).
The following method is useful in order to determine the coefficients of the series expansion for $P$ which satisfies the differential equation reported above.

\[ P'' = 2a_2 + 6a_3 + 12a_4 x^2 + 20a_5 x^3 \ldots \]
\[ -2\sqrt{E}P' = -2\sqrt{E}a_1 - 4\sqrt{E}a_2 x - 6\sqrt{E}a_3 x^2 - 8\sqrt{E}a_4 x^3 \ldots \]
\[ \frac{2}{k + x}P = \frac{2}{k}a_1 x + \frac{2}{k}a_2 x^2 + \frac{2}{k}a_3 x^3 \]
\[ \ldots \]

\[ 2a_2 - 2\sqrt{E}a_1 = 0, \]
\[ 6a_3 - 4\sqrt{E}a_2 + \frac{2}{k}a_1 = 0, \]
\[ 12a_4 - 6\sqrt{E}a_3 + \frac{2}{k}a_2 - \frac{2}{k^2}a_1 = 0, \]
\[ 20a_5 - 8\sqrt{E}a_4 + \frac{2}{k}a_3 - \frac{2}{k^2}a_2 + \frac{2}{k^3}a_1 = 0; \]

\[ a_2 = \sqrt{E}a_1, \]
\[ 3a_3 = 2\sqrt{E}a_2 - \frac{1}{k}a_1, \]
\[ 6a_4 = 3\sqrt{E}a_3 - \frac{1}{k}a_2 + \frac{1}{k^2}a_1, \]
\[ 10a_5 = 4\sqrt{E}a_4 - \frac{1}{k}a_3 + \frac{1}{k^2}a_2 - \frac{1}{k^3}a_1. \]

\[ a_2 = \sqrt{E}a_1, \]
\[ a_3 = \frac{2}{3}\sqrt{E}a_2 - \frac{1}{3} \frac{a_1}{k}, \]
\[ a_4 = \frac{2}{4}\sqrt{E}a_3 - \frac{1}{6} \left( \frac{a_2}{k} - \frac{a_1}{k^2} \right), \]
\[ a_5 = \frac{2}{5}\sqrt{E}a_4 - \frac{1}{10} \left( \frac{a_3}{k} - \frac{a_2}{k^2} + \frac{a_1}{k^3} \right), \]
\[ \ldots \]
\[ a_n = \frac{2}{n}\sqrt{E}a_{n-1} - \frac{2}{n(n-1)} \left( \frac{a_{n-2}}{k} - \frac{a_{n-3}}{k^2} + \frac{a_{n-4}}{k^3} \ldots \right). \]
\[ a_2 = \sqrt{E} a_1, \]
\[ a_3 = \frac{2}{3} E a_1 - \frac{1}{3} \frac{a_1}{k} = a_1 \left( \frac{2}{3} E - \frac{1}{3} \frac{1}{k} \right), \]
\[ a_4 = \frac{1}{3} e^2 a_1 - \frac{1}{6} \frac{a_1}{k} \sqrt{E} - \frac{1}{6} \frac{\sqrt{E} a_1}{k} + \frac{1}{6} \frac{a_1}{k^2} \]
\[ = a_1 \left( \frac{1}{3} e^2 - \frac{1}{3} \frac{E^{\frac{1}{2}}}{k} + \frac{1}{6} \frac{1}{k^2} \right). \]

\[ P \quad \approx \quad x + a x^{1/\sqrt{E}}, \]
\[ P' \quad \approx \quad 1 + \frac{a}{\sqrt{E}} x^{1/\sqrt{E}-1}, \]
\[ P'' \quad \approx \quad a \frac{1}{\sqrt{E}} \left( \frac{1}{\sqrt{E}} - 1 \right) x^{1/\sqrt{E}-2}. \]

\[ P'' - 2\sqrt{E} P' + \frac{2}{k + x} P \approx a \frac{1}{\sqrt{E}} \left( \frac{1}{\sqrt{E}} - 1 \right) x^{1/\sqrt{E}-2} - 2\sqrt{E} \]
\[ -2a x^{1/\sqrt{E}-1} + \frac{2x}{k + x} + \frac{2ax^{1/\sqrt{E}}}{k + x}. \]

\[ y'' = \left( E - \frac{2}{r} \right) y = \left( E - \frac{2}{k + x} \right) y \]
\[ = \left( E - \frac{2}{k} + 2 \frac{x}{k^2} - 2 \frac{x^2}{k^3} + \ldots \right) y. \]

Zeroth approximation:
\[ y'' = \left( E - \frac{2}{k} \right) y. \]

First approximation:
\[ y'' = \left( E - \frac{2}{k} + 2 \frac{x}{k^2} \right) y. \]

\[ x_1 = \left( \frac{k^2}{2} \right)^{2/3} \left( E - \frac{2}{k} + 2 \frac{x}{k^2} \right), \quad dx_1 = \left( \frac{2}{k^2} \right)^{1/3} dx. \]
\[
\frac{d^2 y}{dx_1^2} = \left( \frac{k^2}{2} \right)^{2/3} \left( E - \frac{2}{k} + \frac{2x}{k^2} \right) y = x_1 y.
\]

\[x = 0: \quad x_1 \approx -2.33;\]

\[-2.33 \approx \left( \frac{x^2}{2} \right)^{2/3} \left( E - \frac{2}{k} \right),\]

\[E - \frac{2}{k} \approx -2.33 \left( \frac{2}{k^2} \right)^{2/3},\]

\[E \approx \frac{2}{k} - 2.33 \left( \frac{2}{k^2} \right)^{2/3} = \frac{2}{k} - \frac{2.33 \cdot 2^{2/3}}{k^{4/3}} \approx \frac{2}{k} - 3.70 \frac{1}{k^{4/3}} + \ldots.\]

### 3.9.2 Second Method

\[y = e^{-\int u \, dx},\]

\[y' = -u \, y,\]

\[y'' = (u^2 - u')y.\]

\[u^2 - u' = -\frac{2}{r} + E,\]

\[u^2 - u' - E + \frac{2}{x} = 0.\]

\[u = \sqrt{E} - \frac{a_1}{x} - \frac{a_2}{x^2} - \frac{a_3}{x^3} - \frac{a_4}{x^4},\]

\[a_0 = -\sqrt{E} = -1/n.\]

\[u = -a_0 - \frac{a_1}{x} - \frac{a_2}{x^2} - \frac{a_3}{x^3} - \ldots,\]

\[u^1 = \frac{a_1}{x^2} + 2\frac{a_2}{x^3} + 3\frac{a_3}{x^4} + \ldots,\]

\[u^2 = a_0^2 + (a_0 a_1 + a_1 a_0) \frac{1}{x} + (a_0 a_2 + a_1 a_1 + a_2 a_0) \frac{1}{x^2} + \ldots\]

\[a_0 = -\sqrt{E} = -\frac{1}{n}.\]
\[ 2a_0a_1 + 2 = 0, \quad a_1 = -\frac{1}{a_0} = \frac{1}{\sqrt{E}} = n, \]

\[ a_0a_r + a_1a_{r-1} + \ldots + a_ra_0 - (r - 1)a_{r-1} = 0, \quad r \neq 0, 1. \]

For \( r > 0 \) it is:

\[
\begin{align*}
ar_{r+1} &= a_1a_r + a_2a_{r-1} + \ldots + a_ra_1 - ra_r \\
&= \frac{n}{2}(a_1a_r + a_2a_{r-1} + \ldots + a_ra_1 - ra_r).
\end{align*}
\]

\[
\begin{align*}
a_0 &= -\frac{1}{n}, \\
a_1 &= n, \\
a_2 &= \frac{n^3}{2} - \frac{n^2}{2}, \\
a_3 &= \frac{n^5}{2} - n^4 + \frac{n^3}{2}.
\end{align*}
\]

\[
u' = u^2 - E - \frac{2}{x}.
\]

\[
t = xE, \quad x = \frac{t}{E},
\]

\[
u = p\sqrt{E}, \quad p = \frac{u}{\sqrt{E}};
\]

\[
\frac{dp}{dt} = \frac{1}{E^{3/2}} \frac{du}{dx}.
\]

\[
\frac{dp}{dt} = \frac{1}{\sqrt{E}}(p^2 - 1) + \frac{2}{t\sqrt{E}},
\]

\[
\frac{dp}{dt} = n(p^2 - 1) + \frac{2n}{t},
\]

\[
p^2 - 1 + \frac{2}{t} = \sqrt{E}\frac{dp}{dt}.
\]
First approximation:

\[ p^2 - 1 + \frac{2}{t} = 0; \]
\[ p = \sqrt{1 - \frac{2}{t}}. \]

\[ ^{[24]} \]

3.10. ATOMIC EIGENFUNCTIONS I

In this part, the author searches for solutions of the Schrödinger equation with a screened Coulomb potential, likely to be applied to specific atomic problems, although it is not very clear what particular atom the author has in mind (probably he refers to the 1s term of lithium). See also the next Section.

In the following we give detailed comments of the mathematical passages reported which, otherwise, would result of unclear interpretation.

The equation:

\[ \chi'' + 2 \left[ E - V - \frac{k(k + 1)}{x^2} \right] \chi = 0 \]

can be solved by setting:

\[ \chi = x^{k+1} e^{-\int u \, dx}, \]
\[ \chi' = \left[ (k + 1)x^k - u x^{k+1} \right] e^{-\int u \, dx} = \left( \frac{k + 1}{x} - u \right) x^{k+1} e^{-\int u \, dx}, \]
\[ \chi'' = \left[ k(k + 1)x^{k-1} - 2(k + 1)u x^k - u' x^{k+1} + u^2 x^{k+1} \right] e^{-\int u \, dx} = \left[ \frac{k + 1}{x^2} - \frac{2(k + 1)u}{x} - u' + u^2 \right] x^{k+1} e^{-\int u \, dx}. \]

We then have the following equation for \( u \):

\[ u' = 2(E - V) - \frac{2(k + 1)}{x} u + u^2. \]

\[ ^{24} \text{This Section was left incomplete by the author.} \]
1st application. Let us consider the following form for the potential:

\[ V = a - \frac{b}{x}. \]

We thus have:

\[ u' = 2E - 2a + \frac{2b}{x} - \frac{2(k + 1)}{x} u + u^2, \]

and for \( u' = 0 \) we get:

\[ u = \frac{b}{k + 1}. \]

The energy eigenvalue is:

\[ E = -\frac{1}{2} \left[ \frac{b^2}{(k + 1)^2} - a \right]. \]

2nd application. For \( k = 0 \), let us consider a screened potential of the form \( V = -Z_V/x \), with

\[ Z_V = 9 - 24.3x + 0x^2, \]

and try a solution of the form:

\[ u = 9 - ax + bx^2. \]

By this substitution we have:

\[-a + 2bx \simeq 81 - 18ax + 2E - 48.6 + 2a - 2bx, \]

so that:

\[ a = -\frac{2}{3}E - 10.8, \quad b = -\frac{9}{2}a. \]

More in general, the equation:

\[ u' = u^2 + 2E + 2\frac{Z_V - u}{x}, \]

with:

\[ Z_V \sim 8.5 - 15x, \]

becomes:

\[ u' \sim u^2 + 2E + 30 + 2\frac{8.5 - u}{x}. \]
For \( u \sim 8.5 \) we get \( E \sim -21 \); other detailed results are reported in the following table\(^{25}\)\(^{26}\):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( Z_V )</th>
<th>( E = -20 )</th>
<th>( E = -21 )</th>
<th>( E = -20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9</td>
<td>9.000</td>
<td>-2.533</td>
<td>9.000</td>
</tr>
<tr>
<td>0.05</td>
<td>7.85</td>
<td>8.87</td>
<td>-2.1</td>
<td>8.83</td>
</tr>
<tr>
<td>0.10</td>
<td>6.92</td>
<td>8.81</td>
<td>-0.3</td>
<td>8.70</td>
</tr>
<tr>
<td>0.15</td>
<td>6.20</td>
<td>8.88</td>
<td>3.1</td>
<td>8.65</td>
</tr>
</tbody>
</table>

For very small \( x \), we have to push on the approximation; for example, for \( 0 < x < 0.05 \) we could use \( Z_V = 9 - 24.3x + 580x^3 \). We thus consider:

\[
Z_V = 9 - 24.3x + kx^3, \\
u = 9 - ax - bx^2 + cx^3,
\]

and substituting these expressions in the above differential equation for \( u \), we get the unknown coefficients:

\[
a = -\frac{2}{3}E - 10.8, \quad b = -\frac{9}{2}a, \quad c = \frac{1}{4}(a^2 - 81a + 2k).
\]

In such an approximation, for the function \( \chi \) defined above and satisfying now (for \( k = 0 \)) the equation

\[
\chi'' = -2 \left( \frac{Z_V}{x} + E \right) \chi,
\]

we obtain the values reported in the following tables:

<table>
<thead>
<tr>
<th>( Z_V )</th>
<th>( x )</th>
<th>( E = -21.4 )</th>
<th>( \chi )</th>
<th>( \chi' )</th>
<th>( \chi'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>1.000</td>
<td>-18</td>
<td></td>
</tr>
<tr>
<td>7.85</td>
<td>0.05</td>
<td>0.0320</td>
<td>0.357</td>
<td>-8.68</td>
<td></td>
</tr>
<tr>
<td>7.36</td>
<td>0.075</td>
<td>0.0385</td>
<td>0.177</td>
<td>-5.91</td>
<td></td>
</tr>
<tr>
<td>6.92</td>
<td>0.10</td>
<td>0.0413</td>
<td>0.055</td>
<td>-3.95</td>
<td></td>
</tr>
<tr>
<td>6.54</td>
<td>0.125</td>
<td>0.0416</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.20</td>
<td>0.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.90</td>
<td>0.175</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.63</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.14</td>
<td>0.25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.70</td>
<td>0.30</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.35)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.90</td>
<td>0.40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.45)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^{25}\) The table was evaluated by the author by successive iterations, as can be deduced from the numerical calculations reported in the original manuscript.
<table>
<thead>
<tr>
<th>$Z_V$</th>
<th>$x$</th>
<th>$E = -21.7$</th>
<th>$E = -22$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0</td>
<td>$\chi$ 1.000 $\chi'$ -18</td>
<td>$\chi$ 0.000 $\chi'$ -18</td>
</tr>
<tr>
<td>7.85</td>
<td>0.05</td>
<td>0.0320 0.358 -8.66</td>
<td>0.0320 0.359 -8.64</td>
</tr>
<tr>
<td>7.36</td>
<td>0.075</td>
<td>0.0385 0.178 -5.89</td>
<td>0.0386 0.180 -5.88</td>
</tr>
<tr>
<td>6.92</td>
<td>0.10</td>
<td>0.0414 0.057  -3.93</td>
<td>0.0415 0.059 -3.92</td>
</tr>
<tr>
<td>6.54</td>
<td>0.125</td>
<td>0.0418</td>
<td>0.0419</td>
</tr>
</tbody>
</table>

In the considered interval $0 < x < 0.05$ we could also use a screening factor $X_V = 9 - 23.2x$ and try for a solution of the form:

$$\chi = \sum_n c_n x^n.$$

Substituting it in the following equation:

$$\chi'' = -\left(\frac{18}{x} - 46.4 + 2E\right)\chi,$$

we get the following iterative expression for the coefficients:

$$n(n-1)c_n = -18c_{n-1} + (46.4 - 2E)c_{n-2},$$

$$c_n = \frac{-18}{n(n-1)}c_{n-1} + \frac{46.4 - 2E}{n(n-1)}c_{n-2}.$$

The first coefficients are\(^\text{27}^\text{27}\):

- $c_0 = 0$,
- $c_1 = 1$,
- $c_2 = -9$,
- $c_3 = 27 + \frac{46.4 - 2E}{6}$,
- $c_4 = \frac{-81}{2} - (46.4 - 2E)$,
- $c_5 = \frac{729}{20} + \frac{9}{4} (46.4 - 2E) + \frac{1}{120} (46.4 - 2E)^2$.

\(^{27}\) The original manuscript features some numerical calculations (whose interpretation seems unclear) that are apparently related to the solution here investigated.
3.11. ATOMIC EIGENFUNCTIONS II

The author looks for expressions for the atomic wavefunctions, obtained as solutions of the radial Schrödinger equation. An explicit series solution for a lithium wavefunction is reported.

\[ \chi'' + 2(E - V)\chi = 0. \]

For the 2s term of lithium:

\[ -V = \frac{1}{r} + \left( \frac{2}{r} + \frac{43}{8} \right) e^{-\frac{43}{8}r/8}. \]

\[ \chi = P e^{-\sqrt{-2E}} r = P e^{-r/n}, \]

\( (n = n^*). \)

\[ \chi' = \left( P' - \sqrt{-2E} P \right) e^{-\sqrt{-2E}} r, \]

\[ \chi'' = \left( P'' - 2\sqrt{-2E} P' - 2EP \right) e^{-\sqrt{-2E}} r. \]

\[ P'' - 2\sqrt{-2E} P' - 2VP = 0. \]

\[ \sqrt{-2E} = \frac{1}{n}, \quad n = n^*. \]

\[ P'' - \frac{2}{n} P' + \left[ \frac{2}{r} + \left( \frac{4}{r} + \frac{43}{4} \right) e^{-\frac{43}{8}r/8} \right] P = 0. \]

\[ P = \sum_{s=1}^{\infty} a_s r^s, \quad a_1 = 1, \quad a_0 = 0. \]

\[ s(s-1)a_s - \frac{2}{n}(s-1)a_{s-1} + 2a_{s-1} + 4 \sum_{\ell=0}^{s-2} \frac{(-43/8)^\ell}{\ell!} a_{s-1-\ell} \]

\[ + \frac{43}{4} \sum_{\ell=0}^{s-3} \frac{(-43/8)^\ell}{\ell!} a_{s-2-\ell} = 0. \]
\[ s(s-1)a_s - \frac{2}{n}(s-1)a_{s-1} + 2a_{s-1} + \sum_{\ell=0}^{s-2} (4 - 2\ell) \frac{(-43/8)^\ell}{\ell!} a_{s-1-\ell} = 0. \]

\[ \chi'' + \left[ -\frac{1}{n^2} + \frac{2}{r} + \left( \frac{4}{r} + \frac{43}{4} \right) e^{-43r/8} \right] \chi = 0, \quad E = -\frac{1}{2n^2}. \]

\[ \chi = \sum_{s=1}^{\infty} l_s r^s, \quad b_0 = 0, \quad b_1 = 1. \]

\[ s(s-1)b_s + 2b_{s-1} - \frac{1}{n^2}b_{s-2} + \sum_{\ell=0}^{s-2} (4 - 2\ell) \frac{(-43/8)^\ell}{\ell!} b_{s-1-\ell} = 0. \]

\[ \begin{array}{|c|c|c|c|}
\hline
n^{-2} & 0.34 & n^{-2} & 0.35 & n^{-3} & 0.36 \\
\hline
b_1 & 1.000000 & 1.000000 & 1.000000 \\
b_2 & -3.000000 & -3.000000 & -3.000000 \\
b_3 & 4.848333 & 4.850000 & 4.851667 \\
\hline
\end{array} \]

\[ y'' + \left( 2E + \frac{2Z}{x} - \frac{\ell(\ell+1)}{x^2} \right) y = 0. \]

\[ \lambda = -2E. \]

\[ y'' + \left( -\lambda + \frac{2Z}{x} - \frac{\ell(\ell+1)}{x^2} \right) y = 0. \]

\[ y = e^{-\int udx}, \quad y' = -u y, \quad y'' = (u^2 - u')y. \]

28@ In the original manuscript the following expression is not explicitly equated to 0.
29@ As in the previous footnote.
30@ In the original manuscript the author evidently intended to evaluate (from the previous iterative formula) also the coefficients \(b_4, b_5, b_6\), even for different values of \(n^{-2}\).
\[ u' = u^2 - \lambda + \frac{2Z}{x} - \frac{\ell(\ell + 1)}{x^2}. \]

\[ u \sim -\frac{\ell + 1}{x} \quad \text{for } x \to 0. \]

\[ y(0) = y(x_1) = y(x_2) = \ldots = y(x_n) = 0. \]

\[ U = x(x - x_1)(x - x_2)\ldots(x - x_n)u = P\, u, \]

\[ P = x(x - x_1)\ldots(x - x_n). \]

\[ u = \frac{U}{P}, \quad u' = \frac{U'P - UP'}{P^2}; \]

\[ \lim_{x \to \infty} \frac{U}{P} = \sqrt{\lambda}. \]

\[ U'P - UP' = U^2 - \lambda P^2 + \frac{2Z}{x} P^2 - \frac{\ell(\ell + 1)}{x^2} P^2. \]

For \( n = 0 \):

\[ P = x, \quad U = \sqrt{\lambda} \ x + a, \]
\[ P' = 1, \quad U' = \sqrt{\lambda}. \]

\[ U'P - UP' = -a, \]
\[ U^2 = \lambda x^2 + 2a\sqrt{\lambda} \ x + a^2. \]

\[ -a = 2a\sqrt{\lambda} \ x + a^2 + 2Zx - \ell(\ell + 1). \]

\[ a\sqrt{\lambda} + Z = 0, \quad a^2 + a - \ell(\ell + 1) = 0; \]

\[ \lambda = \frac{Z^2}{(\ell + 1)^2}, \quad a = -(\ell + 1). \]
\[
y = e^{b/x+a(x + a)^n} x^3 e^{-x/n}.
\]

\[
y = e^{- \int udx}, \quad y' = -u y, \quad y'' = (-u' + u^2)y.
\]

\[
\frac{1}{n^2} - u^2 + u' + \frac{6}{x^2} = -2V.
\]

\[
u = \frac{b}{(x + \alpha)^2} + \frac{3 - n}{x + a} - \frac{3}{x} + \frac{1}{n}.
\]

For \(x \to 0\):
\[
u = -\frac{3}{x} + \frac{1}{x} + \frac{3 - n}{a} + \frac{b}{a^2} - \left(\frac{3 - n}{a^2} + \frac{2b}{a^3}\right)x + \ldots,
\]
\[
u' = \frac{3}{x^2} - \left(\frac{3 - n}{a^2} + \frac{2b}{a^3}\right) + \ldots,
\]
\[
u^2 = \frac{9}{x^2} - \frac{6}{x} \left(\frac{1}{n} + \frac{3 - n}{a} + \frac{b}{a^2}\right) + \left(\frac{1}{n} + \frac{3 - n}{a} + \frac{b}{a^2}\right)^2
\]
\[+ 6 \left(\frac{3 - n}{a^2} + \frac{2b}{a^3}\right) + \ldots.
\]

### 3.12. ATOMIC ENERGY TABLES

*Energy unit: \(Ze^2/a_0 = 2Z \text{ Rh.}\)*

<table>
<thead>
<tr>
<th>Energy</th>
<th>1s</th>
<th>2s</th>
<th>2p(_1)</th>
<th>3p(_0)</th>
<th>2p(_{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Electrostatic energy</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1s</td>
<td>5</td>
<td>17</td>
<td>83</td>
<td>83</td>
<td>83</td>
</tr>
<tr>
<td>2s</td>
<td>17</td>
<td>512</td>
<td>83</td>
<td>83</td>
<td>83</td>
</tr>
<tr>
<td>2p(_1)</td>
<td>83</td>
<td>237</td>
<td>447</td>
<td>237</td>
<td></td>
</tr>
<tr>
<td>2p(_0)</td>
<td>83</td>
<td>237</td>
<td>447</td>
<td>237</td>
<td></td>
</tr>
<tr>
<td>2p(_{-1})</td>
<td>83</td>
<td>237</td>
<td>447</td>
<td>237</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Energy</th>
<th>1s</th>
<th>2s</th>
<th>2p(_1)</th>
<th>3p(_0)</th>
<th>2p(_{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exchange energy</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1s</td>
<td>16</td>
<td>-</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>2s</td>
<td>16</td>
<td>729</td>
<td>-</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>2p(_1)</td>
<td>15</td>
<td>512</td>
<td>-</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>2p(_0)</td>
<td>15</td>
<td>512</td>
<td>27</td>
<td>-</td>
<td>27</td>
</tr>
<tr>
<td>2p(_{-1})</td>
<td>15</td>
<td>512</td>
<td>27</td>
<td>27</td>
<td>-</td>
</tr>
</tbody>
</table>
The author considered the polarization forces in alkali elements (in particular, in hydrogen and hydrogen-like atoms), obtaining some approximate expressions for the corresponding correction to the atomic energy levels.
\( \nabla_1^2 \psi(q_1) + 2(E_1 - V_1)\psi(q_1) = 0, \)
\( \nabla_2^2 \varphi(q_2) + 2(E_2 - V_2)\varphi(q_2) = 0, \)

where \( \psi \) describes fast movements (short periods), while \( \varphi \) slow ones (large periods), and \( \psi, \varphi \) are separated.

\[
q_1 = (x_1, y_1, r_1), \quad q_2 = (x_2, y_2, r_2),
\]

\[
\nabla_1^2 = \sum \frac{\partial^2}{\partial x_1^2}, \quad \nabla_2^2 = \sum \frac{\partial^2}{\partial x_2^2}.
\]

\[
x_1 \frac{2x_1}{r^3}, \quad y_1 \frac{y_1}{r^3}, \quad z_1 \frac{z_1}{r^3} \quad \begin{cases} 
2x_1x_2 & \frac{2x_1x_2}{r^3} \\
y_1y_2 & \frac{y_1y_2}{r^3} \\
z_1z_2 & \frac{z_1z_2}{r^3} \\
\end{cases} = V.
\]

For \( s \) terms, \( r \to \infty. \)

\( \psi(q_1) \to \psi'(q_1, q_2): \)

\[
\nabla_1^2 \psi'(q_1, q_2) + 2(E_1 + \delta E_1 - V_1 - V)\psi'(q_1, q_2) = 0,
\]

\[
\delta E_1 = \int V\bar{\psi}\psi'\,d\tau_1, \quad \delta E_1 = \delta E_1(q_2).
\]

At first approximation:

\[
\psi'(q_1, q_2) = -\psi(q_1) - 2x_2Z_x(q_1) - y_2Z_y(q_1) - z_2Z_z(q_1).
\]

\[
\int Z_x(q_1)Z_y(q_1)dx_1dy_1dz_1 = 0.
\]

\( Z_x, Z_y, Z_z \) are infinitesimals for \( r \to \infty. \)
\[ Z_x \text{ is symmetric around } x, \]
\[ Z_y \text{ is symmetric around } y, \quad \int \psi(q_1) Z_x(q_1) d\tau_1 = 0, \]
\[ Z_z \text{ is symmetric around } z, \]
\[ Z_x = f(x_1, y_1^2 + z_1^2), \quad Z_x(x_1, y_1^2 + z_1^2) = -Z_x(-x_1, y_1^2 + z_1^2), \]
\[ Z_y = f(y_1, z_1^2 + x_1^2), \quad \ldots \]
\[ Z_r = f(z_1, x_1^2 + y_1^2), \quad f(x_1, y_1^2 + z_1^2) = -f(-x_1, y_1^2 + z_1^2). \]

\[ \delta E_1 \equiv -(4x_1^2 + y_1^2 + r_2^2) \int \frac{x_1}{r^3} \psi(q_1) r_x d\tau_1 \equiv \frac{1}{2} \int V \psi' \psi' d\tau_1. \]

\[ \varphi(q_2) \longrightarrow \varphi'(q_2): \]
\[ \nabla_2^2 \varphi'(q_2) + 2(E_2 + \delta E_2 - V_2 - \delta E_1) \varphi'(q_2) = 0, \]
\[ \delta E_2 = \int \delta E_1 \varphi(q_2) \varphi'(q_2) d\tau_2 \equiv \int \delta E_1 \varphi^2(q_2) d\tau_2. \]
\[ \psi = \psi'(q_1, q_2) \varphi'(q_2). \]

\[ \nabla_1 = \frac{\partial}{\partial x_1} \hat{i}_1 + \frac{\partial}{\partial y_1} \hat{j}_1 + \frac{\partial}{\partial r_1} \hat{k}_1, \quad \nabla_2 = \frac{\partial}{\partial x_2} \hat{i}_2 + \frac{\partial}{\partial y_2} \hat{j}_2 + \frac{\partial}{\partial r_2} \hat{k}_2. \]

\[ (\nabla_1^2 + \nabla_2^2) \psi + 2(E_1 + E_2 + \delta E_2 - V_1 - V_2 - V) \psi \]
\[ = \nabla_1^2 \psi + 2(E_1 + \delta E_1 - V_1 - V) \psi + \nabla_2^2 \psi + 2(E_2 + \delta E_2 - V_2 - \delta E_1) \psi \]
\[ = \varphi'(q_2) \nabla_2^2 \psi'(q_1, q_2) + 2 \nabla_2 \varphi'(q_2) \cdot \nabla_2 \psi'(q_1, q_2) \]
\[ \cong \varphi'(q_2) \nabla_2^2 [\psi(q_1) + 2x_2 Z_x(q_1) - y_2 Z_y(q_1) - z_2 Z_z(q_1)] \]
\[ + 2 \nabla_2 \varphi'(q_2) \cdot \nabla_2 [\psi(q_1) + 2x_2 Z_x(q_1) - y_2 Z_y(q_1) - z_1 Z_z(q_1)] \]
\[ = 0 + 4 \frac{\partial \varphi'(q_2)}{\partial x_2} Z_x(q_1) - 2 \frac{\partial \varphi'(q_2)}{\partial y_2} Z_y(q_1) - 2 \frac{\partial \varphi'(q_2)}{\partial z_2} Z_z(q_1). \]

\[ \cong 4 \frac{\partial \varphi(q_2)}{\partial x_2} Z_x(q_1) - 2 \frac{\partial \varphi(q_2)}{\partial y_2} Z_y(q_2) - 2 \frac{\partial \varphi(q_2)}{\partial z_2} Z_z(q_1). \]
\[ \delta W \approx \delta E_2 - \int \psi'(q_1, q_2) \varphi'(q_2) \left[ 2 \frac{\partial \varphi'(q_2)}{\partial x_2} Z_x(q_1) - \frac{\partial \varphi'(q_2)}{\partial y_2} Z_y(q_1) \right. \\
\left. - \frac{\partial \varphi'(q_2)}{\partial z_2} Z_z(q_1) \right] d\tau_1 d\tau_2 \]
\[ \approx \delta E_2 - \left\{ 4 \int Z_x^2(q_1) d\tau_1 \int x_2 \varphi'(q_2) \frac{\partial \varphi'(q_2)}{\partial x_2} d\tau_2 \\
+ \int Z_y^2(q_1) d\tau_1 \int y_2 \varphi'(y_2) \frac{\partial \varphi'(q_2)}{\partial y_2} d\tau_2 \\
+ \int Z_z^2(q_1) d\tau_1 \int z_2 \varphi'(z_2) \frac{\partial \varphi'(q_2)}{\partial z_2} d\tau_2 \right\} \]
\[ \approx \delta E_2 - 6 \int Z_x^2(q_1) d\tau_1 \int x_2 \varphi(q_2) \frac{\partial (q_2)}{\partial x_2} d\tau_2. \]

\[ \int x_2 \varphi \frac{\partial \varphi}{\partial x_2} dx_2 dy_2 dz_2 = \frac{1}{2} \int x_2 ^2 \frac{\partial \varphi^2}{\partial x_2} dx_2 dy_2 dz_2 \\
= \frac{1}{2} \int \frac{\partial (x_2 \varphi^2)}{\partial x_2} d\tau_2 - \frac{1}{2} \int \varphi^2 d\tau_2 = -\frac{1}{2}. \]

\[ dW \approx dE_2 + 3 \int Z x_1^2(q_1) d\tau_1. \]

\[ x_1 \psi_1 = \sum_{1}^{\infty} a_k \psi^k, \]

\[ -Z_x = \frac{1}{r^3} \sum_{1}^{\infty} \frac{a_k}{E_1^1 - E_k^1}, \]

\[ \sum a_k^2 = \int x_1^2 \psi_1^2 d\tau_1. \]

\[ dE_2 \approx -6 \int x_2^2 \varphi^2 d\tau_2 \frac{1}{r^6} \sum_k \frac{a_k^2}{E_1^1 - E_k^1}, \]

\[ dE_2 = - \frac{6}{r^6} \sum \frac{a_k^2}{E_1^1 - E_k^1} \int x_2^2 \varphi^2 d\tau_2. \]
\[
\begin{align*}
dW &= dE_2 + 3 \int Z_x^2(q_1)d\tau_1 \\
&= -\frac{6}{r^6} \sum \frac{a_k^2}{E_1^k - E_1^k} \int x_2^2 \varphi^2 d\tau_2 + \frac{3}{r^6} \sum \frac{a_k^2}{(E_1^k - E_1^k)^2}.
\end{align*}
\]

On denoting with \(\alpha \psi\) the electric susceptibility,

\[
\alpha \psi = 2 \int r^3 \psi Z_x d\tau_1 = 2 \sum \frac{a_k^2}{E_1^k - E_1^k},
\]

and with \(\alpha\) the susceptibility of the first atom, we get:

\[
dE_2 = -\frac{3\alpha}{r^6} \int x_2^2 \varphi^2 d\tau_2.
\]

\[
\sum a_k^2 = \int x_1^2 \psi^2 d\tau,
\]

\[
-\sum \frac{\alpha_k^2}{E_1^k - E_1^k} = \frac{\alpha}{2} = \frac{\int x_1^2 \psi^2 d\tau}{W}.
\]

For hydrogen, \(W = 0.444 \frac{\epsilon^2}{\alpha_0}\).

\[
\sum \frac{\alpha_k^2}{(E_1^k - E_1^k)^2} > \int \frac{x_1^2 \psi^2 d\tau}{W^2} = \frac{\int x_1^2 \psi^2 d\tau}{WW_1}
\]

\((W_1\) is slightly lower than \(W\)). At a very approximate level:

\[
\sum \frac{\alpha_k^2}{(E_1^k - E_1^k)^2} \approx \int \frac{x_1^2 \psi^2 d\tau}{W^2}.
\]

\[
dE_2 = -\frac{6}{W_1 r^6} \int x_1^2 \psi^2 d\tau_1 \int x_2^2 \varphi^2 d\tau^2
\]

\((\text{for hydrogen this equals to } 13.5 \frac{r^6}{\epsilon^2})\).

\[
\begin{align*}
dW &= dE_2 + 3 \int Z_x^2(q_1)d\tau_1 \\
&\approx -\frac{6}{W_1 r^6} \int x_1^2 \psi^2 d\tau_1 \int x_2^2 \varphi^2 d\tau_2 + \frac{3}{r^6 WW_1} \int x_1^2 \psi^2 d\tau_1 \\
&\approx -\frac{6}{W_1 r^6} \int x_1^2 \psi^2 d\tau_1 \int x_2^2 \varphi^2 d\tau_2 \left(1 - \frac{1}{2W_1 \int x_2^2 \varphi^2 d\tau_2}\right).
\end{align*}
\]
For hydrogen-like atoms \((Z_1 \geq Z_2)\):

\[
\psi, \quad Z_1, \quad E_1 = \frac{1}{2} Z_1^2;
\]

\[
\varphi, \quad Z_2, \quad E_2 = -\frac{1}{2} Z_2^2.
\]

\[W = 0.444 Z_1^2, \quad W_1 < 0.444 Z_1^2.\]

\[
\int x_1^2 \psi \, d\tau_1 = \frac{1}{Z_1^2}, \quad \int x_2^2 \varphi^2 \, d\tau_2 = \frac{1}{Z_2^2}.
\]

\[
\delta E_2 = -\frac{13.5}{r^6 Z_1^4 Z_2^2}, \quad \delta W = -\frac{13.5}{r^6 Z_1^4 Z_2^2} \left( 1 - \frac{Z_2^2}{2W_1} \right).
\]

\[2W_1 < 2W, \quad 2W_1 \approx 0.87 Z_1^2.\]

\[
\delta W \approx -\frac{13.5}{r^6 Z_1^4 Z_2^2} \left( 1 - \frac{Z_2^2}{0.87 Z_1^2} \right), \quad \delta E_2 = -\frac{13.5}{r^6 Z_1^4 Z_2^2}.
\]

\[
\frac{Z_2}{Z_1} = p,
\]

\[
\delta W = -\frac{13.5}{r^6 Z_1^3 Z_2^3} \left( p - \frac{p^3}{0.87} \right).
\]

\[
q = \frac{1}{p + \frac{1}{p}} = \frac{p}{p^2 + 1};
\]

\[p + \frac{1}{p} = \frac{1}{q}, \quad p^2 - \frac{1}{q} p + 1 = 0,
\]

\[
p = \frac{1}{2q} - \sqrt{\frac{1}{4q^2} - 1} = \frac{1 - \sqrt{1 - 4q^2}}{2q} = \frac{2q^2 + 2q^4 + \ldots}{2q},
\]

\[p = q + q^3 + \ldots, \quad p^3 = q^3 + \ldots.
\]

\[
\delta W \approx -\frac{13.5}{r^6 Z_1^3 Z_2^3} \left( q + q^3 - \frac{q^3}{0.87} \right).
\]
By extrapolating to any value of $p$:

$$\frac{1}{W_1} - 1 \simeq 0.15,$$

$$\delta W = -\frac{13.5}{r^6 Z_1^3 Z_2^3} (q - 0.15q^3).$$

For $Z_1 = Z_2$, $q = 1/2$:

$$\delta W = -\frac{13.5}{r^6 Z_1^3 Z_2^3} (0.5 - 0.15 (0.5)^3)) = -\frac{13.5 \cdot 0.481}{r^6 Z_1^3 Z_2^3} = \frac{6.49}{r^6 Z_1^3 Z_2^3}.$$  

### 3.14. COMPLEX SPECTRA AND HYPERFINE STRUCTURES

In this Section, Majorana studied the problem of the hyperfine structure of the energy spectra of complex atoms. The starting point was the (non-relativistic) Landé formula for the hyperfine splitting, which is then generalized to the case (which the author calls the “non Coulomb field” case) when the complex atom may be regarded as made of an inner part with an average effective nuclear charge $Z_1$, and an outer one with an effective nuclear charge $Z_e$, and a principal quantum number $n^*$ [see, for comparison, the papers by E. Fermi and E. Segrè, Mem. Accad. d’Italia 4 (1933) 131 and S. Goudsmith, Phys. Rev. 43 (1933) 636].

The hyperfine separations between a given group of energy levels were considered in the framework introduced by Houston [see W.V. Houston, Phys. Rev. 33 (1929) 297 and especially E.U. Condon and G.H. Shortley, Phys. Rev. 35 (1930) 1342], where $X$ stands for the exchange perturbation energy (which is effective in the Russell-Saunders or $L - S$ configuration) and $A$ is the perturbation integral measuring the spin energy (which is, instead, effective in the $j - j$ coupling. It is interesting to note that Majorana considered also a generalization of the two mentioned couplings, where both $X$ and $A$ play a role.)

The Landé formula for the hyperfine structures (without relativistic corrections) is

$$\delta W = \frac{\mu_0^2}{1840} \ i \ g(i) \ \cos(i, j) \ \frac{2\ell(\ell + 1)}{j + 1} \ \left(\frac{1}{r^3}\right),$$

$$\cos(i, j) = \frac{i(i + 1) + j(j + 1) - (\ell + 1)}{2ij}.$$
For the $s$ terms:

$$\ell \left( \frac{1}{r^3} \right) = 2\pi \psi^2(0),$$

$$\delta W = \frac{\mu_0^2}{1840} i g(i) \cos(i, j) \frac{8\pi}{3} \psi^2(0).$$

In a Coulomb field:

$$\left( \frac{1}{r^3} \right) \frac{Z^3}{a_0^3} \frac{1}{n^3 \ell (\ell + \frac{1}{2}) (\ell + 1)},$$

and for the $s$ terms:

$$\psi^2(0) = \frac{Z^3}{a_0^3} \frac{1}{\pi n^3}.$$

$$\delta W = \frac{\mu_0^2}{1840} i g(i) \cos(i, j) \frac{Z^3}{a_0^3} \frac{4}{n^3 (j + 1)(2\ell + 1)}$$

$$= \frac{\alpha^2 \text{Rh}}{1840} i g(i) \cos(i, j) \frac{2Z^3}{n^3 (j + 1)(2\ell + 1)},$$

which is valid also for $s$ terms. The Rydberg corrections are

$$2\text{Rh} \frac{Z^2}{n^3} \left( i g(i) \cos(i, j) \frac{Z}{(j + 1)(2\ell + 1)} \right).$$

In a non-Coulomb field, an expression analogous to Landé formula holds:

$$\delta W = \frac{\alpha^2 \text{Rh}}{1840} i g(i) \cos(i, j) \frac{2Z_1 Z^2}{n^* n^3 (j + 1)(2\ell + 1)}.$$
are reported in the following table:

<table>
<thead>
<tr>
<th>n</th>
<th>s</th>
<th>$p_{\frac{1}{2}}$</th>
<th>$p_{\frac{3}{2}}$</th>
<th>$d_{\frac{3}{2}}$</th>
<th>$d_{\frac{5}{2}}$</th>
<th>$f_{\frac{5}{2}}$</th>
<th>$f_{\frac{7}{2}}$</th>
</tr>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{15}$</td>
<td>$\frac{1}{25}$</td>
<td>$\frac{1}{35}$</td>
<td>$\frac{1}{49}$</td>
<td>$\frac{1}{63}$</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{60}$</td>
<td>$\frac{1}{100}$</td>
<td>$\frac{1}{140}$</td>
<td>$\frac{1}{196}$</td>
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</tr>
<tr>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>2</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{81}$</td>
<td>$\frac{2}{243}$</td>
<td>$\frac{2}{405}$</td>
<td>$\frac{2}{675}$</td>
<td>$\frac{2}{945}$</td>
<td>$\frac{2}{1323}$</td>
<td>$\frac{2}{1701}$</td>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{96}$</td>
<td>$\frac{1}{288}$</td>
<td>$\frac{1}{480}$</td>
<td>$\frac{1}{800}$</td>
<td>$\frac{1}{1120}$</td>
<td>$\frac{1}{1568}$</td>
<td>$\frac{1}{2016}$</td>
</tr>
</tbody>
</table>

$$\frac{3/2}{(j + 1)(2\ell + 1)} = 1, \frac{1}{3}, \frac{1}{5}, \frac{3}{25}, \frac{3}{35}, \frac{3}{49}, \frac{1}{21}. \quad [31]$$

<table>
<thead>
<tr>
<th>n</th>
<th>s</th>
<th>$p_{\frac{1}{2}}$</th>
<th>$p_{\frac{3}{2}}$</th>
<th>$d_{\frac{3}{2}}$</th>
<th>$d_{\frac{5}{2}}$</th>
<th>$f_{\frac{5}{2}}$</th>
<th>$f_{\frac{7}{2}}$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<td>1</td>
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<tr>
<td></td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{24}$</td>
<td>$\frac{1}{40}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{27}$</td>
<td>$\frac{1}{81}$</td>
<td>$\frac{1}{135}$</td>
<td>$\frac{1}{225}$</td>
<td>$\frac{1}{225}$</td>
<td>$\frac{1}{315}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{64}$</td>
<td>$\frac{1}{192}$</td>
<td>$\frac{1}{320}$</td>
<td>$\frac{1}{1600}$</td>
<td>$\frac{1}{2240}$</td>
<td>$\frac{1}{3136}$</td>
<td>$\frac{1}{1344}$</td>
</tr>
</tbody>
</table>

By using the Houston formula (Goudsmith method), for the terms $^3p_{012}$, $^1p_1$ we have:

---

[31] The values in the following table were obtained by multiplying those in the previous one by $3/2$. 
and, in general, for the terms $^3L_{\ell-1, \ell, \ell+1}$, $^1L_{\ell}$:

\[
\begin{align*}
S_\frac{1}{2} L_{\ell+\frac{1}{2}} & \left\{ 
\begin{array}{l}
j = L + 1 \\
j = L
\end{array}
\right. \\
S_\frac{1}{2} L_{\ell-\frac{1}{2}} & \left\{ 
\begin{array}{l}
j = L \\
j = L - 1
\end{array}
\right.
\end{align*}
\]

In the Russell-Saunders approximation ($A = 0$) the energy of the given levels are as follows:

\[
\begin{align*}
\text{singlet: } & X, \\
\text{triplet: } & 0;
\end{align*}
\]

\[
j = \ell + 1, \quad j = \ell, \quad j = \ell, \quad j = \ell - 1,
\]

\[
E = \begin{cases} 0, & X, & 0, & 0. \end{cases}
\]

For the $j$-$j$ coupling, the energy of the given levels are instead as follows:

\[
\begin{align*}
S_{1/2} L_{\ell+1/2} : & A\ell, \\
S_{1/2} L_{\ell-1/2} : & -A(\ell + 1);
\end{align*}
\]
\[ j = \ell + 1, \quad j = \ell, \quad j = \ell, \quad j = \ell - 1, \]

\[ E = A\ell, \quad A\ell, \quad -A(\ell + 1), \quad -A(\ell + 1). \]

\[ E_{\ell+1} = A\ell, \quad E_{\ell-1} = -A(\ell + 1). \]

\[ E^2 + a_1 E + a_2 = 0, \]

\[ \begin{aligned} a_1 &= c_1 X + c_2 A, \\
a_2 &= c_3 X^2 + c_4 A^2 + c_5 X A. \end{aligned} \]

\[ A = 0 \quad X = 0 \quad A = 0 \quad X = 0 \]

\[ \begin{aligned} a_1 &= -X, & a_1 &= +A, & a_1 &= c_1 X, & a_1 &= c_2 A, \\
a_2 &= 0, & a_2 &= -A^2 \ell(\ell + 1); & a_2 &= c_3 X^2, & a_2 &= c_4 A^2; \\
c_1 &= -1, & c_2 &= +1, & c_3 &= 0, & c_4 &= -\ell(\ell + 1). \end{aligned} \]

\[ E^2 + (A - X) E + [c_5 AX - \ell(\ell + 1)A^2] = 0. \]

Adopting \( A \) as energy unit, and measuring \( X \) in \( A \) units (instead of considering \( X/A \)):

\[ E^2 - (X - 1) E + [c_5 X - \ell(\ell + 1)] = 0. \]

For \( X \to \infty \), the two roots of the previous equation are

\[ E' = X, \quad E'' = -1; \]

\[ E' E'' = -X, \quad E' E'' = c_5 X, \quad c_5 = -1. \]

\[ E^2 - (X - 1) E - [X + \ell(\ell + 1)] = 0. \]

\[ E = \frac{X - 1}{2} \pm \sqrt{\left(\frac{X + 1}{2}\right)^2 + \ell(\ell + 1)}. \]
\[
\begin{align*}
\ell^1 & = \frac{X-1}{2} + \sqrt{\left(\frac{X+1}{2}\right)^2 + \ell(\ell+1)}, \\
\ell^{3} & = \ell, \\
\ell^3 & = \frac{X-1}{2} - \sqrt{\left(\frac{X+1}{2}\right)^2 + \ell(\ell+1)}, \\
\ell^{3} & = -(\ell+1).
\end{align*}
\]

For the $L - S$ coupling:
\[
\sum f(r_i) \, s_i \cdot \ell_i = a \, S \cdot L.
\]
\[
\Psi_{mm'} = \frac{1}{\sqrt{g}} \sum_{r=1}^{g} \psi_m^r \varphi_{m'}^r;
\]
\[
m = L, L - 1, \ldots, -L; \quad m' = S, S - 1, \ldots, -S.
\]

For $g = 4$:

\[
\begin{array}{cccc}
\varphi^1 & \varphi^2 & \varphi^3 & \varphi^4 \\
\varphi^1 & a_{11}S & a_{12}S & a_{13}S & a_{14}S \\
\varphi^2 & a_{21}S & a_{22}S & a_{23}S & a_{24}S \\
\varphi^3 & & & & \\
\varphi^4 & & & & \\
\end{array}
\quad
\begin{array}{cccc}
\varphi^1 & \varphi^2 & \varphi^3 & \varphi^4 \\
b_{11}L & b_{12}L & b_{13}L & b_{14}L \\
b_{21}L & b_{22}L & & \\
& & & b_{44}L
\end{array}
\]

\[
H_{\psi_{m}^{r} \varphi_{m}^{r}} = \sum_{i=1}^{4} A_{i}^{i} B_{i}^{i} L \, S = \sum_{i,m_1,m_1',s,t} L_{mm_1} S_{m_1'm_1'} A_{i}^{i} B_{i}^{i} \psi_{m}^{r} \varphi_{m_1}^{t};
\]

\[\tag{32}\]

\[
H_{\Psi_{mm'}} = \frac{H}{\sqrt{g}} \sum_{r=1}^{g} \Psi_{mm'} = \frac{1}{\sqrt{g}} \sum_{i,m_1,m_1',r,s,t} L_{mm_1} S_{m_1'm_1'} A_{i}^{i} B_{i}^{i} \psi_{m}^{r} \varphi_{m_1}^{t};
\]

\[\tag{32}\]

In the original manuscript, the factor $1/\sqrt{g}$, appearing before the second sum in the following expression, is omitted.
\[ H\Psi_{mm'} = \sum H_{mm',m_1m_1'} \Psi m_1m_1', \]

\[ \langle H\Psi_{mm'}|\Psi_{ab} \rangle = H_{mm',ab}, \]

\[ H_{mm',ab} = \left( \frac{1}{g} \sum_{i,r,t} A^i_{rt} B^i_{rt} \right) L_{ma} S_{m'b}. \]

\[ E^2 - (X - 1)E - [X + \ell(\ell + 1)] = 0. \]

For an atom in a magnetic field \( H \) there is an additional contribution to the energy of the form \( H\mu_0mg \); redefining \( H\mu_0m \rightarrow H \) we have:\(^{33}\)

\[ E^2 - (X - 1 + pH)E - [X + \ell(\ell + 1)] + qXH + tH = 0. \]

Since the considered unperturbed energy levels have different multiplicities \( g' \) and \( g'' \), the contribution of \( H \) is twofold, \( g'H \) and \( g''H \):

\[
\begin{align*}
g' + g'' &= p, \\
qX + t &= pX - 1 + (g'' - g')\sqrt{(X + 1)^2 + \ell(\ell + 1)}. 
\end{align*}
\]

Transitions between three energy levels A,B,C: \(^{34}\)

\[^{33}\text{In the following expression, as reported in the original manuscript, the factor } E \text{ in the second term and the equating to zero is lacking.}\]

\[^{34}\text{In the following, } E \text{ denotes the electric field, } q_{AC}, q_{BC} \text{ the electric dipole moments and } \nu_{AC}, \nu_{BC} \text{ the frequencies of the given transitions.}\]
\[ M_{AB} = \frac{q_{AC}}{h\nu_{AC}} E \cdot q_{AC} + \frac{q_{BC}}{h\nu_{BC}} E \cdot q_{BC} \]

\[ M_{AC} = \sum_c \frac{1}{h\nu_{AC}} q_{AC}|q_{BC}| = \sum_c \frac{1}{\nu_{AC}} \sqrt{P_{AC}P_{BC}}. \]

Transition \(^2P - ^2D\) \(^{35}\):

\[ \begin{array}{cccc}
\frac{1}{2} & \frac{3}{2} & \sqrt{5} \cdot 9 + \sqrt{1} \cdot 1 & 100 \\
\frac{3}{2} & \frac{1}{2} & \sqrt{2} \cdot 5 & 5 \\
\frac{1}{2} & \frac{3}{2} & \sqrt{5} \cdot 1 & 5 \\
\frac{1}{2} & \frac{1}{2} & \sqrt{5} \cdot 5 & 25 \\
\end{array} \]

Transition \(^2P - ^2F\):

\[ \begin{array}{cccc}
\frac{1}{2} & \frac{3}{2} & \sqrt{5} \cdot 9 + \sqrt{1} \cdot 1 & 100 \\
\frac{3}{2} & \frac{1}{2} & \sqrt{2} \cdot 5 & 5 \\
\frac{1}{2} & \frac{3}{2} & \sqrt{5} \cdot 1 & 5 \\
\frac{1}{2} & \frac{1}{2} & \sqrt{5} \cdot 5 & 25 \\
\end{array} \]

\(^{35}\) The numbers in the following tables indicate the amplitudes (third column) and intensities (fourth column) of a spectral line associated with a given transition between two energy levels (specified in the first two columns).
\[\begin{align*}
2P_\frac{3}{2} - 2F_\frac{7}{2} & : \sqrt{9 \cdot 20} = 180 \\
2P_\frac{3}{2} - 2F_\frac{5}{2} & : \sqrt{7 \cdot 1 + 1 \cdot 14} = 45 \\
2P_\frac{1}{2} - 2F_\frac{7}{2} & : 0 \\
2P_\frac{1}{2} - 2F_\frac{5}{2} & : \sqrt{5 \cdot 14} = 70
\end{align*}\]

Relative intensity between \(P_\frac{3}{2}\) and \(P_\frac{1}{2}\): \(\frac{225}{70} = 3.2\).

### 3.15. CALCULATIONS ABOUT COMPLEX SPECTRA

[36]

Eigenvalues of \(\eta\): \(j(j+1) - j'(j'+1) - 6\).

\[j = j' + 2\]

\[j' = j - 2, \quad j'(j' + 1) = (j - 2)(j - 1) = j^2 - 3j + 2;\]

\[\eta = 4j' = 4j - 8, \quad -\eta = 8 - 4j.\]

\[
\begin{vmatrix}
-4j + 4m & A & 0 & 0 & 0 \\
A & -4j + 2m + 6 & B & 0 & 0 \\
0 & B & -4j + 8 & C & 0 \\
0 & 0 & C & -4j - 2m + 6 & D \\
0 & 0 & 0 & D & -4j - 4m
\end{vmatrix}
\]

where:

\[A = 2\sqrt{(j - m)(j + m - 3)}, \quad B = 6(j - m - 1)(j + m - 2),\]

---

36 It appears here the reference to an unknown “second appendix of the §10” [see, probably, E. Fermi and E. Segrè, Mem. Accad. d’Italia 4 (1933) 131].

37 The symbols \(A, B, C, D\) do not appear in the original manuscript, but have been introduced here for obvious typographic reasons (the matrix is much too large).
\[ C = \sqrt{6(j - m - 2)(j + m - 1)}, \quad D = 2\sqrt{(j - m - 3)(j + m)}. \]  

\[ j \left( j - \frac{1}{2} \right) (j - 1) \left( j - \frac{3}{2} \right), \]

\[ 2j(2j - 1)(2j - 2)(2j - 3). \]

\[ j = 5: \]

\[ -4(j - m - 3)(j + m) + 4(j + m)(4j + 2m - 6) \]
\[ -4j^2 + 4m^2 + 12j + 12m + 16j^2 + 24jm + 8m^2 - 24j - 24m \]
\[ = 12j^2 + 24jm + 12m^2 - 12j - 12m \]
\[ = 12(j + m - 1)(j + m). \]

\[ 24\sqrt{6(j - m - 1)(j - m)(j + m - 3)(j + m - 2)} (j + m - 1)(j + m) \]
\[ 48\sqrt{6(j - m - 1)(j - m)(j + m - 1)(j + m)} \]
\[ 144(j - m - 1)(j - m)(j + m - 1)(j + m) \]
\[ 48\sqrt{6(j - m - 2)(j + m - 1)} (j - m - 1)(j - m)(j + m) \]
\[ 24\sqrt{6(j - m - 3)(j - m - 2)(j + m - 1)(j + m)} (j - m - 1)(j - m) \]

\[ \sqrt{(j + m - 3)(j + m - 2)(j + m - 1)(j + m)} \]
\[ 2\sqrt{(j - m)(j + m - 2)(j + m - 1)(j + m)} \]
\[ \sqrt{6(j - m - 1)(j + m - 1)(j + m)} \]
\[ 2\sqrt{(j - m - 2)(j - m - 1)(j - m)(j + m)} \]
\[ \sqrt{(j - m - 3)(j - m - 2)(j - m - 1)(j - m)} \]

\( m = 0 \) (this is imposed since the eigenvalues do not depend on \( m \))

\[ 2(j - 3)(j - 2)(j - 1)j \]
\[ 8j(j - 2)(j - 1)j \]
\[ 6(j - 1)j(j - 1)j \]

\[ 2j(j - 1)[(j - 3)(j - 2) + 4(j - 2)j + 3(j - 1)j] \]

\[ \text{The original manuscript continues with some calculations aimed at finding the four non-vanishing eigenvalues of the matrix above (whose determinant is equal to 0), the product of which, apart from a numerical factor, is set far below in the text (framed expression). Only for the present case, we have chosen to reproduce those calculations, in this footnote, since the method followed by the author is particularly interesting. For the other cases, with different matrices, appearing in this Section we do not report the analogous calculations.} \]
$$10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$ 

\begin{align*}
m &= 0 & m &= 1 & m &= 5 \\
120 & & 360 & & 5040 \\
1200 & & 1920 & & 0 \\
2400 & & 2160 & & 0 \\
1200 & & 576 & & 0 \\
120 & & 24 & & 0 \\
5040 & & 5040 & & 5040
\end{align*}

\[ j = j + 1 \]

\[ \eta = j(j + 1) - (j - 1)j - 6 = 2j - 6, \quad -\eta = -2j + 6. \]

\[
\begin{vmatrix}
-2j + 4m - 2 & A & 0 & 0 & 0 \\
A & -2j + 2m + 4 & B & 0 & 0 \\
0 & B & -2j + 6 & C & 0 \\
0 & 0 & C & -2j - 2m + 4 & D \\
0 & 0 & 0 & D & -2j - 4m - 2
\end{vmatrix}
\]

where:\[\text{39}\]

\[ A = 2\sqrt{(j - m + 1)(j + m - 2)}, \quad B = \sqrt{6(j - m)(j + m - 1)}, \]

\[ C = \sqrt{6(j - m - 1)(j + m)}, \quad D = 2\sqrt{(j - m - 2)(j + m + 1)}. \]\[\text{40}\]

\[ 2j(2j - 1)(2j - 2)\frac{2j + 2}{4}, \]

\[\text{39}^@\] The symbols \(A, B, C, D\) do not appear in the original manuscript, but, once more, they have been introduced here for obvious typographic reasons (the matrix is much too large).

\[\text{40}^@\] The original manuscript continues with some calculations aimed at finding the four non-vanishing eigenvalues of the matrix above (whose determinant is equal to 0), the product of which, apart from a numerical factor, is given below in the text (framed expressions).
\[ 2j(2j-1)(2j-2) \frac{j+1}{2}, \]

\[ 2(j-1)j(j+1)(2j-1). \]

\( j = 5: \)

\[ 2 \cdot 4 \cdot 5 \cdot 6 \cdot 9 = 2160. \]

\[
\begin{array}{ccc}
m = 0 & m = 1 & m = 5 \\
360 & 600 & 720 \\
720 & 480 & 1440 \\
0 & 144 & 0 \\
720 & 768 & 0 \\
360 & 168 & 0 \\
2160 & 2160 & 2160
\end{array}
\]

\[ j = j' \]

\[ \eta = j(j+1) - j'(j'+1) - 6 = -6. \]

\[
\begin{vmatrix}
4m - 2 & A & 0 & 0 & 0 \\
A & 2m + 4 & B & 0 & 0 \\
0 & B & 6 & C & 0 \\
0 & 0 & C & -2m + 4 & D \\
0 & 0 & 0 & D & -4m - 2
\end{vmatrix}
\]

where:\(^{41}\)

\[ A = 2 \sqrt{(j - m + 2)(j + m - 1)}, \quad B = \sqrt{6(j - m + 1)(j + m)}, \]

\[ C = \sqrt{6(j - m)(j + m + 1)}, \quad D = 2 \sqrt{(j - m - 1)(j + m + 2)}. \]

\(^{41}\) The symbols \( A, B, C, D \) do not appear in the original manuscript, but, once again, they have been introduced here for obvious typographic reasons (the matrix is much too large).

\(^{42}\) The original manuscript continues with some calculations aimed at finding the four non-vanishing eigenvalues of the matrix above (whose determinant is equal to 0), the product of which, apart from a numerical factor, is given below in the text (framed expressions).
\[
\frac{4}{6} j(j + 1)(2j - i)(2j + 3),
\]

\[
2j(2j - 1)\frac{(2j + 2)(2j + 3)}{6}.
\]

\(j = 5:\)

\[
\begin{array}{ccc}
m = 0 & m = 1 & m = 5 \\
840 & 900 & 180 \\
30 & 30 & 810 \\
600 & 486 & 1350 \\
30 & 252 & 0 \\
840 & 672 & 0 \\
2340 & 2340 & 2340
\end{array}
\]

3.16. RESONANCE BETWEEN A \(p (\ell = 1)\) ELECTRON AND AN ELECTRON WITH AZIMUTHAL QUANTUM NUMBER \(\ell'\)

Complex spectra are again considered, now evaluating resonance terms between electrons belonging to different shells.

Exchange energy:

\[
K(n, 1, m_\ell; n', l', m_{\ell'}) = \sum b_k G_k,
\]

\[
G_k = e^r (4\pi)^2 \int_0^\infty \int_0^\infty R(n, 1, r)R(n', l', r)R(n, 1, r')R(n', l', r')
\times \frac{r_n^{k+1} - r^{k+1} - r'^{k+1}}{r_{\ell} r^{k+1} r'^{k+1}} dr dr',
\]

where:
Only the coefficients $b_{\ell' - 1}$ and $b_{\ell' + 1}$ are non vanishing.

### 3.16.1 Resonance Between A $d$ Electron And A $p$ Shell I

<table>
<thead>
<tr>
<th>$m_\ell$</th>
<th>$m'_\ell$</th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell = 1$</td>
<td>$\ell' = 1$</td>
<td>$\pm 1$</td>
<td>$\pm 1$</td>
<td>1</td>
<td>0</td>
<td>1/25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3/25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>$\mp 1$</td>
<td>0</td>
<td>0</td>
<td>6/25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4/25</td>
</tr>
<tr>
<td>$\ell = 1$</td>
<td>$\ell' = 2$</td>
<td>$\pm 1$</td>
<td>$\pm 2$</td>
<td>0</td>
<td>2/5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>$\pm 1$</td>
<td>0</td>
<td>1/5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>0</td>
<td>0</td>
<td>1/15</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>$\mp 1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>$\mp 2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>$\pm 2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\pm 1$</td>
<td>0</td>
<td>1/5</td>
<td>0</td>
<td>24/245</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4/15</td>
<td>0</td>
</tr>
</tbody>
</table>

Where:

43@ The symbols $A, B, C, D, E, F, G, H, R, S$ do not appear in the original manuscript, but have been introduced here for typographic reasons. Note that in the last row the author gave the sum of all the terms in the corresponding column (for example, $S = A + F + E$, or $S = B + G + D$, etc.). He proceeded similarly with respect to the last column (for example, $R = A + B + C + D + E$, etc.).
\[ A = \frac{2}{5} G_1 + \frac{3}{245} G_3, \quad B = \frac{1}{5} G_1 + \frac{9}{245} G_3, \]
\[ C = \frac{1}{15} G_1 + \frac{18}{245} G_3, \quad D = \frac{30}{245} G_3, \]
\[ E = \frac{45}{245} G_3, \quad F = \frac{15}{245} G_3, \]
\[ G = \frac{1}{5} G_1 + \frac{24}{245} G_3, \quad H = \frac{4}{15} G_1 + \frac{27}{245} G_3, \]
\[ S = \frac{2}{5} G_1 + \frac{63}{245} G_3, \quad R = \frac{2}{3} G_1 + \frac{24}{49} G_3. \]

### 3.16.2 Eigenfunctions Of \( d_{\frac{5}{2}} \), \( d_{\frac{3}{2}} \), \( p_{\frac{3}{2}} \) And \( p_{\frac{1}{2}} \) Electrons

The eigenfunctions are expressed by means of the notation \((n', \ell', m'_j, m'_s)\). We replace \((n', \ell', m'_j, m'_s)\) simply with \((m'_\ell, m'_s)\).

For \( d_{\frac{5}{2}} \):

<table>
<thead>
<tr>
<th>( j' )</th>
<th>( m' )</th>
<th>( \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5}{2} )</td>
<td>( \frac{5}{2} )</td>
<td>( (2, \frac{1}{2}) )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \sqrt{\frac{4}{5}} (1, \frac{1}{2}) + \sqrt{\frac{1}{5}} (2, -\frac{1}{2}) )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \sqrt{\frac{3}{5}} (0, \frac{1}{2}) + \sqrt{\frac{2}{5}} (1, -\frac{1}{2}) )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( -\frac{1}{2} )</td>
<td>( \sqrt{\frac{2}{5}} (-1, \frac{1}{2}) + \sqrt{\frac{3}{5}} (0, -\frac{1}{2}) )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( -\frac{3}{2} )</td>
<td>( \sqrt{\frac{1}{5}} (-2, \frac{1}{2}) + \sqrt{\frac{4}{5}} (-1, -\frac{1}{2}) )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( -\frac{5}{2} )</td>
<td>( (-2, -\frac{1}{2}) )</td>
</tr>
</tbody>
</table>
For $d_{\frac{3}{2}}$:

<table>
<thead>
<tr>
<th>$j'$</th>
<th>$m'$</th>
<th>[\sqrt{\frac{1}{5}} (1, \frac{1}{2}) - \sqrt{\frac{4}{5}} (2, -\frac{1}{2})]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>[\sqrt{\frac{2}{5}} (0, \frac{1}{2}) - \sqrt{\frac{3}{5}} (1, -\frac{1}{2})]</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>[\sqrt{\frac{3}{5}} (-1, \frac{1}{2}) - \sqrt{\frac{2}{5}} (0, -\frac{1}{2})]</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>[\sqrt{\frac{4}{5}} (-2, \frac{1}{2}) - \sqrt{\frac{1}{5}} (-1, -\frac{1}{2})]</td>
</tr>
</tbody>
</table>

For $p_{\frac{3}{2}}$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m$</th>
<th>[1, \frac{1}{2}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>[(1, \frac{1}{2})]</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>[\sqrt{\frac{2}{3}} (0, \frac{1}{2}) + \sqrt{\frac{1}{3}} (1, -\frac{1}{2})]</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>[\sqrt{\frac{1}{3}} (-1, \frac{1}{2}) + \sqrt{\frac{2}{3}} (0, -\frac{1}{2})]</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$-\frac{3}{2}$</td>
<td>[(-1, -\frac{1}{2})]</td>
</tr>
</tbody>
</table>

For $p_{\frac{1}{2}}$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m$</th>
<th>[\frac{1}{3} (0, \frac{1}{2}) - \sqrt{\frac{2}{3}} (1, -\frac{1}{2})]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>[\sqrt{\frac{1}{3}} (0, \frac{1}{2}) - \sqrt{\frac{2}{3}} (1, -\frac{1}{2})]</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>[\sqrt{\frac{2}{3}} (-1, \frac{1}{2}) - \sqrt{\frac{1}{3}} (0, -\frac{1}{2})]</td>
</tr>
</tbody>
</table>
### 3.16.3 Resonance Between A d Electron And A p Shell II

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m$</th>
<th>$m' = 5/2$</th>
<th>$m' = 3/2$</th>
<th>$m' = 1/2$</th>
<th>$m' = -1/2$</th>
<th>$d_{5/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{5/2}$</td>
<td>$3/2$</td>
<td>$3/2$</td>
<td>$A$</td>
<td>$B$</td>
<td>$C$</td>
<td>$D$</td>
</tr>
<tr>
<td></td>
<td>$3/2$</td>
<td>$1/2$</td>
<td>$E$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$3/2$</td>
<td>$-1/2$</td>
<td>$F$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$3/2$</td>
<td>$-3/2$</td>
<td>$0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean values</td>
<td>$S_1$</td>
<td>$T_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{1/2}$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$G$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1/2$</td>
<td>$-1/2$</td>
<td>$H$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean values</td>
<td>$S_2$</td>
<td>$T_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean values</td>
<td>$S_1$</td>
<td>$S_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean values</td>
<td>$S$</td>
<td>$T$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where:

\[
A = \frac{2}{5} G_1 + \frac{3}{245} \quad B = \frac{4}{25} G_1 + \frac{36}{1125} G_3, \quad C = \ldots, \\
D = \ldots, \quad E = \frac{10}{245} G_3, \quad F = \frac{15}{245} G_3, \\
G = \frac{5}{245} G_3, \quad H = \frac{30}{245} G_3, \\
S_1 = \frac{2}{5} G_1 + \frac{28}{245} G_3, \quad T_1 = \frac{1}{10} G_1 + \frac{7}{245} G_3, \\
S_2 = \frac{35}{245} G_3, \quad T_2 = \frac{35}{490} G_3, \\
S = \frac{2}{5} G_1 + \frac{63}{245} G_3, \quad T = \frac{1}{15} G_1 + \frac{21}{490} G_3.
\]

\[\text{See the previous footnote. Notice also that } S = S_1 + S_2, \text{ and analogously for the } T \text{ terms.}
\]

Here the manuscript is corrupted and we have represented by dots the expressions we cannot easily interpret.
<table>
<thead>
<tr>
<th>$j$</th>
<th>$m$</th>
<th>$m' = 5/2$</th>
<th>$m' = 3/2$</th>
<th>$m' = 1/2$</th>
<th>$m' = -1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{\frac{3}{2}}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$A$</td>
<td>$B$</td>
<td>$C$</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{-1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\frac{-1}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{-1}{2}$</td>
<td>$\frac{3}{2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{mean \ values}{mean \ values}$</td>
<td>$\frac{S_1}{S_1}$</td>
<td>$\frac{T_1}{T_1}$</td>
<td>$\frac{S_2}{S_2}$</td>
<td>$\frac{T_2}{T_2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{mean \ values}{mean \ values}$</td>
<td>$S$</td>
<td>$\frac{1}{15} G_1 + \frac{63}{245} G_3$, $T_1 = \frac{1}{60} G_1 + \frac{63}{980} G_3$, $S_1 = \frac{1}{15} G_1 + \frac{63}{245} G_3$, $T_2 = \frac{1}{6} G_1$, $S_2 = \frac{1}{5} G_1 + \frac{63}{245} G_3$.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where.\(^{45}\)

\[ A = \ldots, \quad B = \ldots, \quad C = \ldots, \quad D = \ldots, \]

Mean values:

\[
\begin{align*}
\frac{d_3 p_{\frac{3}{2}}}{\frac{3}{2}} : & \quad \frac{1}{15} G_1 + \frac{21}{490} G_3 + \frac{1}{6} \left( \frac{1}{5} G_1 - \frac{21}{245} G_3 \right) = \frac{1}{10} G_1 + \frac{7}{245} G_3, \\
\frac{d_3 p_{\frac{1}{2}}}{\frac{3}{2}} : & \quad \frac{1}{15} G_1 + \frac{21}{490} G_3 - \frac{1}{3} \left( \frac{1}{5} G_1 - \frac{21}{245} G_3 \right) = \frac{1}{14} G_3, \\
\frac{d_3 p_{\frac{3}{2}}}{\frac{1}{2}} : & \quad \frac{1}{15} G_1 + \frac{21}{490} G_3 - \frac{1}{4} \left( \frac{1}{5} G_1 - \frac{21}{245} G_3 \right) = \frac{1}{60} G_1 + \frac{63}{980} G_3, \\
\frac{d_3 p_{\frac{1}{2}}}{\frac{1}{2}} : & \quad \frac{1}{15} G_1 + \frac{21}{490} G_3 + \frac{1}{2} \left( \frac{1}{5} G_1 - \frac{21}{245} G_3 \right) = \frac{1}{6} G_1.
\end{align*}
\]

\(^{45}\) See the previous footnote.
If $G_1 = 1$ and $G_3 = \frac{1}{2}$:

\[
\begin{align*}
  d_{\frac{5}{2}} p_{\frac{3}{2}} : & \quad 0.0881 + \frac{1}{6} \cdot 0.1571 = 0.1143, \\
  d_{\frac{5}{2}} p_{\frac{1}{2}} : & \quad 0.0881 - \frac{1}{3} \cdot 0.1571 = 0.0357, \\
  d_{\frac{3}{2}} p_{\frac{3}{2}} : & \quad 0.0881 - \frac{1}{4} \cdot 0.1571 = 0.0488, \\
  d_{\frac{3}{2}} p_{\frac{1}{2}} : & \quad 0.0881 + \frac{1}{2} \cdot 0.1571 = 0.1667.
\end{align*}
\]

3.17. MAGNETIC MOMENT AND DIAMAGNETIC SUSCEPTIBILITY FOR A ONE-ELECTRON ATOM (RELATIVISTIC CALCULATION)

The following notes are aimed at evaluating the magnetic moment of an hydrogen-like atom by starting from the Dirac equation for an electron in an electromagnetic potential field $(\varphi, C)$.

In the non-relativistic case:

\[
\sigma_\mu = -\frac{e^2}{6mc^2} \frac{3a_0^2}{Z^2}.
\]

\[
\left[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) + \rho_1 \mathbf{\sigma} \cdot \left( \mathbf{p} + \frac{e}{c} C \right) + \rho_3 mc \right] \psi = 0,
\]

\[
\varphi = +\frac{Ze}{r},
\]

\[
A = (\psi_1, \psi_2), \quad B = (\psi_3, \psi_4),
\]

\[
\left( \frac{W}{c} + \frac{Ze^2}{rc} + mc \right) A + \mathbf{\sigma} \left( \mathbf{p} + \frac{e}{c} C \right) B = 0,
\]

\[
\left( \frac{W}{c} + \frac{Ze^2}{rc} - mc \right) B - \mathbf{\sigma} \left( \mathbf{p} + \frac{e}{c} C \right) A = 0.
\]
\[ C_x = -\frac{1}{2} y H, \quad C_y = \frac{1}{2} x H, \quad C_z = 0; \]
\[ H_x = 0, \quad H_y = 0, \quad H_z = H. \]

\[ W = -\frac{Ze^2}{r} - \rho_3 mc^2 - c \rho_1 \sigma \cdot p - \frac{eH}{r} \rho_1 (x\sigma_y - y\sigma_x). \]

\[ \frac{\partial W}{\partial H} = -\mu_z. \]

\[ \left[ W + \frac{Ze^2}{r} + \rho_3 mc^2 + c \rho_1 \sigma \cdot p + \frac{e}{2} H \rho_1 (x\sigma_y - y\sigma_x) \right] \psi_i = 0, \]

\[ \psi = \psi_0 + H \psi_1 + H^2 \psi_2 + \ldots, \]

\[ W = W_0 + HW_1 + H^2 W_2 + \ldots. \]

\[ \left[ W_0 + \frac{Ze^2}{r} + \rho_3 mc^2 + c \rho_1 \sigma \cdot p \right] \psi_0 = 0, \]

\[ \left[ W_0 + \frac{Ze^2}{r} + \rho_3 mc^2 + c \rho_1 \sigma \cdot p \right] \psi_1 + \left[ W_1 + \frac{e}{2} \rho_1 (x\sigma_y - y\sigma_x) \right] \psi_0 = 0, \]

\[ \left[ W_0 + \frac{Ze^2}{r} + \rho_3 mc^2 + c \rho_1 \sigma \cdot p \right] \psi_2 
+ \left[ W_1 + \frac{e}{2} \rho_1 (x\sigma_y - y\sigma_x) \right] \psi_1 + W_2 \psi_0 = 0. \]

\[ W_1 = -\frac{e}{2} \int \tilde{\psi}_0 \rho_1 (x\sigma_y - y\sigma_x) \psi_0 d\tau. \]
\[
W_2 = \frac{e}{2} \int \bar{\psi}_0 \rho_1 (x\sigma_y - y\sigma_x) \psi_1 d\tau - W_1 \int \bar{\psi}_0 \psi_1 d\tau \\
= - \int \bar{\psi}_0 \left[ W_1 + \frac{e}{2} \rho_1 (x\sigma_y - y\sigma_x) \right] \psi_1 d\tau \\
= - \int \bar{\psi}_1 \left[ W_1 + \frac{e}{2} \rho_1 (x\sigma_y - y\sigma_x) \right] \psi_0 d\tau.
\]

\[\psi_0 = (A_0, B_0):\]

\[
\left( W_0 + \frac{Ze^2}{r} + mc^2 \right) A_0 + c\sigma \cdot p B_0 = 0, \\
\left( W_0 + \frac{Ze^2}{r} - mc^2 \right) B_0 - c\sigma \cdot p A_0 = 0.
\]

\[A_0 = f_0 S_{-1}^m, \quad B_0 = g_0 S_{1}^m,\]

\[(m = \pm 1/2, k = 1)\]

\[S_k^m = S_{1}^{\pm 1/2}.\]

\[
\left( W_0 + \frac{Ze^2}{r} + mc^2 \right) f_0 + c \frac{h}{2\pi i} \frac{d}{dr} g_0 = 0, \\
\left( W_0 + \frac{Ze^2}{r} - mc^2 \right) g_0 - c \frac{h}{2\pi i} \left( \frac{d}{dr} + \frac{2}{r} \right) f_0 = 0.
\]

\[f_0 = \frac{u_0}{r}, \quad g_0 = \frac{iv_0}{r} :\]

\[
\left( W_0 + mc^2 + \frac{Ze^2}{r} \right) u_0 + c \frac{h}{2\pi} \left( \frac{d}{dr} - \frac{1}{r} \right) v_0 = 0, \\
\left( W_0 - mc^2 + \frac{Ze^2}{r} \right) v_0 + c \frac{h}{2\pi} \left( \frac{d}{dr} + \frac{1}{r} \right) u_0 = 0.
\]
\[ p^2 = \frac{W^2}{c^2} - m^2 c^2, \]

\[ -W_0^2 + m^2 c^4 = \frac{4\pi^2 e^4}{h^2 c^2} m^2 c^4 Z^2, \]

\[ \sqrt{-W_0^2 + m^2 c^4} = \frac{2\pi e^2}{hc} m c Z = \frac{2\pi e^2}{h} m Z, \]

\[ \alpha = \frac{2\pi e^2}{hc}, \]

\[ W_0 = mc^2 \sqrt{1 - Z^2 \alpha^2}, \quad a_0 = \frac{\hbar^2}{4\pi^2 mc^2}. \]

\[
\begin{align*}
\frac{d}{dr} - \frac{1}{r} v_0 &= v_0 \left( \frac{\sqrt{1 - Z^2 \alpha^2}}{r} - \frac{Z}{a_0} - \frac{1}{r} \right) = v_0 \left( \frac{W - mc^2}{r mc^2} - \frac{Z}{a_0} \right),
\end{align*}
\]

and substituting in the equation above:

\[
\begin{align*}
&v_0 \left( \frac{W - mc^2}{r mc^2} - \frac{Z}{a_0} \right) + \frac{2\pi}{hc} \left( W_0 + mc^2 + \frac{Ze^2}{r} \right) u_0 = 0,
\end{align*}
\]

\[
\begin{align*}
u_0 &= -\frac{W_0 - mc^2 - \frac{r}{a_0} mc^2 Z}{r(W_0 + mc^2) + Ze^2} \frac{hc}{2\pi mc^2} v_0 \\
&= \frac{mc^2 - W_0 + \frac{mc^2 Z}{a_0} \frac{r}{Ze^2 + (W_0 + mc^2)r}}{Ze^2 + (W_0 + mc^2)r} \frac{h}{2\pi mc^2} u_0.
\end{align*}
\]
3.18. THEORY OF INCOMPLETE \( P' \) TRIPLETS

On pages 61-68 and 90-116 of Quaderno 7, the author elaborated the theory of incomplete \( P' \) triplets, as published by him in E. Majorana, Nuovo Cim. 8 (1931) 107. In the following, we reproduce only few topics that were not included in the published paper (which may be consulted for further reference).

3.18.1 Spin-Orbit Couplings And Energy Levels

\[
\begin{align*}
\ell_1 &= 1 \quad s_1 = 1/2 \quad j_1 \quad s_1 \cdot \ell_1 \\
3/2 & \quad 1/2 \\
1/2 & \quad -1
\end{align*}
\]

\[
c = \frac{2}{3} \delta, \quad \delta = \frac{3}{2} c.
\]

The quantity \( \ell \cdot s \) for \( \ell = 1, s = 1/2 \) is as follows:

\[
\begin{align*}
\ell \cdot s &= s_x \ell_x + s_y \ell_y + s_z \ell_z \\
&= \frac{1}{2} (s_x + is_y) (\ell_x - i\ell_y) + \frac{1}{2} (s_x - is_y) (\ell_x + i\ell_y) + \sigma_z \ell_z.
\end{align*}
\]

[See the table on page 234.]
For atoms with one $p$ electron in the inner shells and one $s$ in the outer one (like neon), denoting with $I$ the exchange energy, we have:

[See the tables on page 235.]

For high $Z$ and $I = 1$:

[See the figure on page 236.]

<table>
<thead>
<tr>
<th>$\ell_z$</th>
<th>$s_z$</th>
<th>$1 \frac{1}{2}$</th>
<th>$0 \frac{1}{2}$</th>
<th>$1 - \frac{1}{2}$</th>
<th>$-1 \frac{1}{2}$</th>
<th>$0 - \frac{1}{2}$</th>
<th>$-1 - \frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j_z$</td>
<td></td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{3}{2}$</td>
</tr>
<tr>
<td>$1 \frac{1}{2}$</td>
<td></td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$0 \frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{2}</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{2}</td>
<td>$\sqrt{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$1 - \frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>$\sqrt{2}$</td>
<td>-\frac{1}{2}</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$0 - \frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$-\frac{3}{2}$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|c|}
\hline
m = 2 & \ell_z & s_z^1 & s_z^2 \\
\hline
0 & 0 & 1 & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
m = 1 & \ell_z & s_z^1 & s_z^2 \\
\hline
0 & 0 & 1 & \frac{1}{2} \\
1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
m = 0 & \ell_z & s_z^1 & s_z^2 \\
\hline
0 & 0 & 1 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array}
\]
3.18.2 Spectral Lines For Mg And Zn

<table>
<thead>
<tr>
<th></th>
<th>Zn</th>
<th>Mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>(22')</td>
<td>2086.72</td>
<td>2779.93</td>
</tr>
<tr>
<td>(12')</td>
<td>2070.11</td>
<td>2776.80</td>
</tr>
<tr>
<td>21'</td>
<td>2104.34</td>
<td>2783.08</td>
</tr>
<tr>
<td>11'</td>
<td>2087.27</td>
<td>2779.93</td>
</tr>
<tr>
<td>01'</td>
<td>2079.10</td>
<td>2778.38</td>
</tr>
<tr>
<td>10'</td>
<td>2096.88</td>
<td>2781.52</td>
</tr>
</tbody>
</table>

The wavelengths of the following spectral lines are expressed in angstroms.

---

46 The wavelengths of the following spectral lines are expressed in angstroms.
3.18.3 Spectral Lines For Zn, Cd And Hg

As above, the wavelengths of the following spectral lines are expressed in angstroms.

In the following table the author reported the wavelength (in angstroms) and the frequency (in cm⁻¹) for the spectral lines in the first and the second column, respectively, for each element. As pointed out by the author himself, these values do not take into account the correction induced by propagation of light in air.
3.19. HYPERFINE STRUCTURE: RELATIVISTIC RYDBERG CORRECTIONS

A relativistic formula for the Rydberg corrections of the hyperfine structures was derived in the following calculations. Some particular cases, including s-orbit terms, were considered in detail. Probably, the present calculations were at the basis of what discussed in an appendix of E. Fermi and E. Segre, Mem. Accad. d’Italia 4 (1933) 131 on the same topic, as acknowledged by the authors themselves.

By using electronic units: \( \gamma = \sqrt{k^2 - \alpha^2} \), \( \alpha = Z/c \).

\[
\begin{align*}
\mu_0 = \gamma, & \quad \alpha = \sqrt{k^2 - \gamma^2}, & \quad \mu_0 + k = k + \gamma. \\
A = \mu_0 + k = k + \gamma, & \quad B = n_r + \gamma, & \quad L = \sqrt{(n_r + \gamma)^2 + \alpha^2}.
\end{align*}
\]

\[
E = c \frac{B}{L} - c^2,
\]

\[
\frac{E}{c} + 2c = c \frac{B}{L} + c.
\]

\[
\beta = \frac{\alpha c}{L}, \quad \alpha \beta = \frac{\alpha^2 c}{L}.
\]

\[
\frac{dE}{dn_r} = \alpha c^2 \frac{1}{[(n_r + \gamma)^2 + \alpha^2]^{3/2}}.
\]

\[
a_{\mu_0-1}^+ = \frac{\alpha}{c} \frac{1}{1 - 2\gamma}.
\]

\[
\beta(\mu_0 + k) = \alpha c \frac{A}{L}, \quad -\alpha \frac{E}{c} = \alpha c - \alpha c \frac{B}{L}.
\]

\[
b_{\mu_0-1}^+ = -\frac{A - 1}{c} \frac{1}{1 - 2\gamma},
\]

\[
b_{\mu_0+1} = \frac{\alpha c}{L} - 2\alpha c \frac{B}{L(1 + 2\gamma)} - \frac{\alpha c}{A(1 + 2\gamma)} - \alpha c \frac{B}{AL(1 + 2\gamma)}.
\]
\[ 2\gamma C = \left( A c + A \frac{c B}{L} + A c^2 \frac{1}{L} \right) \frac{A - 1}{c(1 - 2\gamma)} \]

\[ -\alpha \frac{A}{L} + 2\alpha \frac{A B}{L(1 + 2\gamma)} - \frac{\alpha}{1 + 2\gamma} + \frac{\alpha B}{1 - 2\gamma} \]

\[ = \alpha \frac{AB}{L} \left( \frac{A^2 - A}{1 - 2\gamma} + \frac{AB - B}{1 - 2\gamma} - A + \frac{2AB}{1 + 2\gamma} + \frac{B}{1 + 2\gamma} \right) + \alpha \left( \frac{A}{1 - 2\gamma} - \frac{A - 1}{1 - 2\gamma} + \frac{1}{1 + 2\gamma} \right). \]

\[ -C = \frac{\alpha}{\sqrt{(nr + \gamma)^2 + \alpha^2}} \frac{1}{2\gamma(4\gamma^2 - 1)} \]

\[ \cdot \left[ 4k(nr + \gamma) + 2\sqrt{(nr + \gamma)^2 + \alpha^2} \right] \]

\[ -\frac{dE}{d\varepsilon} = \frac{z^2 \alpha}{(nr + \gamma)^2 + \alpha^2} \frac{1}{2\gamma(4\gamma^2 - 1)} \]

\[ \cdot \left[ 4k(nr + \gamma) + 2\sqrt{(nr + \gamma)^2 + \alpha^2} \right] = -\int \frac{uv}{r^2} \, dr. \]

For \( Z \to 0 \) (\( \alpha^2 \to 0, \gamma = k, nr + \gamma = n \)):

\[ -C = \frac{\pm \alpha}{2k(k - 1/2)}, \]

\[ -\frac{dE}{d\varepsilon} = \frac{\pm Z^2 \alpha}{2n^3 k (k - 1/2)}. \]

In particular \((2j + 1 = |2k|)\), for \( k = \ell + 1, j = \ell + 1/2 \):

\[ -C = \frac{\alpha}{2(\ell + 1) (\ell + 1/2)}, \]

\[ -\frac{dE}{d\varepsilon} = \frac{Z^2 \alpha}{2n^3 (\ell + 1/2) (\ell + 1)}. \]
while, for \( k = -\ell, j = \ell - 1/2 \):

\[
-C = \frac{-\alpha}{2\ell (\ell + 1/2)},
\]

\[
-dE/d\varepsilon = \frac{-Z^2\alpha}{2n^3\ell (\ell + 1/2)}.
\]

The ratio \( R \) between the Rydberg corrections for the hyperfine structures in the relativistic form and those in the classical (non-relativistic) form is then given by:

\[
R = \frac{(2j + 1) (k - 1/2)}{\gamma(4\gamma^2 - 1)} \left( 2k \frac{n_r + \gamma}{\sqrt{(n_r + \gamma)^2 + \alpha^2}} + 1 \right).
\]

For \( n_r \to \infty \):

\[
R = \frac{(j + 1/2)(4k^2 - 1)}{\gamma(4\gamma^2 - 1)}.
\]

For \( j = 1/2 \):

\[
R = \frac{1}{\gamma(4\gamma^2 - 1)} \left[ 2 \frac{n_r + \gamma}{\sqrt{(n_r + \gamma)^2 + \alpha^2}} + 1 \right],
\]

and, for \( n = 1, 2, \ldots \):

\[
1s : \quad R = \frac{1}{2\gamma^2 - \gamma} = \frac{2\gamma + 1}{\gamma(4\gamma^2 - 1)},
\]

\[
2s : \quad R = \frac{1 + \sqrt{2 + 2\gamma}}{\gamma(4\gamma^2 - 1)},
\]

\[
\ldots
\]

\[
\infty s : \quad R = \frac{3}{\gamma(4\gamma^2 - 1)}.
\]
The corrections $T$ on the absolute value of the hyperfine structures are instead:

$$T = R \frac{n^3}{[n_r + \gamma^2 + \alpha^2]^{3/2}}.$$  

For the $s$ terms we have $n_r = n - 1$, $\gamma^2 + \alpha^2 = 1$:

$$T = R \frac{n^3}{[n^2 - 2(n - 1)(1 - \gamma)]^{3/2}}.$$  

In particular:

1s  
$$T = \frac{1}{2\gamma^2 - \gamma},$$

2s  
$$T = \frac{8[(2 + 2\gamma) + \sqrt{2 + 2\gamma}]}{\gamma(2\gamma - 1)(2\gamma + 1)(2 + 2\gamma)^2},$$

$$\ldots$$

$$\frac{T_{1s}}{T_{2s}} = \frac{(2\gamma + 1)(2 + 2\gamma)^2}{2 + 2\gamma + \sqrt{2 + 2\gamma}} = \frac{(2\gamma + 1)(2 + 2\gamma)}{1 + 1/\sqrt{2 + 2\gamma}}.$$  

For $\gamma = 0.74$:

$$\frac{T_{1s}}{T_{2s}} = \frac{2.48 \cdot 3.48}{1 + 1/\sqrt{3.48}} = \frac{8.63}{1.536} = 5.62.$$  

### 3.20. Non-Relativistic Approximation of Dirac Equation for a Two-Particle System

After having obtained the usual non-relativistic decomposition of the Dirac wavefunction (at a first as well as at a second approximation), the author considered a particular expression for of the electromagnetic interaction between a system of two identical charged particle (probably electrons in an atom). Then, he obtains the radial equations for the Dirac components in a central field $\varphi$. 
3.20.1 Non-Relativistic Decomposition

\[ \alpha = \rho_1 \sigma, \quad \psi = (A, B); \]

\[ \rho_1 \psi = \rho_1 (A, B) = (B, A), \quad \rho_3 \psi = \rho_3 (A, B) = (A, -B); \]

\[ \sigma \psi = (\sigma A, \sigma B), \quad \rho_1 \sigma \psi = (\sigma B, \sigma A); \]

\[ \bar{\psi} \alpha \psi = \bar{A} \sigma B + \bar{B} \sigma A \]

\[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) \psi + \rho_1 \left( \sigma, p + \frac{e}{c} U \right) \psi + \rho_3 \: m c \: \psi = 0. \]

\[ \left( \left[ \frac{W}{c} + \frac{e}{c} \varphi \right] A, \left[ \frac{W}{c} + \frac{e}{c} \varphi \right] B \right) + \left( \left[ \sigma, p + \frac{e}{c} U \right] B, \left[ \sigma, p + \frac{e}{c} U \right] A \right) + (m c A, -m c B) = 0. \]

\[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) A + \left( \sigma, p + \frac{e}{c} U \right) B + m c \: A = 0, \]

\[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) B + \left( \sigma, p + \frac{e}{c} U \right) A - m c \: B = 0. \]

For \( U = 0 \):

\[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) A + (\sigma, p) \: B + m c \: A = 0, \]

\[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) B + (\sigma, p) \: A - m c \: B = 0. \]

Since \( (\sigma, p) \: (\sigma, p) = p^2 \):

\[ \left( \frac{W}{c} + \frac{e}{c} \varphi \right) B - \frac{1}{2 m c} \: p^2 - m c \: B = 0, \]

\[ W = m c^2 - e \varphi + \frac{1}{2 m} \: p^2. \]
In a first approximation:

\[ A = -\frac{1}{2mc} (\sigma, p) B, \]

while, in the second approximation:

\[ A = -\frac{1}{2mc} (\sigma, p) B + \frac{W + e\varphi}{4m^2c^3} (\sigma, p) B. \]

3.20.2 Electromagnetic Interaction Between Two Charged Particles

*By considering the total interaction:*

\[ \frac{e^2}{r_{12}} \left( 1 - (\alpha, \alpha') \right), \]

the magnetic interaction term is:

\[ -\frac{e^2}{r_{12}} (\alpha, \alpha') = -\frac{e^2}{r_{12}} \rho_1 \rho'_1 (\sigma, \sigma'). \]

*The 4 components \( A_{ij} \) of the wavefunction may be written as:*

\[
\begin{array}{cccc}
  A_{11} & A_{12} & A_{21} & A_{22} \\
  \psi_1 \psi_2 & \psi_1 \psi_2 & \psi_3 \psi_4 & \psi_3 \psi_4 \\
  \psi'_1 \psi'_2 & \psi'_3 \psi'_4 & \psi'_1 \psi'_2 & \psi'_3 \psi'_4 \\
\end{array}
\]

*The complete expression for the energy is:*

\[ W = -e \varphi(q) - e \varphi(q') - c\rho_1 (\sigma, p) - c\rho'_1 (\sigma' \cdot p') \]

\[ -\rho_3 mc^2 - \rho_3' mc^2 + \frac{e^2}{r_{12}} - \frac{e^2}{r_{12}} \rho_1 \rho'_1 (\sigma \cdot \sigma'). \]

In first approximation:

\[ A_{12} = -\frac{1}{2mc} (\sigma, p) A_{22}, \]

\[ A_{21} = -\frac{1}{2mc} (\sigma', p') A_{22}. \]
3.20.3 Radial Equations

\( A = (\psi_1, \psi_2), B = (\psi_1, \psi_4): \)

\[
\begin{pmatrix}
\frac{W}{e} + \frac{e}{c} \varphi \\
\frac{W}{e} + \frac{e}{c} \varphi
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
+ (\sigma, p) \begin{pmatrix}
B + mc A = 0, \\
B - mc A = 0
\end{pmatrix}
\]

By introducing the two-valued Pauli spherical function \( L \) corresponding to \((\ell, j)\), and \( L_1 = \sigma_z L \) corresponding to \((\ell_1, j)\) (with \( \ell_1 = 2j - \ell \)):

\[
\begin{align*}
B &= g(r)L, & A &= f(r)\sigma_r L = f(r)L_1 \\
\end{align*}
\]

(it having been put \( L = \sigma_z L_1 \)).

\[
(\sigma, p) A = (\sigma, p) f(r)\sigma_r L
= (\sigma_x p_x + \sigma_y p_y + \sigma_z p_z) f(r) \left( \frac{x}{r} \sigma_x + \frac{y}{r} \sigma_y + \frac{z}{r} \sigma_z \right) L
\]

\[
= p_x f(r) \frac{x}{r} L + p_y f(r) \frac{y}{r} L + p_z f(r) \frac{z}{r} L
\]

\[
+ i \left[ p_x f(r) \frac{y}{r} - p_y f(r) \frac{x}{r} \right] \sigma_z L
\]

\[
+ i \left[ p_y f(r) \frac{z}{r} - p_z f(r) \frac{y}{r} \right] \sigma_x L
\]

\[
+ i \left[ p_z f(r) \frac{x}{r} - p_x f(r) \frac{z}{r} \right] \sigma_y L.
\]

\[
\begin{align*}
p_x f(r) \frac{x}{r} L &= \frac{x^2}{r^2} L p_r f(r) + \frac{\hbar}{2\pi i} \frac{r^2 - x^2}{r^3} f(r)L + f(r) \frac{x}{r} p_x L, \\
p_y f(r) \frac{z}{r} \sigma_x L &= \frac{z}{r} \sigma_x L \frac{y}{r} p_r f(r) - \frac{\hbar}{2\pi i} \frac{yr}{r^3} f(r)\sigma_x L + f(r) \frac{z}{r} \sigma_x p_y L
\end{align*}
\]

\[
\left[ p_y f(r) \frac{z}{r} - p_z f(r) \frac{y}{r} \right] \sigma_x L = - \frac{f(r)}{r} (yp_z - zp_y) \sigma_x L.
\]

\[
(\sigma, p) A = L \left\{ p_r f(r) + \frac{z}{r} \frac{\hbar}{2\pi i} f(r) - i \frac{f(r)}{r} \frac{\hbar}{2\pi} (\sigma, \ell) \right\},
\]

\[
(\sigma, p) A = L \left\{ \frac{\hbar}{2\pi} \frac{\partial f}{\partial r} + \frac{h}{2\pi i r} f(r) + \frac{h}{2\pi i} \frac{f(r)}{r} (k - 1) \right\} f(r).
\]
\[ (\sigma, \ell) = \begin{cases} \ell & = k - 1, \\ -\ell - 1 & = k - 1. \end{cases} \]

\[ (\sigma, \ell_1) = \begin{cases} -\ell - 2 & = -(k + 1). \end{cases} \]

\[ (W + mc^2 + e\varphi) f(r) + c \frac{h}{2\pi i} \left\{ \frac{d}{dr} - \frac{k - 1}{r} \right\} g(r) = 0, \]

\[ (W - mc^2 + e\varphi) g(r) + c \frac{h}{2\pi i} \left\{ \frac{d}{dr} + \frac{k + 1}{r} \right\} f(r) = 0. \]

By setting \( r \cdot g(r) = v, r \cdot f(r) = i u \):\(^{49}\)

\[ (W + mc^2 + e\varphi) u - c \frac{h}{2\pi} \left( \frac{d}{dr} - \frac{k}{r} \right) v = 0, \]

\[ (W - mc^2 + e\varphi) v + c \frac{h}{2\pi} \left( \frac{d}{dr} + \frac{k}{r} \right) u = 0. \]

### 3.21. HYPERFINE STRUCTURES AND MAGNETIC MOMENTS: FORMULAE AND TABLES

In the following the author reported some final formulae concerning his studies on hyperfine structures and the atomic magnetic moments (as in the previous Section, he set \( E = W - mc^2, e\varphi = -V \)). Related calculations are developed in the next Section.

\[ (E - V + 2mc^2) u - c \frac{h}{2\pi} \left( \frac{d}{dr} - \frac{k}{r} \right) v = 0, \]

\[ (E - V) v + c \frac{h}{2\pi} \left( \frac{d}{dr} + \frac{k}{r} \right) u = 0, \]

\(^{49}\)In the original manuscript, the second equation in the following is written incorrectly as:

\[ (W - mc^2 - e\varphi) v + c \frac{h}{2\pi} \left( \frac{d}{dr} + \frac{k}{r} \right) u = 0. \]
\[
k = \begin{cases}
\ell + 1 & \left(j = \ell + \frac{1}{2}\right), \\
-\ell & \left(j = \ell - \frac{1}{2}\right),
\end{cases}
\]
\[
k = \left(j + \frac{1}{2}\right)^2 - \ell(\ell + 1), \quad |k| = j + \frac{1}{2},
\]
\[
k(k - 1) = \ell(\ell + 1).
\]

Atomic magnetic moment:
\[
\mu_0 = \frac{e\hbar}{4\pi mc},
\]
\[
-\mathcal{M} = j \, g(j) \, \mu_0 = -e \frac{k}{j + 1} \int r \, u \, v \, dr, \tag{1}
\]
\[
\mu_0 \, g(j) = -\frac{k}{j} \frac{e}{j + 1} \int r \, u \, v \, dr. \tag{1'}
\]

Magnetic field at the origin:
\[
 j \, C = H = \frac{2k}{j + 1} \frac{e}{r^2} \int u \, v \, dr, \tag{2}
\]
\[
 C = \frac{2k}{j(j + 1)} \frac{e}{r^2} \int u \, v \, dr. \tag{2'}
\]

Nuclear magnetic moment:
\[
\mathcal{M}_n = i \, g(i) \frac{\mu_0}{1840} = i \, g(i) \, \mu,
\]
\[
\mu = \frac{e\hbar}{4\pi M_n c} = \frac{\mu_0}{1840}. \tag{2}\]

Hyperfine structure formula:
\[
\delta W = -(\mathcal{M}_n, H) = -(i, j) \, g(i) \, \mu \, C
\]
\[
= -(i, j) \, g(i) \, \mu \, \frac{2k}{j(j + 1)} \frac{e}{r^2} \int u \, v \, dr, \tag{3}
\]

\[\text{[50]}
\]
\[\text{[50]} \text{ In the following the author introduced the sum } f = i + j.\]
\[ (i, j) = \frac{f(f + 1) - i(i + 1) - j(j + 1)}{2} \]

In first approximation:

\[
\int ru \, v \, dr = - \left( k + \frac{1}{2} \right) \frac{h}{4\pi mc} = - \left( k + \frac{1}{2} \right) \frac{\mu_0}{e},
\]

\[
\int \frac{u \, v}{r^2} \, dr = -(k - 1) \frac{h}{4\pi mc} \frac{T}{r^3} = -(k - 1) \frac{T}{r^3} \frac{\mu_0}{e}.
\]

Atomic magnetic moment:

\[
\frac{-M}{\mu_0} = \frac{k(k + 1/2)}{j + 1},
\]

\[
g(j) = \frac{k(k + 1/2)}{j(j + 1)} = \begin{cases} 
\frac{2\ell + 2}{2\ell + 1} \left( j = \ell + \frac{1}{2} \right), \\
\frac{2\ell}{2\ell + 1} \left( j = \ell - \frac{1}{2} \right),
\end{cases}
\]

Magnetic field at the origin:

\[
H = j \, C = - \frac{2k(k - 1)}{j + 1} \frac{T}{r^3} = -2 \frac{\ell(\ell + 1)}{j + 1} \frac{T}{r^3},
\]

\[
C = -\frac{2k(k - 1)}{j(j + 1)} \frac{T}{r^3} = -2 \frac{\ell(\ell + 1)}{j(j + 1)} \frac{T}{r^3}.
\]

Hyperfine structure formula:

\[
\delta W = \frac{\mu_0^2}{1840} (i, j) \, g(i) \frac{2k(k - 1)}{j(j + 1)} \frac{T}{r^3} = \frac{\mu_0^2}{1840} (i, j) \, g(i) \frac{2\ell(\ell + 1)}{j(j + 1)} \frac{T}{r^3}.
\]

For s-terms:

\[
\int \frac{u \, v}{r^2} \, dr = -2\pi \psi^2(0) \frac{h}{4\pi mc} = -2\pi \psi^2(0) \frac{\mu_0}{e}.
\]
\[ H = j \ C = - \frac{8\pi}{3} \ \psi^2(0) \ \mu_0, \]
\[ C = - \frac{16\pi}{3} \ \psi^2(0) \ \mu_0. \]

\[ \delta W = \frac{\mu_0^2}{1840} \ (i, j) \ g(i) \ \frac{16\pi}{3} \ \psi^2(0) = \frac{\mu_0^2}{1840} \ (2i + 1) \ g(i) \ \frac{8\pi}{3} \ \psi^2(0). \]

In first approximation, with a Coulomb field:

\[ \frac{T}{r^3} = \frac{Z^3}{a_0^3} \ \frac{1}{n^3 \ \ell(\ell + 1/2)(\ell + 1)} \]

and, for \( s \)-terms,

\[ \psi^2(0) = \frac{Z^3}{a_0^3} \ \frac{1}{\pi n^3}. \]

\[ \left\{ \begin{array}{l}
\frac{2\ell(\ell + 1)}{j(j + 1)} \ \frac{T}{r^3} \\
s\text{-terms:} \ \frac{16\pi}{3} \ \psi^2(0)
\end{array} \right\} = \frac{Z^3}{a_0^3} \ \frac{4}{n^3 \ j(j + 1)(2\ell + 1)} \]

\[ \delta W = \frac{\mu_0^2}{1840} \ (i, j) \ g(i) \ \frac{Z^3}{a_0^3} \ \frac{4}{n^3 \ j(j + 1)(2\ell + 1)} \]

(which holds also for \( s \)-terms).

\[ \frac{\mu_0^2}{a^3} = \frac{1}{2} \ \alpha^2 \ \text{Rh}, \quad \alpha = \frac{2\pi e^2}{\hbar c}, \quad \alpha^2 R/c = 5.83 \ \text{cm}^{-1}. \]

\[ \delta W = \frac{2\alpha^2 \text{Rh}}{1840} \ (i, j) \ g(i) \ \frac{Z^3}{n^3 \ j(j + 1)(2\ell + 1)}, \]

\[ \frac{\delta W}{hc} = \delta n = 0.00634 \ (i, j) \ g(i) \ \frac{Z^3}{n^3 \ j(j + 1)(2\ell + 1)} \ \text{cm}^{-1}. \]

The term \( \delta n_1 \) corresponds to the particular case \( f = i + j \), that is, \( \cos \ \hat{i} \hat{j} = 1 \) and \( (i, j) = ij \):
\[ \delta n_1 = 0.00634 \frac{Z^3 \ i \ g(i)}{n^3 \ (j + 1)(2\ell + 1)} \ \text{cm}^{-1}. \]

**a + b = c:**

\[ \cos \tilde{a}b = \frac{(a, b)}{a b}, \]

\[ (a, b) = \frac{c(c + 1) - a(a + 1) - b(b + 1)}{2}. \]

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### 3.22. HYPERFINE STRUCTURES AND MAGNETIC MOMENTS: CALCULATIONS

Some calculations concerning atomic systems with magnetic moment are presented in the following, by using similar notations as in the previous Section. The Dirac equation for the $u$ and $v$ wavefunctions underlies such study. Explicit iterative formulae for the perturbative calculation of the wavefunctions are given, as well as the relevant self-consistent relations (left unsolved).

#### 3.22.1 First Method

On using electronic units:\(^{51}\)

\[
\alpha = \frac{Z}{c}, \quad \mu_0 = \frac{1}{2c}.
\]

\[
\left( E + \frac{Z}{r} + 2c^2 \right) u - c \left( \frac{d}{dr} - \frac{k}{r} \right) v = 0,
\]

\[
\left( E + \frac{Z}{r} \right) v + c \left( \frac{d}{dr} + \frac{k}{r} \right) u = 0.
\]

\(^{51}\) In the original manuscript, an unidentified reference (see pages 15 and 25) appears here.
\[
\left( E' \frac{Z}{r} + 2c^2 \right) y_1 - c \left( \frac{d}{dr} - \frac{k}{r} \right) y_2 = \varepsilon r y_2,
\]
\[
\left( E' + \frac{Z}{r} \right) y_2 + c \left( \frac{d}{dr} + \frac{k}{r} \right) y_1 = 0
\]
for \( \varepsilon \to 0 \).

\[
y_1 = P' e^{-\beta' r}, \quad y_2 = Q' e^{-\beta' r},
\]
\[
P' = \sum a'_\mu r^\mu, \quad Q' = \sum b'_\mu r^\mu,
\]
\[
a'_\mu = a_\mu + \varepsilon a^*_\mu, \quad \text{etc.} \quad 52
\]

Remembering that \( \alpha = Z/c \):

\[
(\mu + k) a'_\mu + \alpha b'_\mu = \beta a'_{\mu-1} - \frac{E'}{c} b'_{\mu-1},
\]
\[
-\alpha a'_\mu + (\mu - k) b'_\mu = \left( \frac{E'}{c} + 2c \right) a'_{\mu-1} + \beta b'_{\mu-1} - \frac{E}{c} b'_{\mu-2}.
\]

Note that it is unnecessary to vary \( \beta \).

\[
(\mu + k) a^*_\mu + \alpha b^*_\mu = \beta a^*_{\mu-1} - \frac{E}{c} b^*_{\mu-1} + \beta^* a_{\mu-1} - \frac{E^*}{c} b_{\mu-1},
\]
\[
-\alpha a^*_\mu + (\mu - k) b^*_\mu = \left( \frac{E}{c} + 2c \right) a^*_{\mu-1} + \beta b^*_{\mu-1} + \frac{E^*}{c} a_{\mu-1}
\]
\[
+ \beta^* b_{\mu-1} - \frac{1}{c} b_{\mu-2}.
\]

\[
\left[ \beta (\mu + k) - \alpha \frac{E}{c} \right] a^*_\mu + \left[ \alpha \beta + \frac{E}{c} (\mu - k) \right] b^*_\mu
\]
\[
= \left( \beta \beta^* + \frac{E}{c} \frac{E^*}{c} \right) a_{\mu-1} + \left( -\beta \frac{E^*}{c} + \frac{E}{c} \beta^* \right) b_{\mu-1} - \frac{E}{c} \frac{1}{c} b_{\mu-2}.
\]

Let us set

\[
\nu = \mu_0 + n_r
\]

52@ That is: \( b'_\mu = b_\mu + \varepsilon b^*_\mu, \beta' = \beta + \varepsilon \beta^*, E' = E + \varepsilon E^* \).
and assume that

\[ b_\nu = 0 \quad \text{but} \quad a_\nu \neq 0 : \]

(b\textsubscript{*}\nu = 0)

\[
\left[ \beta (\nu + k) - \alpha \frac{E}{c} \right] a^*_\nu = \left( \beta^{*} \beta + \frac{E}{E^*} \frac{E^*}{c} \right) a_{\nu-1}
+ \left( -\beta \frac{E^*}{c} + \frac{E}{c} \beta^* \right) b_{\nu-1} - \frac{E}{c} \frac{b_{\nu-2}}{c}. \]  

(1)

\[
\begin{cases}
(\nu + k + 1) a^{*}_{\nu+1} + \alpha b^{*}_{\nu+1} = \beta a^{*}_{\nu} + \beta^* a_{\nu} - \frac{E^*}{c} b_{\nu} \\
-\alpha a^{*}_{\nu+1} + (\nu + 1 - k) b^{*}_{\nu+1} = \left( \frac{E}{c} + 2c \right) a^{*}_{\nu} + \frac{E^*}{c} a_{\nu}
+ \beta^* b_{\nu} - \frac{1}{c} b_{\nu-1}
\end{cases}
\]  

(2)

\[
\begin{cases}
(\nu + k + 2) a^{*}_{\nu+2} + \alpha b^{*}_{\nu+2} = \beta a^{*}_{\nu+1} - \frac{E}{c} b^{*}_{\nu+1} \\
-\alpha a^{*}_{\nu+2} + (\nu + 2 - k) b^{*}_{\nu+2} = \left( \frac{E}{c} + 2c \right) a^{*}_{\nu+1}
+ \beta b^{*}_{\nu+1} - \frac{1}{c} b_{\nu}
\end{cases}
\]  

(3)

\[ \beta a^{*}_{\nu+2} - \frac{E}{c} b^{*}_{\nu+2} = 0. \]  

(4)

We can set \( \beta^* = 0 \) or, rather:

\[ \beta^* = \beta \frac{E^*}{E}. \]
It follows that:

\[ \beta \beta^* + \frac{E}{c} \frac{E^*}{c} = \frac{E^*}{E} \left( \beta^2 + \frac{E^2}{c^2} \right) = -2E^*, \]
\[ -\beta \frac{E^*}{c} + \frac{E}{c} \beta^* = 0, \]
\[ \beta^* a_\nu - \frac{E^*}{c} b_\nu = 0, \]
\[ \frac{E^*}{c} a_\nu + \beta^* b_\nu = \frac{E^*}{E} \left( \frac{E}{c} a_\nu + \beta b_\nu \right) = -2c \frac{E^*}{E} a_\nu. \]

\[ \left[ \beta (\nu + k) - \alpha \frac{E}{c} \right] a_\nu^* = -2E^* a_{\nu-1} - \frac{E}{c^2} b_{\nu-2}, \quad (1') \]

\[
\begin{cases}
(\nu + k + i) a_{\nu+1}^* + \alpha b_{\nu+1}^* = \beta a_\nu^* \\
-\alpha a_{\nu+1}^* + (\nu + 1 - k) b_{\nu+1}^* = \left( \frac{E}{c} + 2c \right) a_\nu^* - 2c \frac{E^*}{E} a_\nu - \frac{1}{c} b_{\nu-1}
\end{cases} \quad (2')
\]

Equations (1'), (2'), (3) and (4) are six homogeneous equations in \( a_\nu^*, a_{\nu+1}^*, b_{\nu+1}^*, a_{\nu+2}^*, b_{\nu+2}^* \) and \(-1\).

### 3.22.2 Second Method

\[
\begin{cases}
\left( E' + \frac{Z}{r} + 2c^2 \right) y_1 - c \left( \frac{d}{dr} - \frac{k}{r} \right) y_2 = \varepsilon ry_2, \\
\left( E' + \frac{Z}{r} \right) y_2 + c \left( \frac{d}{dr} + \frac{k^*}{r} \right) y_2 = \varepsilon ry_1.
\end{cases}
\]

\[ E^* = Z \int ruv \, dr. \]
With the previous notations:

\[
\begin{cases}
(\mu + k) a_\mu^* + \alpha b_\mu^* = \beta a_{\mu-1} - \frac{E}{c} b_{\mu-1} + \beta^* a_{\mu-1} \\
- \frac{E^*}{c} b_{\mu-1} + \frac{1}{c} a_{\mu-2}, \\
-\alpha a_\mu^* + (\mu - k) b_\mu^* = \left( \frac{E}{c} + 2c \right) a_{\mu-1} + \beta b_{\mu-1} + \frac{E^*}{c} a_{\mu-1} \\
+ \beta^* b_{\mu-1} - \frac{1}{c} b_{\mu-2}.
\end{cases}
\]

\[\nu = \mu_0 + \nu_r.\]

\[b_\nu = 0, \quad a_\nu \neq 0.\]

Note that it is unnecessary to vary \(\beta\).

\[
\begin{bmatrix}
\beta (\nu + k) - \alpha \frac{E}{c}
\end{bmatrix}
\begin{bmatrix}
a_\nu
\end{bmatrix}
= \left( \beta \beta^* + \frac{E}{c} \frac{E^*}{c} \right) a_{\nu-1} - \left( \frac{\beta E^*}{c} \right)
- \frac{E}{c} \beta^* \right) b_{\nu-1} + \frac{\beta}{c} a_{\nu-2} - \frac{E}{c^2} b_{\nu-2}
\]

\[(b_\nu^* = 0).\]

\[
\begin{bmatrix}
\beta a_{\nu+2} - \frac{E}{c} b_{\nu+2}
\end{bmatrix}
= 0.
\]

\[
\begin{cases}
(\nu + k + 2) a_{\nu+2} + \alpha b_{\nu+2} = \beta a_{\nu+1} - \frac{E}{c} b_{\nu+1} + \frac{1}{c} a_\nu, \\
-\alpha a_{\nu+2} + (\nu + 2 - k) b_{\nu+2} = \left( \frac{E}{c} + 2c \right) a_{\nu+1} + \beta b_{\nu+1} - \frac{1}{c} b_\nu.
\end{cases}
\]

Note that \(a_{\nu+2}^*/b_{\nu+2}^*\) is different from what obtained by Eq. (4), so that \(a_{\nu+2}^* = b_{\nu+2}^* = 0:\)

\[
\begin{cases}
a_{\nu+2}^* = 0, \\
\beta a_{\nu+1}^* - \frac{E}{c} b_{\nu+1}^* + \frac{1}{c} a_\nu = 0.
\end{cases}
\]
\[
\begin{align*}
(\nu + k + 1) a_{\nu+1}^* + \alpha b_{\nu+1}^* &= \beta a_{\nu}^* + \beta^* a_{\nu} - \frac{E^*}{c} b_{\nu} + \frac{1}{c} a_{\nu-1}, \\
-\alpha a_{\nu+1}^* + (\nu + 1 - k) b_{\nu+1}^* &= \left(\frac{E}{c} + 2c\right) a_{\nu}^* + \frac{E^*}{c} a_{\nu} \\
&+ \beta^* b_{\nu} - \frac{1}{c} b_{\nu-1}.
\end{align*}
\]

(2)

[See the equations on pages 257 and 258.]

\[
\left| \begin{array}{cccccc}
 a_{11} & 0 & 0 & 0 & 0 & a_{16} \\
 -\beta & a_{22} & \alpha & 0 & 0 & 0 \\
 0 & 0 & 0 & \beta & -\frac{E}{c} & 0 \\
 a_{41} & -\alpha & a_{43} & 0 & 0 & a_{46} \\
 0 & -\beta & \frac{E}{c} & a_{34} & \alpha & 0 \\
 0 & a_{62} & -\beta & -\alpha & a_{65} & a_{66}
\end{array} \right| = 0,
\]

For a suitable value of \(\beta^*\), from (3) and (4) we get:

\[
a_{\nu+1}^* = 0, \quad b_{\nu+1}^* = \frac{a_{\nu}}{E},
\]

\[
\begin{align*}
\frac{\alpha}{E} a_{\nu} &= \beta a_{\nu}^* + \beta^* a_{\nu} - \frac{E^*}{c} b_{\nu} + \frac{1}{c} a_{\nu-1}, \\
\frac{\nu + 1 - k}{E} a_{\nu} &= \left(\frac{E}{c} + 2c\right) a_{\nu}^* + \frac{E^*}{c} a_{\nu} + \beta^* b_{\nu} - \frac{1}{c} b_{\nu-1}.
\end{align*}
\]

(2')

Equations (1) and (2') are homogeneous equations in \(a_{\nu}^*, \beta^*\) and 1, so that:

[See equation on page 259.]

\[53\text{Note that the second determinant differs from the first one with respect the ordering of the rows (1,2,3,4,5,6 in the first, and 1,2,6,3,4,5 in the second matrix), as pointed out by the author himself in the original manuscript.}\]
\[
\begin{array}{cccccc}
\beta (\nu + k) - \alpha \frac{E}{c} & 0 & 0 & 0 & 0 & -2E^* a_{\nu-1} - \frac{E}{c^2} b_{\nu-2} \\
-\beta & (\nu + k + 1) & \alpha & 0 & 0 & 0 \\
-\left( \frac{E}{c} + 2c \right) & -\alpha & \nu + 1 - k & 0 & 0 & -2c \frac{E^*}{E} a_{\nu} - \frac{1}{2} b_{\nu-1} \\
0 & -\beta & \frac{E}{c} & \nu + k + 2 & \alpha & 0 \\
0 & -\left( \frac{E}{c} + 2c \right) & -\beta & -\alpha & \nu + 2 - k & -\frac{1}{c} b_{\nu} \\
0 & 0 & 0 & \beta & -\frac{E}{c} & 0 \\
\end{array}
\]
\[
\beta (\nu + k) - \alpha \frac{E}{c}
\]

\[
-\beta \quad 0 \quad \frac{(E + 2c)}{c}
\]

\[
\frac{E}{c} - \beta \quad 0 \quad 0
\]

\[
-2E^*a_{\nu - 1} - \frac{E}{c^2}b_{\nu - 2}
\]

\[
0 \quad 0 \quad 0 \quad 0
\]

\[
-2cE^*a_{\nu} - \frac{1}{b_{\nu - 1}}\frac{E}{c}
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\beta \quad 0 \quad 0
\]

\[
-\alpha \quad \nu + k + 2 \quad 0 \quad 0 \quad 0 \quad \frac{E}{c} - \beta
\]

\[
\frac{1}{b_{\nu}} - \beta \quad 0 \quad 0 \quad 0
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \nu + 2 - k
\]
\[
\begin{vmatrix}
\beta & a_\nu & -\frac{E^*}{c} b_\nu + \frac{1}{c} a_{\nu-1} - \frac{\alpha}{E} a_\nu \\
\left( \frac{E}{c} + 2c \right) & b_\nu & \frac{E^*}{c} a_\nu - \frac{1}{c} b_{\nu-1} - \frac{\nu + 1 - k}{E} a_\nu \\
\beta(\nu + k) - \alpha \frac{E}{c} - \left( \beta a_{\nu-1} + \frac{E}{c} b_{\nu-1} \right) & -\frac{E^*}{c} \left( \frac{E}{c} a_{\nu-1} - \beta b_{\nu-1} \right) - \frac{\beta}{c} a_{\nu-2} + \frac{E}{c^2} b_{\nu-2} \\
\end{vmatrix} = 0
\]
4

MOLECULAR PHYSICS

4.1. THE HELIUM MOLECULE

4.1.1 The Equation For $\sigma$-electrons In Elliptic Coordinates

We assume the nuclei to be fixed at a distance $r$ one from the other (in electronic units); the nuclei are supposed to have positive charges, of magnitude $Z \leq 2$, taking approximatively into account the screening action of the other electrons.

$$\nabla^2 \psi + 2 \left( E + \frac{Z}{r_1} + \frac{Z}{r_2} \right) \psi = 0.$$

By measuring the energy (denoted with $W$) in Rh we have $W = 2E$, from which:

$$\nabla^2 \psi + W \psi + 2Z \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \psi = 0.$$

Putting:

$$u = \frac{r_1 + r_2}{2}, \quad v = \frac{r_1 - r_2}{2},$$

$$r_1 = u + v, \quad r_2 = u - v,$$

$$r_1^2 = u^2 + 2uv + v^2, \quad r_2^2 = u^2 - 2uv + v^2, \quad r_1r_2 = u^2 - v^2,$$

we have

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial u^2} |\nabla u|^2 + \frac{\partial^2 \psi}{\partial v^2} |\nabla v|^2 + \frac{\partial \psi}{\partial u} \nabla \cdot u + \frac{\partial \psi}{\partial v} \nabla \cdot v,$$
and, since
\[
|\nabla u|^2 = \frac{1 + \cos(r_1, r_2)}{2} = \frac{1}{2} + \frac{r_1^2 + r_2^2 - 4}{4r_1r_2}
\]
\[
|\nabla v|^2 = \frac{1 - u^2 - v^2 - 2}{2(u^2 - v^2)} = \frac{1 - v^2}{u^2 - v^2},
\]
\[
\nabla^2 u = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{u + v} + \frac{1}{u - v} = \frac{2u}{u^2 - v^2};
\]
\[
\nabla^2 v = \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{u + v} - \frac{1}{u - v} = \frac{2v}{u^2 - v^2};
\]
it follows that
\[
\nabla^2 \psi = \frac{u^2 - 1}{u^2 - v^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{1 - v^2}{u^2 - v^2} \frac{\partial^2 \psi}{\partial v^2} + \frac{2u}{u^2 - v^2} \frac{\partial \psi}{\partial u} - \frac{2v}{u^2 - v^2} \frac{\partial \psi}{\partial v};
\]
\[
\frac{u^2 - 1}{u^2 - v^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{1 - v^2}{u^2 - v^2} \frac{\partial^2 \psi}{\partial v^2} + \frac{2u}{u^2 - v^2} \frac{\partial \psi}{\partial u} - \frac{2v}{u^2 - v^2} \frac{\partial \psi}{\partial v}
\]
\[
+ W \psi + \frac{2Z_1}{u + v} \psi + \frac{2Z_2}{u - v} \psi = 0,
\]
where, for the sake of generality, we have distinguished \(Z_1\) from \(Z_2\) (while we take the half-distance between the nuclei equal to 1). On multiplying the previous equation by \((u^2 - v^2)\):
\[
(u^2 - 1) \frac{\partial^2 \psi}{\partial u^2} + 2u \frac{\partial \psi}{\partial u} + 2u(Z_1 + Z_2) \psi + (1 - v^2) \frac{\partial^2 \psi}{\partial v^2} - 2v \frac{\partial \psi}{\partial v}
\]
\[
- 2v(Z_1 - Z_2) \psi + u^2 W \psi - v^2 W \psi = 0. \tag{1}
\]
By setting
\[
\psi = P_1(u)P_2(v),
\]
and again \(Z_1 = Z_2 = Z\), we have the following separated equations:
\[
(u^2 - 1)P_1'' + 2uP_1' + 4uZP_1 + u^2 WP_1 - \lambda P_1 = 0, \tag{2}
\]
\[
(1 - v^2)P_2'' - 2vP_2' - v^2 WP_2 + \lambda P_2 = 0. \tag{3}
\]
These equations have to be solved together in order to determine \(W\) and \(\lambda\). It is useful to deduce firstly a relation between \(W\) and \(\lambda\) from the second equation, which does not depend on \(Z\) (but depends on the distance between the nuclei, which we have definitively chosen to be
equal to 2; with a similarity transformation we can always turn back to this case). Such a relation between $W$ and $\lambda$ depends only on the azimuthal quantum number, related to $P_2$, and not on the radial one, corresponding to $P_1$.

### 4.1.2 Evaluation Of $P_2$ For $s$-electrons: Relation Between $W$ And $\lambda$

The quantity $P_2$ does not change sign if we replace $v$ with $-v$; $v$ varies between $-1$ and $1$; singular points are at $v = -1$ and $v = 1$. Let us set $P_2(-1) = 1$, so that $P'_2(-1)$ is determined:

$$2P'_2(-1)W + \lambda = 0,$$

$$P'_2(-1) = \frac{W - \lambda}{2}.$$

Quantity $\lambda$ results as determined as the smallest value for which $P'_2(0) = 0$. In Eq. (2) we put, for the moment,

$$v = x - 1 = -1 + x, \quad x = v + 1;$$

it follows:

$$(2x - x^2)P''_2 + (2 - 2x)P'_2 - (1 - x)^2WP_2 + \lambda P_2 = 0;$$

and, setting:

$$P_2 = 1 + \frac{W - \lambda}{2}x + bx^2 + cx^3 + \ldots,$$
$$P'_2 = \frac{W - \lambda}{2} + 2bx + 3cx^2 + \ldots,$$
$$P''_2 = 2b + 6cx + \ldots,$$

after some algebra\(^1\)

$$b = \frac{(W - \lambda)^2}{16} - \frac{W + \lambda}{8},$$
$$c = \frac{b}{3} + \frac{W - \lambda}{18}b + \frac{W}{18} - \frac{W(W - \lambda)}{18}.$$

\(^1\)@ \text{In the original manuscript some scratch calculations are reported, leading to the following expressions for } b \text{ and } c \text{ (obtained by substituting the expansions for } P_2, P'_2, P''_2 \text{ into the differential equation for } P_2 \text{ written above).}
\[ P_2 = 1 + \frac{W - \lambda}{2} x + \left[ \frac{(W - \lambda)^2}{16} - \frac{W + \lambda}{8} \right] x^2 + Cx^3 + \ldots \]

<table>
<thead>
<tr>
<th>( v )</th>
<th>( P_2 )</th>
<th>( -P'_2 )</th>
<th>( P''_2 )</th>
<th>( P_2 )</th>
<th>( -P'_2 )</th>
<th>( P''_2 )</th>
<th>( P_2 )</th>
<th>( -P'_2 )</th>
<th>( P''_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1.000</td>
<td>0.350</td>
<td>0.396</td>
<td>1.000</td>
<td>0.300</td>
<td>0.395</td>
<td>1.000</td>
<td>0.326</td>
<td></td>
</tr>
<tr>
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<td>0.967</td>
<td>0.313</td>
<td>0.365</td>
<td>0.972</td>
<td>0.261</td>
<td>0.379</td>
<td>0.969</td>
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<tr>
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<td>0.937</td>
<td>0.277</td>
<td>0.346</td>
<td>0.948</td>
<td>0.224</td>
<td>0.364</td>
<td>0.942</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.911</td>
<td>0.243</td>
<td>0.331</td>
<td>0.927</td>
<td>0.188</td>
<td>0.355</td>
<td>0.919</td>
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<tr>
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<td>0.211</td>
<td>0.301</td>
<td>0.910</td>
<td>0.153</td>
<td>0.341</td>
<td>0.899</td>
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</tr>
<tr>
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<td>0.189</td>
<td>0.271</td>
<td>0.896</td>
<td>0.125</td>
<td>0.331</td>
<td>0.882</td>
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</tr>
<tr>
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<td>0.166</td>
<td>0.246</td>
<td>0.886</td>
<td>0.095</td>
<td>0.322</td>
<td>0.868</td>
<td></td>
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</tr>
<tr>
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<td>0.812</td>
<td>0.145</td>
<td>0.223</td>
<td>0.877</td>
<td>0.067</td>
<td>0.310</td>
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<td></td>
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</tr>
<tr>
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<td>0.202</td>
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<td>0.181</td>
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<td>0.012</td>
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<td></td>
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</tr>
<tr>
<td>0</td>
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<td>0.161</td>
<td>0.846</td>
<td>0.000</td>
<td>0.280</td>
<td>0.835</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ P''_2 = \frac{2vP'_2 - (v^2 + \lambda)P_2}{1 - v^2} \]
Let us now set:

\[ P_2 = e^{\int z \, dv}, \]
\[ P'_2 = z \, e^{\int z \, dv}, \]
\[ P''_2 = (z' + z^2) \, e^{\int z \, dv}; \]

\[(1 - v^2)z' + (1 - v^2)z^2 - 2vz + \lambda - v^2W = 0. \tag{4}\]

By solving Eq. (4) with respect to \( z' \):

\[ z' = \frac{2v}{1 - v^2}z - \frac{\lambda - v^2W}{1 - v^2} - z^2. \tag{5}\]

\( \lambda \) and \( z \) are infinitesimals with \( W \); we will put:

\[ z = z_1 + z_2 + z_3 + \ldots, \quad \lambda = \lambda_1 + \lambda_2 + \ldots, \]

where \( z_1 \) stands for a first-order infinitesimal, \( z_2 \) for a second-order infinitesimal, etc. We will have:

\[ z'_1 = \frac{2v}{1 - v^2}z_1 - \frac{\lambda_1 - v^2W}{1 - v^2}, \tag{6}\]

from which, by imposing regularity conditions on the boundaries,

\[ z_1 = -\frac{1}{1 - v^2} \int_{-1}^{1} (\lambda_1 - v^2W)dv \]
\[ = -\frac{1}{1 - v} \left( \lambda_1 - \frac{1}{3}W + \frac{1}{3}Wv - \frac{1}{3}Wv^2 \right). \]

**We set:**

\[ q_1 = z_1 = -\frac{1}{3}Wv, \tag{7}\]
\[ \ell_1 = \lambda_1 = \frac{1}{3}W. \tag{8}\]

When determining \( z_2, \) etc., we will set:

\[ q_1 = z_1, \quad q_2 = z_1 + z_2, \quad q_3 = z_1 + z_2 + z_3, \quad \ldots \]
\[ \ell_1 = \lambda_1, \quad \ell_2 = \lambda_1 + \lambda_2, \quad \ell_3 = \lambda_1 + \lambda_2 + \lambda_3, \quad \ldots \]

\( ^3 \text{In the original manuscript the upper limit of the following integrals is not explicitly indicated.} \)
In general we will have:

\[ q'_{n+1} = \frac{2v}{1-v^2} q_{n+1} - \frac{\ell_{n+1} - v^2 W}{1-v^2} - q_n^2. \]

From it:

\[ q_{n+1} = \frac{1}{1-v^2} \int_{-1}^{v} \left[ v^2 W - \ell_{n+1} - (1-v^2) q_n^2 \right] dv \]

\[ = \frac{1}{1-v^2} \left[ \frac{1}{3} v^3 W + \frac{1}{3} W - v \ell_{n+1} - \ell_{n+1} - \int_{-1}^{v} (1-v^2) q_n^2 dv \right] \]

\[ = \frac{1}{3} v^2 W - vW + W - 3\ell_{n+1} - \frac{1}{1-v^2} \int_{-1}^{v} (1-v^2) q_n^2 dv \]

\[ = -\frac{1}{3} Wv + \frac{1}{1-v}\left(-\ell_{n+1} + \frac{1}{3} W - \frac{1}{1+v} \int_{-1}^{v} (1-v^2) q_n^2 dv \right), \]

and, by imposing the regularity at the point \( v = 1 \), it must be:

\[ \ell_{n+1} = \frac{1}{3} W - \frac{1}{2} \int_{-1}^{1} (1-v^2) q_n^2 dv. \] (9)

By substituting Eq. (9) into previous equation:

\[ q_{n+1} = -\frac{1}{3} Wv + \frac{1}{1-v} \left( \frac{1}{2} \int_{-1}^{1} (1-v^2) q_n^2 dv \right. \]

\[ - \frac{1}{1+v} \int_{-1}^{v} (1-v^2) q_n^2 dv \bigg), \] (10)

or, more easily,

\[ q_{n+1} = -\frac{1}{3} Wv + \frac{1}{1-v^2} \left( \frac{1}{2} v \int_{-1}^{1} (1-v^2) q^2 dv \right. \]

\[ - \int_{0}^{v} (1-v^2) q_n^2 dv \bigg). \] (10')

By taking into account that \( q_{n+1}(v) = -q_{n+1}(-v) \), we also have:

\[ \ell_{n+1} = W - 2q_{n+1}(-1) = W + 2q_{n+1}(1), \] (11)

which can replace Eq. (9). Let us now evaluate \( q_2 \); since \( q_1 = -\frac{1}{3} Wv \), by substitution into Eq. (9'):
\[ q_2 = -\frac{1}{3}Wv + \frac{1}{1 - v^2} \left[ \frac{1}{2}v \int_{-1}^{1} (1 - v^2) \left( -\frac{1}{3}Wv \right)^2 dv \right. \]
\[ \left. - \int_{0}^{v} (1 - v^2) \left( -\frac{1}{3}Wv \right)^2 dv \right], \]

that is:

\[ q_2 = -\frac{1}{3}vW + \frac{W^2}{9} \frac{1}{(1 - v^2)} \left( \frac{2}{15}v - \frac{1}{3}v^3 + \frac{1}{5}v^5 \right), \]

or, more simply:

\[ q_2 = -\frac{1}{3}vW - \left( \frac{1}{45}v^3 - \frac{2}{135}v \right) W^2, \]

\[ l_2 = \frac{1}{3}W - \frac{2}{135}W^2. \]

Recalling that

\[ z = z_1 + z_2 + z_3 + \ldots, \]
\[ \lambda = \lambda_1 + \lambda_2 + \lambda_3 + \ldots, \]
\[ q_n = z_1 + z_2 + \ldots + z_n, \]
\[ \ell_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n, \]

and that \( z_n \) and \( \lambda_n \) are infinitesimals of order \( n \), with this procedure we can obtain any term in the series expansion of \( z \) and \( \lambda \) with increasing powers of \( W \):

\[ z = -\frac{1}{3}vW - \left( \frac{1}{45}v^3 - \frac{2}{135}v \right) W^2 + \ldots, \]

\[ \lambda = \frac{1}{3}W - \frac{2}{135}W^2 + \ldots. \]

From \( z \) we can then obtain \( P_2 \):

\[ P_2 = e^{\int z dv}, \]

by choosing a suitable normalization, in such a way that \( P_2(-1) = P_2(1) = 1 \):

\[ P_2 = e^{\frac{1}{6}(1-v^2)W - \frac{1}{540}(1-4v^2+3v^4)W^2 + \ldots}, \]

\[ P_2(0) = e^{\frac{1}{6}W - \frac{1}{540}W^2 + \ldots}. \]
The expansions (12), (13) and (15) cannot be used for large values of $W$. Then, we now consider asymptotic expansions with decreasing powers of $W$ for $W$ tending to the (negative) infinity. We will set:

$$
\begin{align*}
    z &= y_1 + y_2 + y_3 + \ldots, \\
    \lambda &= m_1 + m_2 + \ldots, \\
    p_n &= y_1 + \ldots + y_n, \\
    L_n &= m_1 + \ldots + m_n,
\end{align*}
$$

where we always assume that $m_{n+1}/m_n$ or $y_{n+1}/y_n$ are infinitesimals for $W \to -\infty$ and consider only infinities of higher order. By substitution into Eq. (5):

$$
    y_1^2 = -\frac{m_1 - v^2 W}{1 - v^2},
$$

so that, by requiring regularity in the singular points,

$$
\begin{align*}
    L_1 &= m_1 = W, \\
    p_1 &= y_1 = \pm \sqrt{-W}.
\end{align*}
$$

Since $p_1(v) = p_1(-v)$ (and $\int_{-1}^{0} p_1(v)dv$ is certainly negative) and $p_1$ has the same sign as $v$,

$$
\begin{align*}
    p_1 &= y_1 = -\sqrt{-W}, \quad v < 0; \\
    p_1 &= y_1 = \sqrt{-W}, \quad v > 0.
\end{align*}
$$

Note that the discontinuity at the point 0 results in a divergence for $z'$ in Eq. (5), which cannot be neglected; however, by replacing the jump with a suitable junction line in the interval $-\varepsilon, +\varepsilon$, $|z'|$ will be of the order of $\sqrt{-W}/\varepsilon$, while the other infinities are of the same order of $W$. Then we can neglect $z'$ provided that:

$$
\varepsilon \sqrt{-W} \gg 1,
$$

and since $W$ tends to the infinity, we may take the limit $\varepsilon = 0$.

For the successive approximations we have to consider:

$$
    p'_n = \frac{2v}{1 - v^2} p_n - \frac{L_{n+1} - v^2 W}{1 - v^2} - p_{n+1}^2,
$$

and, imposing the regularity conditions,

$$
    L_{n+1} = W - 2p_n(-1) = W + 2p_n(1),
$$
one gets

\[
p_{n+1} = -\sqrt{-\frac{L_{n+1} - v^2 W}{1 - v^2}} + \frac{2v}{1 - v^2} p_n - p'_n \quad (v < 0),
\]

\[
p_{n+1} = \sqrt{-\frac{L_{n+1} - v^2 W}{1 - v^2}} + \frac{2v}{1 - v^2} p_n - p'_n \quad (v > 0).
\]

The asymptotic expansions of \( z \) do not yield a continuous curve and cannot be used in any interval around \( v = 0 \) whose extension is of the order of \( \sqrt{-W} \). We will find later an appropriate approximation formula for \( z \).

We now focus directly on the asymptotic expansion of \( \lambda \) as a function of \( W \).

By integrating Eq. (2) from \(-1\) to 0 we obtain:

\[
\lambda = W \int_{-1}^{0} \frac{v^2 P_2 \, dv}{P_2}.
\]

For \( W \to -\infty \) it suffices to integrate over a very small interval, starting at \(-1\) for any order of approximation; this would be an indication of the fact that the asymptotic expansion is never convergent.

By setting

\[
x = 1 + v, \quad v = x - 1,
\]

Equation (2) becomes:

\[
(2x - x^2)P''_2 + (2 - 2x)P'_2 - (1 - x)^2 WP_2 + \lambda P_2 = 0,
\]

and, putting

\[
P_2 = Re^{-\sqrt{-W}x},
\]

\[
P'_2 = (R' - R\sqrt{-W})e^{-\sqrt{-W}x},
\]

\[
P''_2 = (R'' - 2R'\sqrt{-W} - RW)e^{-\sqrt{-W}x},
\]

it follows:

\[
(2x - x^2)R'' - [(2x - x^2)2\sqrt{-W} - (2 - 2x)]R' - (2x - x^2)RW
- (1 - x)^2 RW - 2R(1 - x)\sqrt{-W} + \lambda R = 0,
\]
that is:

\[(2x - x^2)R'' - 2[(2x - x^2) - \sqrt{W} - (1 - x)]R'\]
\[-[W - \lambda + 2(1 - x)\sqrt{-W}]R = 0.\]

For \(x = 0\) we will take \(R = 1\). It follows:

\[2R'(0) = W - \lambda + 2\sqrt{-W},\]

where from:

\[R = 1 + \frac{W - \lambda + 2\sqrt{-W}}{2}x + bx^2 + \ldots,\]
\[R' = \frac{W - \lambda + 2\sqrt{-W}}{2} + 2bx + \ldots,\]
\[R'' = 2b + \ldots.\]

By substitution into the above equation, from the vanishing of first-order terms, one has

\[4b - 2(W - \lambda - 2\sqrt{-W})\sqrt{-W} - (W - \lambda - 2\sqrt{-W}) + 2\sqrt{-W} = 0,\]

where from:

\[b = \frac{(W - \lambda - 2\sqrt{-W} - 1)}{2}\sqrt{-W} + \frac{W - l\lambda - 2\sqrt{-W}}{4}.\]

On the other hand, asymptotically we have:

\[\lambda = W - W\int_0^\infty \frac{(2x - x^2)P_2dx}{\int_0^\infty P_2dx},\]

and, since

\[P_2 = (1 + ax + bx^2 + \ldots) e^{-\sqrt{-W}x},\]
\[(2x - x^2)P_2 = [2x + (2a - 1)x^2 + (2b - a)x^3 + \ldots] e^{-\sqrt{-W}x},\]
we deduce:

\[
\lambda = W - W \frac{2 + 2a - 1 + 2b - a}{(-W)^{3/2} + W^2}, \\
\lambda = W + 2\sqrt{-W} \frac{1 + 2a - 1 + 2b - a}{2\sqrt{-W} - W - 2} \\
\lambda = W + 2\sqrt{-W} - 1 + \ldots.
\]

Summing up, for the moment we know the behavior of the function \(\lambda = \lambda(W)\) for small and large values of \(W\):

\[
W \to 0, \quad \lambda = \frac{1}{3}W - \frac{2}{135}W^2 + \ldots, \tag{25}
\]

\[
W \to -\infty, \quad \lambda = W + 2W^2 + \sqrt{-W} - 1 + \ldots.
\]

Let us put again

\[
P_2 = e^{f(v)} \int_{-1}^{v} r\,dv;
\]

it follows:

\[
z' = \frac{2v}{1 - v^2} z - \lambda - v^2 W - r^2. \tag{5}
\]

As an approximate solution, we take:

\[
z = a \arctan bv. \tag{26}
\]

Substituting it into Eq. (5):

\[
\frac{ab}{1 + b^2v^2} = \frac{2va}{1 - v^2} \arctan bv - \frac{\lambda - v^2 W}{1 - v^2} - u^2 \arctan^2 bv + \ldots. \tag{27}
\]

We require that this equation be satisfied for \(v = 0\); it follows:

\[
ab = -\lambda. \tag{28}
\]

Regularity conditions for \(v = 1\) impose:

\[
2a \arctan b = \lambda - W. \tag{29}
\]
We also require that the equation be satisfied for \( v = 1 \), since:

\[
\lim_{v \to 1} \frac{2av \arctan bv - \lambda + v^2W}{1 - v^2} = -W - a \arctan b - \frac{ab}{1 + b^2}.
\]

It follows:

\[
W + a \arctan b + \frac{2ab}{1 + b^2} + a^2 \arctan^2 b = 0. \tag{30}
\]

From Eqs. (28), (29), (30) we can determine \( a, b \) and \( \lambda \). We can then consider the following equations:

\[
W + \left( \frac{\lambda - W}{2} \right)^2 + \frac{\lambda - W}{2} - \frac{2\lambda}{1 + b^2} = 0, \tag{31}
\]

\[
b = \tan \left( b \frac{W - \lambda}{2\lambda} \right) = \tan \left( b \frac{1 - \frac{\lambda}{W}}{\frac{2\lambda}{W}} \right), \tag{32}
\]

\[
a = -\frac{\lambda}{b}. \tag{33}
\]

By taking a series expansion, for small \( W \) we have:

\[
\lambda = \frac{1}{3} W + KW^2 + \ldots,
\]

\[
\frac{\lambda}{W} = \frac{1}{3} + KW + \ldots.
\]

Equation (32) becomes:

\[
b = \tan \left( b \frac{2}{3} - KW + \ldots \right) = \tan \left( b - \frac{9}{2} bKW + \ldots \right).
\]

On the other hand:

\[
\left( b - \frac{9}{2} bKW \right) = \arctan b = b - \frac{1}{3} b^3 + \ldots,
\]

from which:

\[
-\frac{9}{2} KbW = -\frac{1}{3} b^3 + \ldots,
\]

\[
b^2 = \frac{27}{2} KW + \ldots.
\]
Substituting it into Eq. (31):

\[ 1 + \frac{1}{9}W - \frac{1}{3} + \frac{1}{2}KW - \frac{2}{3} - 2KW + 9KW + \ldots = 0, \]

from which:

\[ \frac{1}{9} + \frac{1}{2}K - 2K + 9K = 0, \quad \frac{1}{9} + \frac{15}{2}K = 0, \]

\[ K = -\frac{2}{135}, \]

\[ \lambda = \frac{1}{3}W - \frac{2}{135}W^2 + \ldots, \]

which agrees with Eq. (25). We have thus an exact result holding in first and second approximation:

\[ b^2 = -\frac{1}{5}W + \ldots. \]  \hfill (34)

For the asymptotic expansion \((W \to -\infty)\), we set:

\[ \lambda = W + 2\sqrt{-W} + \alpha + \ldots. \]

By substituting it into Eq. (31), noting that \(b\) is an infinite of order \(1/2\) and equating to zero higher-order infinities, we have:

\[ \alpha\sqrt{-W} + \sqrt{-W} = 0, \]

from which \(\alpha = -1\) and:

\[ \lambda = W + 2\sqrt{-W} - 1 + \ldots, \]

which again agrees with Eq. (25). We can likely presume that for arbitrary \(W\) a very good approximation for \(\lambda = \lambda(W)\) is obtained.

\[ ^4 \text{It is not very clear how the author obtained the values reported in the following table. Probably, for a given value of } W, \text{ } \lambda \text{ was obtained from the approximate Eq. (25) for } W \to 0 \text{ (in this case, for } -W = 2,3 \text{ we would have } -\lambda = 0.726,1.133), \text{ while } b \text{ is deduced from Eq. (31).} \]

<table>
<thead>
<tr>
<th>(-W)</th>
<th>(-\lambda)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>+0.348</td>
<td>0.47</td>
</tr>
<tr>
<td>2</td>
<td>+0.731</td>
<td>0.72</td>
</tr>
<tr>
<td>3</td>
<td>1.151</td>
<td>0.94</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The following two tables seem the continuation of the table appearing at page 264, but it is not clear how the author obtained the numerical values reported here. Note that, as above, in some places the author omits the notation “0.” in the reported numbers.

Probably, the numerical values for the second derivative of $P_2''$ were deduced in some manner from the following formula (which appears in the manuscript):

\[
P_2'' = \frac{2vP_2' - (2v^2 + \lambda)P_2}{1 - v^2},
\]

for $W = -2$, and

\[
P_2'' = \frac{2vP_2' - (3v^2 + \lambda)P_2}{1 - v^2},
\]

for $W = -3$. 

<table>
<thead>
<tr>
<th>$W = -2$</th>
<th>$W = -3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = -0.72$</td>
<td>$\lambda = -0.74$</td>
</tr>
<tr>
<td>$v$</td>
<td>$P_2$</td>
</tr>
<tr>
<td>$-1$</td>
<td>1.000</td>
</tr>
<tr>
<td>$-0.95$</td>
<td>969.1</td>
</tr>
<tr>
<td>$-0.9$</td>
<td>940.3</td>
</tr>
<tr>
<td>$-0.85$</td>
<td>888.7</td>
</tr>
<tr>
<td>$-0.7$</td>
<td>844.5</td>
</tr>
<tr>
<td>$-0.6$</td>
<td>807.2</td>
</tr>
<tr>
<td>$-0.5$</td>
<td>776.1</td>
</tr>
<tr>
<td>$-0.4$</td>
<td>751.1</td>
</tr>
<tr>
<td>$-0.3$</td>
<td>732.1</td>
</tr>
<tr>
<td>$-0.2$</td>
<td>718.1</td>
</tr>
<tr>
<td>$-0.1$</td>
<td>709.1</td>
</tr>
<tr>
<td>$0$</td>
<td>706.9</td>
</tr>
<tr>
<td>$\lambda = -1.14$</td>
<td>$\lambda = -1.16$</td>
</tr>
<tr>
<td>$v$</td>
<td>$P_2$</td>
</tr>
<tr>
<td>$-1$</td>
<td>1.000</td>
</tr>
<tr>
<td>$-0.95$</td>
<td>955.3</td>
</tr>
<tr>
<td>$-0.9$</td>
<td>914.1</td>
</tr>
<tr>
<td>$-0.85$</td>
<td>876.1</td>
</tr>
<tr>
<td>$-0.8$</td>
<td>841.1</td>
</tr>
<tr>
<td>$-0.7$</td>
<td>780.1</td>
</tr>
<tr>
<td>$-0.6$</td>
<td>729.1</td>
</tr>
<tr>
<td>$-0.5$</td>
<td>687.1</td>
</tr>
<tr>
<td>$-0.4$</td>
<td>654.1</td>
</tr>
<tr>
<td>$-0.3$</td>
<td>629.1</td>
</tr>
<tr>
<td>$-0.2$</td>
<td>611.1</td>
</tr>
<tr>
<td>$-0.1$</td>
<td>600.1</td>
</tr>
<tr>
<td>$0$</td>
<td>597.3</td>
</tr>
</tbody>
</table>
4.1.3 Evaluation Of $P_1$

In the general case $Z_1 \neq Z_2$, equations (1) and (2) become:

\[(u^2 - 1)P_1'' + 2uP_1' + 2u(Z_1 + Z_2)P_1 + u^2 WP_1 - \lambda P_1 = 0, \quad (35)\]
\[(1 - v^2)P_2'' - 2vP_2' - 2v(Z_1 - Z_2)P_2 - v^2 WP_2 + \lambda P_2 = 0. \quad (36)\]

For the moment we focus only on $P_1$ or, better, on the first eigenfunction that $P_1$ can represent. Then the energy $W$ depends on $Z_1 + Z_2$ and $\lambda$ (we suppose that they are given by or depend in a given way on $W$).

Let us consider the ground state $1s\sigma$; for $\sigma$-electrons we know a relation between $W$ and $\lambda$ due to Eq. (2). We have only to fix $Z_1 + Z_2$. The expansion for large $Z = (Z_1 + Z_2)/2$ is:

\[W = Z^2 + Z + \ldots. \quad (37)\]

[6]

4.2. VIBRATION MODES IN MOLECULES

A particular study of the vibration modes in molecules was carried out in the following notes. The main scope was to diagonalize the quadratic forms of kinetic ($T$) and potential energy ($V$) of the coupled oscillators, in order to find the eigenfrequencies and eigendirections of their vibration modes. Several cases were considered, and a particularly careful study was devoted to the vibration modes of the molecule $C_2H_2$ (acetylene) that, due to its geometry, presents three eigenfrequencies, two of which are equal. A possible different (more general) study, suggested by the

\[\text{This Section was probably left incomplete. The corresponding page in the original manuscript reported the following table with practically no entry, pointing out the intention of the author to evaluate } P_1 \text{ and its derivatives for some values of } W \text{ and } \lambda, \text{ in analogy with what was already done for } P_2:\]

\[Z_1 + Z_2 = 4, \quad 1s\sigma\]

\[
\begin{array}{cccccccc}
W = & \lambda = & W = & \lambda = & W = & \lambda = & W = & \lambda = \\
\hline
u & P_1 & P_1' & P_1'' & P_1 & P_1' & P_1'' & P_1 & P_1' & P_1'' \\
1.00 & & & & & & & & & \\
\end{array}
\]

\[\]
just considered molecule of acetylene, was envisaged at the end of this Section.

\[ T = \frac{1}{2} \sum \dot{x}_i^2, \quad V = \frac{1}{2} \sum (x_i - x_{i-1})^2; \]

\[ \frac{\partial T}{\partial \dot{x}_i} = \dot{x}_i, \quad \frac{\partial V}{\partial x_i} = x_i - x_{i-1} - x_{i+1} + x_i = 2x_i - x_{i+1} - x_{i-1}. \]

The equation of motion is then:

\[ \ddot{x}_i = x_{i+1} - 2x_i + x_{i-1}. \]

\[ x_i = c_i \eta: \]

\[ \ddot{x}_i = c_i \ddot{\eta} = x_{i+1} - 2x_i + x_{i-1}, \]

\[ c_i \ddot{\eta} = (c_{i+1} - 2c_i + c_{i-1}) \eta. \]

\[ \ddot{\eta} = -\lambda \eta: \]

\[ -c_i \lambda = (c_{i+1} - 2c_i + c_{i-1}). \]

\[ c_r = k^r: \]

\[ -\lambda = k - 2 + \frac{1}{k}, \]

\[ k^2 - (2 - \lambda)k + 1 = 0; \]

\[ k = \frac{2 - \lambda \pm \sqrt{-4\lambda + \lambda^2}}{2} = 1 - \frac{\lambda}{2} \pm \sqrt{\frac{1}{4} \lambda(\lambda - 4)}. \]

\[ k = e^{i\varphi}, \quad \varphi = \arccos \left( 1 - \frac{\lambda}{2} \right). \]

\[ k_1 = e^{\frac{2\pi i}{N}}, \quad k_2 = e^{\frac{2\pi i}{N}}, \ldots \quad k_r = e^{\frac{2\pi i}{N}}, \ldots \quad k_N = 1; \]

\[ \varphi_1 = \frac{2\pi}{N}, \quad \varphi_2 = 2 \frac{2\pi}{N}, \ldots \quad \varphi_r = r \frac{2\pi}{N}, \ldots \quad \varphi_n = 2\pi. \]

\[ \cos \varphi = 1 - \frac{\lambda}{2}, \quad \frac{\lambda}{2} = 1 - \cos \varphi, \quad \lambda = 4 \sin^2 \frac{\varphi}{2}; \]

\[ \lambda_r = 4 \sin^2 \frac{r}{N} \pi. \]
\[ U = \frac{1}{2} \sum a_{ik} q_i q_k, \quad T = \frac{1}{2} \sum b_{ik} \dot{q}_i \dot{q}_k. \]

\[ \sum a_{ik} q_i q_k = \sum a_{ik} S_{ir} S_{ks} \xi_r \xi_s = \sum A_{rs} \xi_r \xi_s, \]
\[ \sum b_{ik} \dot{q}_i \dot{q}_k = \sum b_{ik} S_{ir} S_{ks} \dot{\xi}_r \dot{\xi}_s = \sum B_{rs} \dot{\xi}_r \dot{\xi}_s, \]

\[ A_{rs} = \sum a_{ik} S_{ir} S_{ks}, \quad A = S^* a S, \]
\[ B_{rs} = \sum b_{ik} S_{ir} S_{ks}, \quad B = S^* b S. \]

\[ B_{rs} = \delta_{rs}, \quad A_{rs} = \lambda_r \delta_{rs}. \]

\[ \lambda_s \delta_{rs} = \sum_{ik} a_{ik} S_{ir} S_{ks}, \]
\[ \delta_{rs} = \sum_{ik} b_{ik} S_{ir} S_{ks}. \]

\[ \left( \sum_{ik} (a_{ik} - \lambda_s b_{ik}) S_{ir} S_{ks} = 0 \right) \cdot \xi_r, \]
\[ \sum_{ik} (a_{ik} - \lambda_r b_{ik}) S_{ir} S_{ks} = 0; \]
\[ \sum_{ik} (a_{ik} - \lambda_s b_{ik}) q_i S_{ks} = 0; \]
\[ \sum_k (a_{ik} - \lambda_s b_{ik}) S_{ks} = 0, \quad i = 1, 2, \ldots, n; \]

\[ \text{In the original manuscript, the potential and kinetic energies are loosely written as } \]
\[ U = \frac{1}{2} \sum a_{ik}, \quad T = \frac{1}{2} \sum b_{ik}. \]

\[ \text{In the original manuscript, the dots (differentiation with respect to time) over the } \xi \]
\[ \text{variables in the last expression were omitted.} \]
\[
\sum_k (a_{ik} - \lambda_t b_{ik}) S_{kt} = 0.
\]
\[
\sum b_{ik} S_{it} S_{ks} = f(t) \delta_{rs}.
\]

### 4.2.1 The Acetylene Molecule

![Diagram of the acetylene molecule]

\[
V = \frac{1}{2} (aq_1^2 + bq_2^2 + aq_3^2).
\]

\[
y_1 + y_2 = q_2,
\]
\[
x_1 + x_2 = q_1 + q_2 + q_3,
\]
\[
x_2 - y_2 = q_3,
\]
\[
y_2 - y_1 = 2y_2 - q_2.
\]
\[
x_2 - x_1 + 12(y_2 - y_1) = 0.
\]
\[
2y_2 + q_3 - q_1 - q_2 + 24y_2 - 12q_2 = 0,
\]
\[
26y_2 - q_1 - 13q_2 + q_3 = 0.
\]

\[
y_2 = \frac{q_1 + 13q_2 - q_3}{26},
\]
\[
y_1 = \frac{-q_1 + 13q_2 + q_3}{26},
\]
\[
x_2 = \frac{q_1 + 13q_2 + 25q_3}{26},
\]
\[
x_1 = \frac{25q_1 + 13q_2 + q_3}{26}.
\]

\[
(26)^2(x_1^2 + x_2^2 + 12y_1^2 + 12y_2^2)
\]
\[
= (q_1 + 13q_2 + 25q_3)^2 + (25q_1 + 13q_2 + q_3)^2
\]
\[
+ 12(q_1 + 13q_2 - q_3)^2 + 12(-q_1 + 13q_2 + q_3)^2.
\]
The following table was aimed to fully evaluate the expression just reported above. The numbers given in the lines 2 through 4 are just the coefficients of the terms indicated in the first line, while those in the sixth line are the corresponding sums. In the last line the author listed these sums divided by 26.

\[
\begin{array}{ccccccc}
q_1^2 & q_2^2 & q_3^2 & q_1q_2 & q_1q_3 & q_2q_3 & q_3q_1 \\
1 & 169 & 625 & 26 & 650 & 50 \\
26 & 625 & 169 & 1 & 650 & 26 & 50 \\
12 & 2028 & 12 & 312 & -312 & -24 \\
12 & 2028 & 12 & -312 & 312 & 24 \\
650 & 4394 & 650 & 676 & 676 & 52 \\
26 & 25 & 169 & 25 & 26 & 26 & 2 \\
\end{array}
\]

\[
26 \left( \dot{x}_1^2 + \dot{x}_2^2 + 12\dot{y}_1^2 + 12\dot{y}_2^2 \right) \\
= 25q_1^2 + 169q_2^2 + 25q_3^2 + 26q_1q_2 + 26q_2q_3 + 2q_3q_1.
\]

\[
U = \frac{1}{2} \begin{vmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & a \\
\end{vmatrix}, \quad T' = 26T = \frac{1}{2} \begin{vmatrix}
25 & 13 & 1 \\
13 & 169 & 13 \\
1 & 13 & 25 \\
\end{vmatrix}, \\
T = \frac{1}{2} \begin{vmatrix}
25 & 1 & 1 \\
1 & 13 & 1 \\
1 & 1 & 25 \\
\end{vmatrix}, \\
T = \frac{1}{2} \begin{vmatrix}
26 & 2 & 26 \\
2 & 2 & 2 \\
26 & 2 & 26 \\
\end{vmatrix},
\]

\[
U - \lambda T = \begin{vmatrix}
a - \frac{25}{26} \lambda & -\frac{1}{2} \lambda & \frac{1}{26} \lambda \\
-\frac{1}{2} \lambda & b - \frac{13}{2} \lambda & -\frac{1}{2} \lambda \\
-\frac{1}{26} \lambda & -\frac{1}{2} \lambda & a - \frac{25}{26} \lambda \\
\end{vmatrix}.
\]
\[ 2V = aq_1^2 + bq_2^2 + aq_3^2, \]
\[ 2T = \frac{25}{26} \dot{q}_1^2 + \frac{13}{2} \dot{q}_2^2 + \frac{25}{26} \dot{q}_3^2 + \dot{q}_1 \ddot{q}_2 + \dot{q}_2 \ddot{q}_3 + \frac{1}{13} \dot{q}_3 \dot{q}_1. \]

\[ u = \frac{q_1 + q_3}{2}, \quad v = \frac{q_1 - q_3}{2}, \]
\[ q_1 = u + v, \quad q_3 = u - v. \]

\[ q_1^2 + q_3^2 = 2u^2 + 2v^2, \quad q_1q_3 = u^2 - v^2, \quad q_1 + q_3 = 2u. \]

\[ 2V' = 2au^2 + bq_2^2, \]
\[ 2T' = 2 \ddot{u}^2 + \frac{13}{2} \dot{q}_2^2 + 2u \ddot{q}_2. \]
\[ U' = \begin{vmatrix} 2a & 0 \\ 0 & b \end{vmatrix}, \quad T' = \begin{vmatrix} 2 & 1 \\ 1 & 13/2 \end{vmatrix}, \]

\[ U' - \lambda T' = \begin{vmatrix} 2a - 2\lambda & -\lambda \\ -\lambda & b - 13/2 \lambda \end{vmatrix}. \]

\[ 12\lambda^2 - (13a + 2b)\lambda + 2ab = 0 \]

\[ T = \frac{1}{2}(a\dot{\varphi}_1^2 + a\dot{\varphi}_2^2 - 2b\dot{\varphi}_1\dot{\varphi}_2), \]

\( b < a. \)

\[ x = \frac{\varphi_1 + \varphi_2}{2}, \quad y = \frac{\varphi_1 - \varphi_2}{2}, \]

\[ \varphi_1 = x + y, \quad \varphi_2 = x - y. \]

\[ \dot{\varphi}_1^2 + \dot{\varphi}_2^2 = 2\dot{x}^2 + 2\dot{y}^2, \quad \dot{\varphi}_1\dot{\varphi}_2 = \dot{x}^2 - \dot{y}^2. \]

\[ T = \frac{1}{2} \left[ 2(a - b)\dot{x}^2 + 2(a + b)\dot{y}^2 \right]; \]

\[ \frac{\partial T}{\partial \dot{x}} = 2(a - b)\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = 2(a + b)\dot{y}. \]

\[ V = -C_1\varphi_1 + C_2(t)\varphi_2 = [-C_1 + C_2(t)] x - [C_1 + C_2(t)] y; \]

\[ 13a + 2b \pm \sqrt{169a^2 - 44ab + 4b^2} \]

\[ \frac{24}{24}, \]

whose physical meaning probably was not clear.
\[ \frac{\partial V}{\partial x} = -C_1 + C_2(t), \quad \frac{\partial V}{\partial y} = -(C_1 + C_2(t)). \]

\[ 2(a - b)x = -C_1 + C_2(t). \]

\[ T = \frac{1}{2} \left( a\ddot{\varphi}_1^2 - 2b\dot{\varphi}_1\dot{\varphi}_2 + c\dot{\varphi}_2^2 \right), \]

\[ V = -C_1\varphi_1 + C_2\varphi_2; \]

\[ \frac{\partial T}{\partial \varphi_1} = a\dot{\varphi}_1 - b\dot{\varphi}_2, \quad \frac{\partial T}{\partial \varphi_2} = c\dot{\varphi}_2 - b\dot{\varphi}_1, \]

\[ \frac{\partial V}{\partial \varphi_1} = -C_1, \quad \frac{\partial V}{\partial \varphi_2} = C_2. \]

\[ a\ddot{\varphi}_1 - b\ddot{\varphi}_2 = C_1, \]

\[ c\ddot{\varphi}_2 - b\ddot{\varphi}_1 = C_2. \]

\[ \varphi_2 = \dot{\varphi}_2 = \ddot{\varphi}_2 = 0: \]

\[ a\ddot{\varphi}_1 = C_1, \quad -b\ddot{\varphi}_1 = C_2; \]

\[ C_2 = \frac{b}{a}C_1. \]

### 4.3. REDUCTION OF A THREE-FERMION TO A TWO-PARTICLE SYSTEM

The following calculations are aimed at studying the system formed by three fermions, the first two being described by the state \( \Psi(q_1, q_2) \), and the third one by \( \Psi(q) \). After some general remarks, the author shows how the study of the system considered may be reduced to that of a suitable two-particle system. Probably, he refers to the \( H_2^+ \) molecule or similar systems.
Let us consider an antisymmetric function of \( q_1, \ldots, q_n, \psi_n(q_1, q_2, \ldots, q_n) \):
\[
\sqrt{n + 1} \psi^{n+1}(q_1, q_2, \ldots, q_{n+1}) = \psi^n(q_1, \ldots, q_n) \psi'(q_{n+1}) + \psi^n(q_2, q_3, \ldots, q_{n+1}) \psi'(q_1) + \psi^n(q_3, q_4, \ldots, q_{n+1}, q_1) \psi'(q_2) + \ldots + \psi^n(q_{n+1}, q_1, \ldots, q_{n-1}) \psi'(q_n),
\]
where the upper signs refer to even \( n \), the lower ones to odd \( n \).

Let us take a set of orthogonal functions \( \varphi_1, \varphi_2, \ldots \):
\[
\sqrt{n!} g^n_{i_1, i_2, \ldots}(q_1, q_2, \ldots, q_n) = |\varphi_{i_1}(q_1) \varphi_{i_2}(q_2) \ldots \varphi_{i_n}(q_n)|
\]
\((i_1 < i_2 < i_3 < \cdots < i_n)\).
\[
\psi^n(q_1, \ldots, q_n) = \sum_i a_i g^n_i(q_1, \ldots, q_n),
\]
\[
\psi'(q) = \sum_r c_r \varphi_r(q),
\]
\[
\sum |a_i^2| = 1, \quad \sum |c_r^2| = 1.
\]
\[
\psi^{n+1}(q_1, \ldots, q_{n+1}) = \sum_{i_1, \ldots, i_n, r} a_{i_1, \ldots, i_n} c_r g^{n+1}_{i_1, \ldots, i_n, r}(q_1, q_2, \ldots, q_{n+1})
\]
\((r \neq i_1, \ldots, i_n)\).

Let us now consider the states \( \psi(n_1, n_2, \ldots, n_S, \ldots, n_A) \) and \( \psi'(n_1, n_2, \ldots, n_S, \ldots, n_A) \) with
\[
n_1, n_2, \ldots, n_A = \begin{cases} 0, \\ 1, \end{cases}
\]
and \( \Psi = \psi \psi' \):
\[
\Psi(n_1, n_2, \ldots, n_S, \ldots, n_A) = \sum_{n'_1, n'_2, \ldots, n'_A} \pm \psi(n'_1, n'_2, \ldots, n'_A) \cdot \psi'(n_1 - n'_1, n_2 - n'_2, \ldots, n_A - n'_A).
\]
\[ \Psi(q_1, q_2) = -\Psi(q_2, q_1), \]

\[ \Psi(q_1, q_2) = \sum a_{ik} \varphi_i(q_1) \varphi_k(q_2), \]

\[ a_{ik} = -a_{ki}, \quad \sum |a_{ik}^2| = 1; \]

\[ \Psi(q) = \sum c_i \varphi_i(q), \]

\[ \sum |c_i^2| = 1. \]

\[ \Psi(q_1, q_2, q) = \frac{\Psi(q_1, q_2)\psi(q) + \Psi(q_2, q)\psi(q_1) + \Psi(q, q_1)\psi(q_2)}{\sqrt{3}} \]

\[ = -\Psi(q_2, q_1, q) = -\Psi(q_1, q, q_2) = -\Psi(q, q_2, q_1) \]

\[ = \Psi(q_2, q, q_1) = \Psi(q, q_1, q_2). \]

\[ \sqrt{3} \Psi(q_1, q_2, q) = \sum_{i, k, r} a_{ik} c_r [\varphi_i(q_1)\varphi_k(q_2)\varphi_r(q) + \varphi_i(q_2)\varphi_k(q)\varphi_r(q_1) \]

\[ + \varphi_i(q)\varphi_k(q_1)\varphi_r(q_2)]. \]

\[ \int \bar{\Psi} \Psi \, d\tau_1 d\tau_2 d\tau = \frac{1}{3} \sum_{i, k, r; \ell, m, s} \bar{a}_{ik} a_{\ell m} \bar{c}_r c_s [\delta_{i\ell}\delta_{km}\delta_{rs} + \delta_{is}\delta_{k\ell}\delta_{rm} \]

\[ + \delta_{im}\delta_{ks}\delta_{r\ell} + \ldots] \]

\[ = \sum_{i, k, r; \ell, m, s} \bar{a}_{ik} a_{\ell m} [\delta_{i\ell}\delta_{km}\delta_{rs} + \delta_{is}\delta_{k\ell}\delta_{rm} \]

\[ + \delta_{im}\delta_{ks}\delta_{r\ell}] \bar{c}_r c_s \]

\[ = \sum_{r, s} A_{rs} \bar{c}_r c_s. \]

\[ A_{rs} = \sum_{i, k} \bar{a}_{ik} a_{ik} \delta_{rs} + \sum_k \bar{a}_{sk} a_{kr} + \sum_i \bar{a}_{is} a_{ri} \]

\[ = \delta_{rs} + \sum_i (\bar{a}_{is} a_{ri} + a_{ir} \bar{a}_{si}) \]

\[ = \delta_{rs} + \sum_i (\bar{a}_{si} a_{ir} + a_{ri} \bar{a}_{is}). \]
\[
A_{rs} = \delta_{rs} - 2 \sum_i \bar{a}_{si} a_{ri}
\]

by using \(a_{ik} = -a_{ki}\).

\[
A_{rs} = \delta_{rs} - L_{rs},
\]

\(L = A\bar{A}\).

Without interaction we have:

\[
\dot{a}_{ik} = -\frac{2\pi i}{\hbar} \sum_{\ell} (H_{i\ell} a_{lk} + H_{k\ell} a_{i\ell}),
\]

\[
\bar{a}_{ik} = \frac{2\pi i}{\hbar} \sum_{\ell} (\bar{H}_{i\ell} \bar{a}_{\ell k} + \bar{H}_{k\ell} \bar{a}_{i\ell}).
\]

\[14\]

\[
\frac{d}{dt} \sum_i (\bar{a}_{si} a_{ri}) = -\frac{2\pi i}{\hbar} \sum_{i,\ell} (\bar{a}_{si} a_{\ell i} H_{r\ell} + \bar{a}_{si} a_{r\ell} H_{i\ell} - \bar{a}_{\ell i} a_{ri} H_{s\ell} - \bar{a}_{s\ell} a_{r\ell} \bar{H}_{i\ell}).
\]

\[
K_{rs} = \sum_i \bar{a}_{si} a_{ri},
\]

\[15\]

\[
\frac{\partial}{\partial t} K_{rs} = -\frac{2\pi i}{\hbar} \sum_{\ell} (K_{\ell s} H_{r\ell} - K_{r\ell} H_{s\ell}),
\]

\[
\dot{K} = -\frac{2\pi i}{\hbar} (KH - HK).
\]

\[14\] In the following expression appearing in the original manuscript, the author pointed out the cancellation of the second and fourth term in the sum.

\[15\] In the original manuscript, some signs in the following expressions were incorrect.
5

STATISTICAL MECHANICS

5.1. DEGENERATE GAS

A degenerate gas of spinless electrons in a box of length $L$ is considered in the following. The electrostatic interaction between the particles is taken into account in a peculiar way.

[1]

For spinless electrons:

$$\psi_{\ell,m,n} = \frac{1}{L^{3/2}} e^{2\pi i (\ell x + my + nz)/\hbar} = \frac{1}{L^{3/2}} e^{2\pi p \cdot q /\hbar},$$

$$T = \frac{\hbar^2}{2L^2m} (\ell^2 + m^2 + n^2),$$

$$p_x = \frac{\hbar \ell}{L}, \quad p_y = \frac{\hbar m}{L}, \quad p_z = \frac{\hbar n}{L}.$$

$$\Psi = \frac{1}{\sqrt{N!}} \sum_{\pm} \psi_1(q_1) \cdots \psi_n(q_n),$$

$$\psi_i = \psi_{\ell_i,m_i,n_i}.$$

$$A_{ik} = \int \frac{e^2}{r_{12}} |\psi_i^2(q_1)| |\psi_k^2(q_2)| \, dq_1 \, dq_2 = A \quad \text{(independent of } i \text{ and } k),$$

$$I_{ik} = \int \frac{e^2}{r_{12}} \bar{\psi}_i(q_1) \bar{\psi}_k(q_2) \psi_i(q_2) \psi_k(q_1) \, dq_1 \, dq_2.$$

[1] In the original manuscript, the unidentified Ref. 8.47 appears here.
\[
\int \Psi H \Psi \, d\tau = \sum_{i=1}^{n} T_i + \frac{1}{2} \sum_{i,k} A_{ik} - \frac{1}{2} \sum_{i,k} I_{ik},
\]

\[A_{ik} = A, \quad I_{ik} = \frac{e^2 h^2}{\pi |p_i - p_k|^2 L^3}.\]

### 5.2. PAULI PARAMAGNETISM

In the following notes the author reported a (preliminary?) study on Pauli paramagnetism. He considered an ensemble of \(N\) degenerate fermions (so that \(N\) is proportional to the third power of the Fermi momentum, or \(V^{3/2}\) where \(V\) is the electrostatic potential) interacting with a magnetic field \(H\) by means of the Pauli term \(\mu_0 H\), and obtained an expression for the magnetic susceptibility \(\chi\). The number of spin-up and spin-down fermions was denoted with \(n'\) and \(n''\), respectively.

\[
\frac{N}{2} = kV^{3/2}
\]

\[
N = n' + n''
\]

\[
n' = k (V + \mu_0 H)^{3/2} \simeq kV^{3/2} + \frac{3}{2} kV^{1/2} \mu_0 H = \frac{N}{2} + \frac{3 \mu_0 H}{2V} \cdot \frac{N}{2},
\]

\[
n'' = k (V - \mu_0 H)^{3/2} \simeq kV^{3/2} - \frac{3}{2} kV^{1/2} \mu_0 H = \frac{N}{2} - \frac{3 \mu_0 H}{2V} \cdot \frac{N}{2}.
\]

\[
n' - n'' = \frac{3 \mu_0 H}{V} N,
\]

\[
\mu_0(n' - n'') = \frac{3 \mu_0^2 H}{V} N.
\]

\[
\chi = \frac{(n' - n'')\mu_0}{H} = \frac{3}{2} \frac{N \mu_0^2}{V}.
\]

\[2] \text{In the original manuscript some numerical calculations appear here, that probably represent an attempt to evaluate the magnetic susceptibility of sodium Na (considered as an}
5.3. FERROMAGNETISM

In this Section, Majorana studied the problem of ferromagnetism in the framework of the Heisenberg model with the exchange interaction. However, it is rather evident that the Majorana approach is seemingly original, since he does not follow neither the Heisenberg formulation (see W. Heisenberg, Z. Phys. 49 (1928) 619) nor the subsequent van Vleck formulation (which followed Dirac) in terms of spin Hamiltonian (see J.H. van Vleck, The Theory of Electric and Magnetic Susceptibilities (Oxford University Press, London, 1932). He considered a system of \( i \) atoms (located at positions \( r_1, r_2, \text{etc.} \)) with spin parallel to the applied magnetic field on a total of \( n \) atoms, and started by writing the Slater determinants \( A \) of the atomic wavefunctions \( \psi \) with respect to the possible combinations of \( i \) spin-up atoms out of the \( n \) total atoms. The Heisenberg exchange interaction (which is of electrostatic origin) \( V_{rs} \) among nearest neighbor atoms (the number of nearest neighbors is denoted with \( a \)) was then introduced and the energy \( E \) of the system evaluated. The subsequent calculations, performed by employing statistical arguments, were aimed to obtain the magnetization of the system (with respect to the saturation value) when a magnetic field \( H \) acts on the magnetic moment \( \mu \) of each atom. For further discussion, see S. Esposito, preprint arXiv:0805.3057 [physics:hist-ph].

\[
A(r_1 \ldots r_i|r_{i+1} \ldots r_n) = \\
\begin{vmatrix}
\psi_{r_1}(q_1)\delta(s_1 - 1) & \ldots & \psi_{r_1}(q_n)\delta(s_n - 1) \\
\vdots & \ddots & \vdots \\
\psi_{r_{i+1}}(q_1)\delta(s_1 - 1) & \ldots & \psi_{r_{i+1}}(q_n)\delta(s_n - 1) \\
\psi_{r_{i+1}}(q_1)\delta(s_1 + 1) & \ldots & \psi_{r_{i+1}}(q_n)\delta(s_n + 1) \\
\vdots & \ddots & \vdots \\
\psi_{r_n}(q_1)\delta(s_1 + 1) & \ldots & \psi_{r_n}(q_n)\delta(s_n + 1)
\end{vmatrix}
\]

ensemble of \( 6 \cdot 10^{23} \) (Avogadro’s number) nucleons, or \( 3 \cdot 10^{22} \) nuclei:

\[
V = p \cdot 1.59 \cdot 10^{-12} \text{ volt},
\]

\[
\chi = \frac{3}{2} \cdot 3 \cdot 10^{22} \cdot 0.85 \cdot 10^{-40} \cdot \frac{1}{p \cdot 1.59 \cdot 10^{-12}} = \frac{3 \cdot 0.85}{2 \cdot 1.59} \cdot 10^{-6}.
\]
\[ A_i(r_1^1, r_2^1 \ldots r_{i+1}^1, r_{i+2}^1 \ldots r_n^1) \]
\[ \ldots \]
\[ A_\tau(r_1^\tau, \ldots, r_i^\tau r_{i+1}^\tau, r_n^\tau) \]
\[ \tau = \frac{n!}{i!(n-i)!} \left( \text{the order of } r_1 \ldots r_i \text{ or } r_{i+1} \ldots r_n \text{ is not important} \right). \]

If \( H \) is the interaction operator acting on each particle, the electrostatic interaction potential \( V_0 \) is given by:
\[ V_0 = \int H \psi_1(q_1) \overline{\psi}_1(q_1) \psi_2(q_2) \overline{\psi}_2(q_2) \ldots \psi_n(q_n) \overline{\psi}_n(q_n) \, dq_1 \ldots dq_n. \]

The exchange energy between \( r \) and \( s \) orbits \( V_{rs} \):
\[ V_{rs} = \int \frac{e^2}{|q_1 - q_2|} \psi_r(q_1) \overline{\psi}_s(q_1) \overline{\psi}_r(q_2) \psi_s(q_2) \, dq_1 dq_2; \]
\[ V_{rs} = V_{sr}. \]

\[ H_{mm} = V_0 - \sum_{r<s} V_{rs} + \sum_{r=r_1^m}^{r_n^m} \sum_{s=r_{i+1}^m}^{r_n^m} V_{rs}, \]
and for \( m \neq n \):
\[ H_{mn} = \begin{cases} -V_{rs}, & \text{for a transition from } A_m \text{ to } A_n \text{ by exchanging} \\
& \text{the opposite intrinsic orientation in the orbits} \\
& \psi_r \text{ and } \psi_s, \\
0, & \text{for the other cases.} \end{cases} \]

In the ferromagnetic case, if each atom has \( n \) neighbor atoms:
\[ V_{rs} = \begin{cases} \varepsilon, & \text{(neighbor atoms),} \\
0, & \text{(distant atoms).} \end{cases} \]
\[ E = H - V_0 + \sum_{r<s} V_{rs} = H - V_0 + \frac{na}{2}\varepsilon. \]

\(^3\text{In the original manuscript, the upper limits of the second sum in the expression for } E_{mm} \text{ are both (incorrectly) written as } r_i^n.\)
\[ E_{mm} = \sum_{r_1^m} \sum_{r_{i+1}^m} V_{rs} = N_m \varepsilon, \]

and for \( m \neq n \):

\[
E_{mn} = \begin{cases} 
-V_{rs}, \\
0.
\end{cases}
\]

Can we consider \( E \) as diagonal, in a statistical sense? Let us assume that it can be.

For any given value of \( N \), \( y \) solutions exist:

\[ y = y(N). \]

In each of the quantities \( A \) we exchange randomly an orbit \( \uparrow \) with a \( \downarrow \) one; the quantities \( A \) change into \( B \):

\[
\begin{align*}
A_1 & \rightarrow B_1 \\
\vdots & \quad \cdot \cdot \cdot \\
A_\tau & \rightarrow B_\tau
\end{align*}
\]

Statistically, the set of \( B \)'s coincides with that of the \( A \)'s.

\[ y_0 = y(N_0), \]

that is, we have \( y_0 \) quantities \( A \) corresponding to \( N_0 \). If we perform the transformation \( \diamond \), the quantities \( B \) corresponding to the \( y_0 \) quantities \( A \) will be distributed between

\[ N_0 - 2a \quad \text{and} \quad N_0 + 2a, \]
and let
\[ p_{2a}, \quad N_0 + 2a, \]
\[ p_{2a-2}, \quad N_0 + 2a - 2, \]
\[ \ldots \]
\[ p_2, \quad N_0 + 2, \]
\[ p_0, \quad N_0, \]
\[ \ldots \]
\[ p_{-2}, \quad N_0 - 2, \]
\[ \ldots \]
\[ p_{-2a}, \quad N_0 - 2a, \]
be the probabilities that one out of the mentioned \( B \) quantities corresponds to \( N = N_0 + 2a \), or \( N_0 + 2a - 2 \), etc.

We can evaluate the average increment:
\[ \Delta N_0 = \sum_{-a}^{a} 2r \cdot p_{2r}. \]

In fact, on average an electron \( \uparrow \) has
\[ \frac{N_0}{i} \text{ electrons } \downarrow \quad \text{and} \quad a - \frac{N_0}{i} \text{ electrons } \uparrow \]
as neighbors, while an electron \( \downarrow \) has
\[ \frac{N_0}{n-i} \text{ electrons } \uparrow \quad \text{and} \quad a - \frac{N_0}{n-i} \text{ electrons } \downarrow \]
as neighbors. By performing the mentioned exchange, we evidently have:
\[ \Delta N_0 = 2a - 2N_0 \left( \frac{1}{i} + \frac{1}{n-i} \right). \]

Let us assume that the probabilities \( p \) obey the following law (which we can call “normal”)
\[ p_{2r} = \left( \frac{1}{2} + \frac{\Delta N_0}{4a} \right)^{a+r} \left( \frac{1}{2} - \frac{\Delta N_0}{4a} \right)^{a-r} \frac{(2a)!}{(a-r)!(a+r)!}. \]

Assuming that, for a restricted range,
\[ y(N_0 + 1) = y_0 e^k, \]
\[ \ldots \]
\[ y(N_0 \pm a) = y_0 e^{\pm ka}, \]
the condition that \( y(N) \) does not change while we pass from \( A \)'s to \( B \)'s can be expressed as:

\[
\sum_{-a}^{a} p_{2r} e^{-2kr} = 1,
\]

which is solved by:

\[
e^k = \frac{1 + \Delta N_0}{1 - \Delta N_0}.
\]

The trivial solution:

\( k = 0 \)

has to be excluded since, although it does not change \( y = y(N) \) for short ranges, it gives rise to a non constant “flux” of “radions”\( ^4 \) through any section \( N = N_0 \) of the curve \( y = y(N) \) when passing from the \( A \)'s to the \( B \)'s.

It follows that, by considering \( y \) as a continuous function of \( N \):

\[
y' = \frac{2 - \frac{N}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right)}{\frac{N}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right)} \log \frac{2 - \frac{N}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right)}{\frac{N}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right)},
\]

and setting

\[
\alpha = \frac{1}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right) = \frac{n}{a \ i \ (n-i)},
\]

\[
y' = \log \left( \frac{2}{\alpha N} - 1 \right),
\]

we have

\[
d \log y = \log \left( \frac{2}{\alpha N} - 1 \right) dN.
\]

\[
t = \frac{2}{\alpha N} - 1, \quad t + 1 = \frac{2}{\alpha N}, \quad N = \frac{2}{\alpha (t + 1)},
\]

\( ^4 @ \) We find the original text quite obscure.
\[ dN = -\frac{2}{\alpha(t+1)^2} dt; \]
\[ d \log y = -\log t \cdot \frac{2}{\alpha(t+1)^2} dt. \]

\[
\log y = \frac{2}{\alpha} \frac{\log t}{(t+1)} - \frac{2}{\alpha} \int \frac{dt}{t(t+1)}
\]
\[
= \frac{2}{\alpha} \log N - \frac{2}{\alpha} \log t + \frac{2}{\alpha} \log(t+1) + \log \left( \frac{2}{\alpha} \right)
\]
\[ + k,
\]
\[
y = c \left( \frac{2}{\alpha N} \right)^{\frac{2}{\alpha}} \left( \frac{2}{\alpha N} - 1 \right)^{-\frac{2}{\alpha} + N}
\]
or
\[
y = c \left( \frac{2}{2-\alpha N} \right)^{\frac{2}{\alpha}} \left( \frac{2}{\alpha N} - 1 \right)^{N}
\]
or
\[
y = c \left( \frac{2}{2-\alpha N} \right)^{\frac{2}{\alpha}} \left( \frac{2}{\alpha N} - 1 \right)^{N}
\]
\[= c \left( t+1 \right)^{\frac{2}{\alpha}} t^{-\frac{2}{\alpha} + N} = c \left( t+1 \right)^{\frac{2}{\alpha}} t^{-\frac{2}{\alpha} + \frac{1}{\alpha(t+1)}};
\]
\[y(0) = c = y \left( \frac{2}{\alpha} \right) = c, \quad y \left( \frac{2}{\alpha} + e^2 \right) = 0.
\]
\[\int y \, dN \cong \int y \left( \frac{1}{\alpha} \right) e^{-\alpha N^2} dN' = \sqrt{\frac{\pi}{\alpha}} y \left( \frac{1}{\alpha} \right) = \sqrt{\frac{\pi}{\alpha}} c \, 2^{\frac{2}{\alpha}}.
\]
Since the number of solutions is \( \binom{n}{i} \), we have:
\[
c = \binom{n}{i} \sqrt{\frac{1}{\alpha \pi} \frac{1}{i(n-i) \left( \frac{1}{2} + \frac{a-2(n-i)}{n} \right)^{\frac{a-2(n-i)}{n}}}} \binom{n}{i} \frac{1}{2} \sqrt{\frac{\alpha}{\pi}}.
\]
\[
y = c \left( \frac{2}{\alpha N} \right)^{\frac{2}{\alpha}} \left( \frac{2}{\alpha N} - 1 \right)^{-\frac{2}{\alpha} + N},
\]

\[
y = c \left( \frac{2}{2 - \alpha N} \right)^{\frac{2}{\alpha}} \left( \frac{2}{\alpha N} - 1 \right)^N.
\]

Numerical example:

\[n = 10, \quad i = 3, \quad a = 4,\]

\[
\binom{n}{i} = 120, \quad \alpha = \frac{5}{42}, \quad \frac{2}{\alpha} = 16.8,
\]

\[c = 0.0002046,\]

\[y = 0.0002046 \left( \frac{16.8}{N} \right)^{16.8} \left( \frac{16.8}{N} - 1 \right)^{-16.8 + N},
\]

\[y = 0.0002046 \left( \frac{16.8}{16.8 - N} \right)^{16.8} \left( \frac{16.8}{N} - 1 \right)^N.
\]

\[\text{Note that the correct value of } c \text{ is 0.0002047.}\]

\[\text{The following table lists some values of } y \text{ for given } N \text{ (for example, 0 or 16.8, 1 or 15.8, etc.) as calculated from the previous expressions. Note, however, that the (complete) correct numerical values should be as:}\]

<table>
<thead>
<tr>
<th>(N)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 - 16.8</td>
<td>0.0002</td>
</tr>
<tr>
<td>1.0 - 15.8</td>
<td>0.0091</td>
</tr>
<tr>
<td>2.0 - 14.8</td>
<td>0.094</td>
</tr>
<tr>
<td>3.0 - 13.8</td>
<td>0.54</td>
</tr>
<tr>
<td>4.0 - 12.8</td>
<td>2.07</td>
</tr>
<tr>
<td>5.0 - 11.8</td>
<td>5.66</td>
</tr>
<tr>
<td>6.0 - 10.8</td>
<td>11.65</td>
</tr>
<tr>
<td>7.0 - 9.8</td>
<td>18.47</td>
</tr>
<tr>
<td>8.0 - 8.8</td>
<td>22.90</td>
</tr>
<tr>
<td>8.4</td>
<td>23.35</td>
</tr>
</tbody>
</table>
\[
\begin{array}{cc}
N & y \\
0 - 16.8 & 0.0002 \\
1 - 15.8 & 0.0088 \\
2 - 14.8 & 0.089 \\
3 - 13.8 & \\
4 - 12.8 & \\
5 - 11.8 & 5.69 \\
6 - 10.8 & \\
7 - 9.8 & 18.45 \\
8 - 8.8 & \\
8.4 & 23.35 \\
\end{array}
\]

We can also write:

\[
y = c \, \left(1 - \alpha^2 \right)^{\frac{1}{\alpha}} \, \left(\frac{1 - \alpha N}{1 + \alpha N} \right)^{N'},
\]

where

\[
N' = N - \frac{1}{\alpha},
\]

which points out the symmetry property.

\[
\log y = k - \frac{1}{\alpha} \log(1 - \alpha^2) + N' \log \frac{1 - \alpha N'}{1 + \alpha N'}.
\]

\[
\log(1 - \alpha N') = -\alpha N' - \frac{1}{2} \alpha^2 N'^2 - \frac{1}{3} \alpha^3 N'^3 - \ldots,
\]

\[
\log(1 + \alpha N') = \alpha N' - \frac{1}{2} \alpha^2 N'^2 + \frac{1}{3} \alpha^3 N'^3 + \ldots;
\]

\[
\log(1 - \alpha^2) = -\alpha^2 N'^2 - \frac{1}{2} \alpha^4 N'^4 - \ldots,
\]

\[
\log \frac{1 - \alpha N'}{1 + \alpha N'} = -2\alpha N' - \frac{2}{3} \alpha^3 N'^3 - \ldots;
\]

\[
\log y = k - \alpha N'^2 - \frac{1}{6} \alpha^3 N'^4 - \frac{1}{15} \alpha^5 N'^6 - \frac{1}{28} \alpha^7 N'^8 - \frac{1}{45} \alpha^9 N'^{10} - \ldots.
\]

\[
e^{-\frac{W}{kT}} = e^{-\frac{N\varepsilon}{kT}} = e^{-LN} = Ce^{-LN'},
\]

\[
L = \frac{\varepsilon}{kT}, \quad C = e^{-\frac{L}{\alpha}}.
\]
\[
\log(y e^{-\frac{W}{kT}}) = k + \log C - LN' - \alpha N'^2 - \frac{1}{6} \alpha^3 N'^4 - \frac{1}{15} \alpha^5 N'^6 - \ldots
\]

\[
= k + \log C - LN' - \frac{1}{\alpha} \log(1 - \alpha N'^2) + N' \log \frac{1 - \alpha N'}{1 + \alpha N'}.
\]

\[
\frac{d}{dN} \log \left( y e^{-\frac{W}{kT}} \right) = -L + \log \frac{1 - \alpha N'}{1 + \alpha N'} = -L + \log \left( \frac{2}{\alpha N} - 1 \right),
\]

\[
\frac{d^2}{dN^2} \log \left( y e^{-\frac{W}{kT}} \right) = -\frac{2}{\alpha N^2} \frac{\alpha N}{2 - \alpha N} = -\frac{2}{N(2 - \alpha N)} = \frac{2\alpha}{(1 - \alpha^2 N'^2)};
\]

\[
\left( y e^{-\frac{W}{kT}} \right)_{\text{max}} = y_0 e^{-\frac{W_0}{kT}},
\]

\[-1 + \frac{2}{\alpha N_0} = e^L,\]

\[
\frac{2}{\alpha N_0} = e^L + 1,
\]

\[
N_0 = \frac{2}{\alpha(e^L + 1)},
\]

\[
\alpha N = \frac{2}{e^L + 1},
\]

\[
2 - \alpha N = \frac{2e^L}{e^L + 1},
\]

\[
\alpha N(2 - \alpha N) = \frac{4e^L}{(e^L + 1)^2},
\]

\[
y_0 = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}} e^{LN_0},
\]

\[
y_0 e^{-\frac{W}{kT}} = e^{-LN_0} c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}} c^{LN_0} = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}}.
\]

\[
y e^{-\frac{W}{kT}} = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}} e^{-\frac{(e^L+1)^2}{4e^L} \alpha(N-N_0)^2}.
\]
\[
\int y \, e^{-\frac{W}{kT}} \, dN = \int y \, dN \cdot \left( \frac{e^L + 1}{2} \right)^{\frac{2}{\alpha}} \sqrt{\frac{4e^L}{(e^L + 1)^2}}
= \int y \, dN \cdot \left( \frac{1 + e^{-L}}{2} \right)^{\frac{2}{\alpha}} \frac{2\sqrt{e^L}}{e^L + 1}
= \binom{n}{i} \left( \frac{1 + e^{-L}}{2} \right)^{\frac{2}{\alpha}} \frac{2\sqrt{e^L}}{e^L + 1}
\]

\[
M = \frac{\mu H}{kT}.
\]

\[
\begin{array}{cccccc}
\uparrow & \uparrow & \uparrow & \cdots & \uparrow \\
1 & 2 & 3 & \cdots & i \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
i + 1 & i + 2 & i + 3 & \cdots & n
\end{array}
\]

The ratio \( S \) between the magnetic moment under the influence of the field \( H \) and the saturation magnetic moment is:

\[
S = \frac{\sum 2i \binom{n}{i} \left( \frac{1 + e^{-L}}{2} \right)^{\frac{2i(n-i)}{n}} e^{M2i} [e^{-Mn}]}{\sum \binom{n}{i} \left( \frac{1 + e^{-L}}{2} \right)^{\frac{2i(n-i)}{n}} e^{M2i} [e^{-Mn}]} - 1.
\]

\[
\log \int e^{-\frac{W}{kT}} e^{M(2i-n)} y \, dN
= \frac{2i(n - i)}{n} \log \frac{1 + e^{-L}}{2} + 2Mi - i \log i - (n - i) \log(n - i) + \text{const}.
\]

By taking the derivative with respect to \( i \) and equating the result to 0:

\[
a \frac{2(n - 2i)}{n} \log \frac{1 - e^{-L}}{2} + 2M - i \log i - A + (n - i) + \log(n - i) + A = 0,
\]

\[
\log \frac{i}{n - i} = 2M + a \frac{2(n - 2i)}{n} \log \frac{1 + e^{-L}}{2}.
\]
\[ S' = \frac{2i - n}{n} = \frac{2i}{n} - 1, \]
\[ \frac{2i}{n} = 1 + S', \]
\[ i \frac{n}{n} = 1 + S'; \]
\[ \frac{n - i}{n} = \frac{1 - S'}{2}. \]

\[ \log \frac{1 + S'}{1 - S'} = 2M - 2aS' \log \frac{1 + e^{-L}}{2}. \]

It follows:

\[ \log \frac{1 + S'}{1 - S'} = 2 \frac{\mu}{kT} H + 2aS' \log \frac{2}{1 + e^{-\frac{\epsilon}{kT}}}. \]

For small \( H \) and large \( T \):

\[ 2S = 2 \frac{\mu}{kT} H + 2a \log \frac{2}{1 + e^{-\frac{\epsilon}{kT}}}. \]

For \( T \) lower\(^7\) than the Curie point: for a given value of \( H \) there exist 2 values of \( S \) which, for not extremely high \( H \), are practically equal and opposite.

From \( \otimes \) it follows:

\[ \frac{a2i(n-i)}{n} \log \frac{1 + e^{-L}}{2} = \left( \log \frac{i}{n-i} - 2M \right) i \frac{n-i}{n-2i}. \]

Substituting in \( \oplus \):

\[ -2Mi \frac{i}{n-2i} + \log i \cdot \frac{i^2}{n-2i} - \log(n-i) \cdot \frac{(n-i)^2}{n-2i}. \]

Let us set \( y > 0 \):

\[ \log \frac{1 + y}{1 - y} = 2ay \log \frac{2}{1 + e^{-\frac{\epsilon}{kT}}}, \]
\[ \log \frac{1 + y + \Delta y}{1 - y - \Delta y} = 2 \frac{\mu}{kT} H + 2a(y + \Delta y) \log \frac{2}{1 + e^{-\frac{\epsilon}{kT}}}, \]

\(^7\)\( @ \) We find the original text to be quite obscure, and our own interpretation is only a probable one.
\[
\frac{\Delta y}{1 - y^2} = \frac{\mu H}{kT} + a \log \frac{2}{1 + e^{-\frac{\varepsilon}{kT}}} \Delta y,
\]

[8]
\[
\Delta y = \frac{\mu H}{kT} \frac{1}{1 - y^2 - a \log \frac{2}{1 - e^{-\frac{\varepsilon}{kT}}}.
\]

The LHS in \( \oplus \) can also be written as:
\[
a \frac{1 - S'^2}{2} n \log \frac{1 + e^{-L}}{2} + M(1 + S') n - n \frac{1 + S'}{2} log n \frac{1 + S'}{2} \\
- n \frac{1 - S'}{2} \log n \frac{1 - S'}{2} + \text{const.}
\]
\[
= \left( a \frac{1 - S'^2}{2} \log \frac{1 + e^{-L}}{2} M S' - \frac{1 + S'}{2} \log \frac{1 + S'}{2} \\
- \frac{1 - S'}{2} \log \frac{1 - S'}{2} \right) + \text{const.}
\]

5.4. FERROMAGNETISM: APPLICATIONS

In the following, the author gives some examples of ferromagnetic materials with different geometries (corresponding to different numbers \( i \) of oriented spins on a total of \( n \), and to different numbers \( a \) of nearest neighbors). Three insert also appear, mainly aimed at evaluating some theoretical quantities related to spontaneous magnetization.

\[
a = 3, \quad i = 3, \quad n - i = 3; \quad \binom{n}{i} = 20.
\]

In the original manuscript, the following formula is incorrectly written as:
\[
\Delta y = \frac{\mu H}{kT} \frac{1}{1 - y^2 - a \log \frac{2}{1 - e^{-\frac{\varepsilon}{kT}}}.
\]
Mean value:

\[
\frac{a_i(n - i)}{n - 1} = \frac{1}{\alpha},
\]

\[
\frac{3 \cdot 3 \cdot 3}{5} = 5.4.
\]
\[
\begin{align*}
\left( \frac{a}{n-1} \right)^2, & \quad \frac{i-1}{n-i} \frac{n-i-1}{n-i} = \frac{ni-i^2-n+1}{i(n-i)}, \\
\frac{a}{n-1} \frac{a-1}{n-2}, & \quad \frac{1}{i} \frac{n-i-1}{n-i} + \frac{i-1}{i} \frac{1}{n-i} = \frac{n-2}{i(n-i)}, \\
\frac{a}{n-1}, & \quad \frac{1}{i(n-i)},
\end{align*}
\]

1) 

2) 

\[
\begin{align*}
& \frac{n-1}{n} \frac{n-2}{n} \frac{n-3}{n} \frac{[i(n-i)-(n-1)]^3}{i(n-i)(n-1)(n-2)(n-3)} \frac{i(n-i)-(n-1)}{i(n-i)} \\
+ & \frac{1}{i(n-i)} \frac{n}{n-1} \frac{a-1}{n-2} \left[ n-2-4i(n-i)-(n-1) \right] \\
+ & \frac{1}{i(n-i)} \frac{a}{n-1} \left[ 1-2i(n-i)-(n-1) \right] \\
= & \frac{1}{i(n-i)(n-1)(n-2)(n-3)} \left\{ a^2n \left[ i(n-i)-(n-1) \right] \\
+ & a(a-1)(n-2)(n-3) - 4a(a-1) \left[ i(n-i)-(n-1) \right] \\
+ & a(n-2)(n-3) - 2a \left[ i(n-i)-(n-1) \right] \right\} \\
= & \frac{(a^2n - 4a^2 + 2a)(i(n-i)-(n-1)) + a^2(n-2)(n-3)}{i(n-i)(n-1)(n-2)(n-3)}. 
\end{align*}
\]

Mean value of the square of the terms in the diagonal:

\[
\begin{align*}
i(n-i) & \frac{(a^2n - 4a^2 + 2a)(i(n-i)-(n-1)) + a^2(n-2)(n-3)}{(n-1)(n-2)(n-3)} \\
= & \ 3 \cdot 3 \cdot \frac{24 \cdot 4 + 108}{60} = \frac{1836}{60} = 30.6 = \frac{612}{20} \\
= & \ \frac{a^2}{(n-1)^2} i^2(n-i)^2 + \frac{4n-6}{(n-2)(n-3)} \frac{a^2}{(n-1)^2} i^2(n-i)^2 \\
- & \ \frac{4n-4}{(n-2)(n-3) (n-1)^2} i^2(n-i)^2 \\
+ & \ \frac{2a}{(n-2)(n-3) n-1} i^2(n-i)^2 - \frac{a^2n}{(n-2)(n-3)} i(n-i) \\
+ & \ \frac{4a^2}{(n-2)(n-3)} i(n-i) - \frac{2a}{(n-2)(n-3)} i(n-i) + \frac{a^2}{n-1} i(n-i).
\end{align*}
\]
terms in the diagonal  eigenvalues

mean value  \( \frac{a(n-i)}{n-1} \)  \( \frac{a(n-i)}{n-1} \)

mean value of the square  \( \frac{a^2i^2(n-i)^2}{(n-1)^2} + k^2 \)  \( \frac{a^2i^2(n-i)^2}{(n-1)^2} + k^2 + \frac{a(n-i)}{n-1} \)

Statistically:

terms in the diagonal  eigenvalues

mean value  \( \frac{a(n-i)}{n} \)  \( \frac{a(n-i)}{n} \)

mean value of the square  \( \frac{a^2i^2(n-i)^2}{n^2} + \frac{2ai^2(n-i)^2}{n^3} \)  \( \frac{a^2i^2(n-i)^2}{n^2} + \frac{2ai^2(n-i)^2}{n^3} + \frac{a(n-i)}{n} \)

\[
\begin{align*}
n &= 24, \\
i &= 6, \\
a &= 2;
\end{align*}
\]

\[
\binom{n}{i} = 134596.
\]

\[\text{In the original manuscript, there appears here the matrix:}\]

\[
\begin{pmatrix}
V_4 + V_5 & -V_5 & -V_4 \\
-V_5 & V_5 + V_6 & -V_4 \\
-V_4 & -V_4 & V_6 + V_4
\end{pmatrix},
\]

whose meaning is unclear to us.
The numbers in the last line of the following table are the mean values of $y$, $yN$ and $y(N-N_0)^2$, respectively, which are obtained by dividing the numbers in the previous line by 134596.

10@ See the previous footnote. The symbols introduced below have the following meaning: according to what is asserted in the original manuscript:

\[ y^\dagger = y \left( \begin{array}{c} 30 \\ 10 \end{array} \right) \cdot 2^{10}, \quad yN^\dagger = yN \left( \begin{array}{c} 60 \\ 20 \end{array} \right) \cdot 2^{10}, \quad y(N-N_0)^2 = y(N-N_0)^2 \left( \begin{array}{c} 30 \\ 10 \end{array} \right) \cdot 2^{10}. \]
\[
\begin{array}{cccccc}
\ y & N & yN & N - N_0 & (N - N_0)^2 & y(N - N_0)^2 \\
1 & 10 & 10 & 1.52 & 2.31 & 2.31 \\
1.071 & 8 & 8.57 & -0.48 & 0.23 & 0.25 \\
0.341 & 6 & 2.05 & -2.48 & 6.15 & 2.10 \\
0.037 & 4 & 0.15 & -4.48 & 20 & 0.74 \\
0.001 & 2 & -6.48 & 42 & 0.04 \\
0 & 0 & -8.48 & 72 \\
2.450 & 20.77 & 5.44 \\
\end{array}
\]

\[
\frac{1 \cdot 2 \cdot 100 \cdot 2500}{216000} = 2.31.
\]

\[
\begin{pmatrix} n \\ i \end{pmatrix} \approx \sqrt{\frac{n}{2\pi}} \frac{n^n}{\sqrt{i}} \frac{(n-i)^{(n-i)}}{(n-i)} = \frac{k}{\sqrt{i} \sqrt{n-i}},
\]

\[
k = \sqrt{\frac{n}{2\pi}} n^{n}.
\]

\(P_i\) solutions with apparent momentum \(n - 2i\),
\(Q_i\) solutions with intrinsic momentum \(n - 2i\).

\(i \leq n/2\).

\[
P_i = \sum_{j=0}^{i} Q_i, \quad Q_i = P_i - P_{i-1}.
\]

\[
p_i = \binom{n}{i}.
\]

\[
q_i = p_i - p_{i-1};
\]

\[
p_{i-1} = p_i \frac{i}{n-i+1},
\]

\[
q_i = p_i - p_{i-1} = p_i \left(1 - \frac{i}{n-i+1}\right) = p_i \frac{n-2i+1}{n-i+1},
\]

\[
q_i = \binom{n}{i} \frac{n+1-2i}{n+1-i}.
\]
\[ n = 4: \]
\[ i \quad p_i \quad n_i \]
\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 4 & 3 \\
2 & 6 & 2 \\
\end{array}
\]

\[
\begin{align*}
\sum_{\sigma} P_{i}^\sigma &= \binom{n}{i} \frac{a \ i (n - i)}{n}, \\
\sum_{\sigma} P_{i}^{\sigma^2} &= \binom{n}{i} \left[ \frac{a^2 i^2 (n - i)^2}{n^2} + \frac{2a \ i^2 (n - i)^2}{n^3} + \frac{a \ i (n - i)}{n} \right], \\
\sum_{\sigma} Q_{i}^\sigma &= \sum_{\sigma} P_{i}^\sigma - \sum_{s} P_{i-1}^\sigma \\
&= \binom{n}{i} \frac{a \ i (n - i)}{n} - \binom{n}{i-1} \frac{a (i - i) (n - i + 1)}{n} \\
&= \binom{n}{i} \frac{a \ i (n - 2i + 1)}{n}, \\
\sum_{\sigma} Q_{i} q_{i} &= \frac{a \ i (n + 1 - i)}{n}.
\end{align*}
\]

Curves of the eigenvalues corresponding to apparent momentum \( N - 2i \):
\( y_{i} = y_{i}(N) \).
For large \( n \), \( y_{i} \) tend to the limiting form:
\[
\log \frac{y_{i}}{n} = n + n f_{i} \left( \frac{2}{a \ n} N \right) = n + n f_{i}(x), \quad x = \frac{2}{a \ n} N.
\]
\[
f_{i}(x)_{\text{max}} = f_{i}(x_{0}^i), \quad x_{0}^i = \frac{2}{a \ n} \frac{a \ i (n - i)}{n} = \frac{2i}{n} \frac{n - i}{n},
\]
\[
f_{i}'(x_{0}^i) = 0,
\]
\[
f_{i}''(x_{0}^i) = -a \left( \frac{4}{n} \frac{n - i}{n} + 8 \frac{i^2 (n - i)^2}{n^2} \right).
\]
\[
\begin{align*}
N: \quad \mu^2 &= \frac{2a \ i^2 (n - i)^2}{n^3} + \frac{a \ i (n - i)}{n}, \\
\chi: \quad \mu^2 &= \frac{8a \ i^2 (n - i)^2}{a \ n^5} + \frac{4i(n - i)}{a \ n^3}.
\end{align*}
\]
5.5. AGAIN ON FERROMAGNETISM

In the following pages, the author probably comes back again to ferromagnetism, but the meaning is quite obscure to us. See also E. Majorana, Nuovo Cim. 8 (1931) 78.

\[
\begin{align*}
\psi_1(q_1) & \psi_1(q_2) \ldots \psi_1(q_n) \\
\psi_2(q_1) & \psi_2(q_2) \ldots \psi_2(q_n) \\
\vdots & \\
\psi_n(q_1) & \psi_n(q_2) \ldots \psi_n(q_n) \\
\end{align*}
\]

\[
\psi_1(q_1) \psi_1(q_2) \ldots \psi_1(q_n) \\
\psi_2(q_1) \ldots \psi_2(q_n) \\
\psi_n(q_1) \ldots \psi_n(q_n) \\
\psi_2(q_3) \ldots \psi_2(q_1) \\
\psi_2(q_1) \ldots \psi_2(q_n) \\
\psi_n(q_3) \ldots \psi_n(q_1)
\]

\[
\pm \psi_1(q_2) \ldots \psi_1(q_2) \\
\psi_2(q_3) \ldots \psi_2(q_1) \\
\psi_3(q_1) \ldots \psi_3(q_1)
\]

\[
\sum_{r=1}^{n} \varphi(q_{r+1}, q_{r+2}, \ldots, q_{n}, q_{1}, \ldots, q_{r-1})\psi(q_r), \quad n = 2p + 1.
\]

\textsuperscript{[12]}

\[
\begin{array}{c}
1 \uparrow \uparrow \uparrow \uparrow \quad 0 \\
2 \uparrow \uparrow \downarrow \quad \varphi_1(\psi_2\psi_3) - \varphi_2(\psi_1\psi_3) \quad (123) \\
3 \uparrow \downarrow \uparrow \uparrow \quad \varphi_3(\psi_1\psi_2) - \varphi_1(\psi_3\psi_2) \quad (132) \\
4 \uparrow \downarrow \downarrow \quad 0 \\
5 \downarrow \uparrow \uparrow \quad \varphi_2(\psi_3\psi_1) - \varphi_3(\psi_2\psi_1) \quad (123) \\
6 \downarrow \uparrow \uparrow \quad 0 \\
7 \downarrow \downarrow \uparrow \quad 0 \\
8 \downarrow \downarrow \downarrow \quad 0
\end{array}
\]

\[\psi, \varphi, u, v.\]

\[\psi_1\psi_2(u_1v_2-u_2v_1)\varphi_3u_3+\psi_2\psi_3(u_2v_3-u_3v_2)\varphi_1u_1+\psi_3\psi_1(u_3v_1-u_1v_3)\varphi_2u_2.\]

\textsuperscript{[12]} In the original manuscript, some pages of scratch calculations appear here: they deal with combinations of several objects grouped in different ways, probably with an eye on the study of ferromagnetism (see below).
PART III
THE THEORY OF SCATTERING

6.1. SCATTERING FROM A POTENTIAL WELL

The author studied here the problem of the scattering of a plane wave from a one-dimensional square potential well. All the physically interesting cases were treated.

e = h/2\pi = m = 1.

\[ \nabla^2 \psi + 2(E - V)\psi = 0. \]

\[ V = 0: \]

\[ y'' + 2Ey = 0. \]

\[ 2E = k^2, \]

\[ y_1 = e^{ikx}, \quad y_2 = e^{-ikx}. \]

\[ y'' + 2(E - U)y = 0, \]

\[ U = -V, \]

\[ y'' + 2(E + V)y = 0. \]

\[ 2(E + V) = \mu^2, \quad 2E = k^2, \]

\[ \mu = k\sqrt{1 + \frac{V}{E}}. \]

By imposing the matching conditions for the wavefunction and its derivative, one obtains:
\[
y = \begin{cases} 
\frac{1 + \sqrt{1 + \frac{V}{E}}}{2} e^{ik(x+a) - i\mu a} + \frac{1 - \sqrt{1 + \frac{V}{E}}}{2} e^{-ik(x+a) - i\mu a}, & x < -a, \\
e^{i\mu x}, & -a < x < a, \\
\frac{1 + \sqrt{1 + \frac{V}{E}}}{2} e^{ik(x-a) + i\mu a} + \frac{1 - \sqrt{1 + \frac{V}{E}}}{2} e^{-ik(x+a) + i\mu a}, & a < x,
\end{cases}
\]

\[E > 0,\]
\[E = \frac{1}{2} \mu^2 - V,\]
\[E = \frac{1}{2} k^2,\]

\[\mu = k \sqrt{1 + \frac{V}{E}}, \quad k = \frac{\mu}{\sqrt{1 + \frac{V}{E}}}.\]

\[\sqrt{g} \text{ gives the ratio of the wave amplitude inside and outside the well.}^1\]

\(E < 0, E > -V:\)

\[\sqrt{1 + \frac{V}{E}} = i \sqrt{\frac{V}{-E} - 1},\]

\[\mu = ik \sqrt{\frac{V}{-E} - 1} = k_1 \sqrt{\frac{V}{-E} - 1}, \quad k_1 = ik.\]

---

1. That is: \(g\) is given by the ratio \(a^2 + b^2 / c^2\) where \(a, b\) is the coefficient of the first [second] wave term in the first or third row, while \(c\) is the coefficient of the wave term in the second row \((c = 1)\). Note that the quantity we call \(g\), here and in what follows, is in the original manuscript denoted by \(y\), the same as the symbol there used for the wave function.
\[
y = \begin{cases} 
1 + i \sqrt{\frac{V}{-E} - 1} \frac{e^{k_1(x+a) - i\mu a}}{2} + \frac{1 - i \sqrt{\frac{V}{-E} - 1}}{2} e^{-k_1(x+a) - i\mu a}, & x < -a, \\
e^{i\mu x}, & -a < x < a, \\
1 + i \sqrt{\frac{V}{-E} - 1} \frac{e^{k_1(x-a) + i\mu a}}{2} + \frac{1 - i \sqrt{\frac{V}{-E} - 1}}{2} e^{-k_1(x-a) + i\mu a}, & a < x,
\end{cases}
\]

\(-V < E < 0,\)

\[
E = \frac{1}{2} \mu^2 - V,
\]

\[
E = -\frac{1}{2} \mu^2,
\]

\[
\mu = k_1 \sqrt{\frac{V}{-E} - 1}, \quad k_1 = \sqrt{\frac{V}{-E} - 1}.
\]

Stationary states: 2

2\@ In the original manuscript, there appear here the following calculations:

\[
\frac{1 + i \sqrt{\frac{V}{-E} - 1}}{2} e^{i\mu a} = e^{i^n} = e^{in\pi/2}.
\]

\[
e^{-i\mu a} \left[ \cos k(x + a) + i \sqrt{1 + \frac{V}{E}} \sin k(x + a) \right],
\]

\[
e^{i\mu a} \left[ \cos k(x + a) - i \sqrt{1 + \frac{V}{E}} \sin k(x + a) \right];
\]

\[-\sin \mu a \cos k(x - a) + \cos \mu a \sqrt{1 + \frac{V}{E}} \sin k(x + a),\]

\[
\cos \mu a \cos k(x - a) - \sin \mu a \sqrt{1 + \frac{V}{E}} \sin k(x + a).
\]
\[
y = \begin{cases}
\sqrt{1 + \frac{V}{E}} \cos \mu a \sin k(x + a) - \sin \mu a \cos k(x + a), & x < -a, \\
\sin \mu x, & -a < x < a,
\end{cases}
\]
\[
y = \begin{cases}
\sqrt{1 + \frac{V}{E}} \cos \mu a \sin k(x - a) + \sin \mu a \cos k(x - a), & a < x,
\end{cases}
\]
\[
\begin{align*}
k &= \frac{\mu}{\sqrt{1 + \frac{V}{E}}}, & \mu &= k \sqrt{1 + \frac{V}{E}}, & E &= \frac{1}{2} k^2 = \frac{1}{2} \mu^2 - V.
\end{align*}
\]

\footnote{In the original manuscript, the following calculations appear at this point:

\[
\left(1 + c \sqrt{1 + \frac{V}{E}}\right) \cos \mu a - \left(c i + i \sqrt{1 + \frac{V}{E}}\right) \sin \mu a = 0,
\]
\[
c \left(\sqrt{1 + \frac{V}{E}} \cos \mu a - i \sin \mu a\right) = -\cos \mu a + i \sqrt{1 + \frac{V}{E}} \sin \mu a,
\]
\[
c = -\frac{\cos \mu a - i \sqrt{1 + \frac{V}{E}} \sin \mu a}{\sqrt{1 + \frac{V}{E}} \cos \mu a - i \sin \mu a}.
\]

[The footnote continues on the next page.]
Reflection

\[
y = \begin{cases} 
2\sqrt{1 + \frac{V}{E}} \cos 2\mu a - i \left( 2 + \frac{V}{E} \right) \sin 2\mu a \, e^{ik(x+a)} \\
+ \frac{V}{E} i \sin 2\mu a \, e^{-ik(x+a)}, & x < -a,
\end{cases}
\]

\[
\left( 1 + \sqrt{1 + \frac{V}{E}} \right) e^{i\mu(x-a)} - \left( 1 - \sqrt{1 + \frac{V}{E}} \right) e^{-i\mu(x-a)},
\]

\[
\left( 1 - \sqrt{1 + \frac{V}{E}} \right) e^{i\mu(x-a)}, & -a < x < a,
\]

\[
2\sqrt{1 + \frac{V}{E}} \, e^{ik(x-a)}, & a < x,
\]

incident energy: \( A = 4 + 4 \frac{V}{E} + \frac{V^2}{E^2} \sin^2 2\mu a, \)

reflected energy: \( A_r = \frac{V^2}{E^2} \sin^2 2\mu a, \)

refracted energy: \( A_R = 4 + 4 \frac{V}{E}, \)

reflecting power: \( \rho = \frac{A_r}{A} = \frac{\frac{V^2}{E^2} \sin^2 2\mu a}{4 + 4 \frac{V}{E} + \frac{V^2}{E^2} \sin^2 2\mu a}; \)

minima: \( \mu a = n \frac{\pi}{2}. \)

\[
\left( \sqrt{1 + \frac{V}{E}} + 1 \right) \cos \mu a - i \left( \sqrt{1 + \frac{V}{E}} + 1 \right) \sin \mu a = \left( \sqrt{1 + \frac{V}{E}} + 1 \right) e^{-i\mu a},
\]

\[
\left( \sqrt{1 + \frac{V}{E}} - 1 \right) \cos \mu a + i \left( \sqrt{1 + \frac{V}{c}} - 1 \right) \sin \mu a = \left( \sqrt{1 + \frac{V}{E}} - 1 \right) e^{i\mu a};
\]

\[
\left( \sqrt{1 + \frac{V}{E}} \right) e^{-i\mu a} / \left( 1 - \sqrt{1 + \frac{V}{E}} \right) e^{i\mu a}.
\]
6.2. SIMPLE PERTURBATION METHOD

In the following few passages, Majorana traced the general lines of a simple perturbation method in order to solve the Schrödinger equation for a particle in a potential field \( V \) in terms of the known eigenstates \( \psi_i \).

\[
\nabla^2 \psi + 2(E - V)\psi = 0.
\]

\[
\nabla^2 \psi_0 + 2E\psi_0 = 0.
\]

\[\psi = \psi_0 + \chi,\]

\[
\nabla^2 \psi_0 + 2(E - V)\psi_0 + \nabla^2 \chi + 2(E - V)\chi = 0,
\]

\[
\nabla^2 \chi + 2(E - V)\chi = 2V\psi_0.
\]

\[
\nabla^2 \psi_i + 2(E_i - V)\psi_i = 0.
\]

\[2V\psi_0 = 2 \sum c_i \psi_i,
\]

\[
\nabla^2 \chi + 2(E - V)\chi = 2 \sum c_i \psi_i.
\]

\[\chi = \sum d_i \psi_i,
\]

\[
\nabla^2 \psi_i + 2(E - V)\psi_i = 2(E - E_i)\psi_i;
\]

\[
\nabla^2 \chi + 2(E - V)\chi = 2 \sum d_i (E - E_i)\psi,
\]

\[d_i = \frac{c_i}{E - E_i}.
\]
6.3. THE DIRAC METHOD

The author applied the perturbation theory to the problem of the scattering of a particle of momentum \( p = h \gamma \) from a potential \( V \); the free-particle wavefunction is denoted with \( \phi_\gamma \). Some approximated expressions for the transition probability were obtained within the framework of the Dirac method, which are subsequently applied to the particular case of Coulomb scattering.

\[
E = \frac{1}{2m} p^2 + V = \frac{\hbar^2}{2m} \gamma^2 + V.
\]

\[
\phi_\gamma = e^{2\pi i (\gamma_x x + \gamma_y y + \gamma_z z)} e^{-2\pi i (h/2m) \gamma^2 t}.
\]

\[
<\gamma'|V|\gamma''> = \int \overline{\phi_{\gamma'}} V \phi_{\gamma''} d\gamma' d\gamma d\gamma z = k_{\gamma'\gamma''} e^{-2\pi i (h/2m) (\gamma''^2 - \gamma^2) t} = k_{\gamma'\gamma''} e^{2\pi i (h/2m) (\gamma'^2 - \gamma''^2) t}.
\]

\[
\psi = \int \alpha_{\gamma} \phi_\gamma d\gamma,
\]

\[
\dot{\alpha}_{\gamma} = -\frac{2\pi i}{\hbar} \int k_{\gamma'\gamma''} e^{2\pi i (h/2m) (\gamma'^2 - \gamma''^2) t} \alpha_{\gamma'} d\gamma'.
\]

For \( t = 0 \):

\[
\alpha_{\gamma} = \delta (\gamma - \gamma_0).
\]

For \( t > 0 \):

1st approximation:

\[
\dot{\alpha}_{\gamma} = -\frac{2\pi i}{\hbar} k_{\gamma\gamma_0} e^{2\pi i (h/2m) (\gamma^2 - \gamma_0^2) t},
\]

\[
\alpha_{\gamma} = -\frac{2m}{\hbar^2 (\gamma^2 - \gamma_0^2)} k_{\gamma\gamma_0} \left( e^{2\pi i (h/2m) (\gamma^2 - \gamma_0^2) t} - 1 \right) + \delta (\gamma - \gamma_0).
\]

\textsuperscript{4} Here the author denotes with \( \gamma_0 = p_0/\hbar \) the momentum (divided by \( \hbar \)) of the free particle, while \( \delta(x) \) signifies the Dirac delta-function.
\[ \dot{\alpha}_{\gamma} = -\frac{2\pi i}{h} k_{\gamma\gamma_0} e^{2\pi i(h/2m)(\gamma^2 - \gamma_0^2)t} + \frac{4\pi i m}{h^3} \int k_{\gamma\gamma'} k_{\gamma',\gamma_0} \frac{1}{\gamma'^2 - \gamma_0^2} \]
\[ \times \left( e^{2\pi i(h/2m)(\gamma^2 - \gamma_0^2)t} - e^{2\pi i(h/2m)(\gamma^2 - \gamma'^2)t} \right) d\gamma', \]
\[ \alpha_{\gamma} = -\frac{2m}{h^2(\gamma^2 - \gamma_0^2)} \frac{k_{\gamma\gamma_0}}{\gamma^2 - \gamma_0^2} \left( e^{2\pi i(h/2m)(\gamma^2 - \gamma_0^2)t} - 1 \right) \]
\[ + \frac{4m^2}{h^4} \int k_{\gamma\gamma'} k_{\gamma',\gamma_0} \left( \frac{e^{2\pi i(h/2m)(\gamma^2 - \gamma_0^2)t} - 1}{(\gamma^2 - \gamma_0^2)(\gamma'^2 - \gamma_0^2)} \right. \]
\[ \left. - e^{2\pi i(h/2m)(\gamma^2 - \gamma'^2)t} - 1 \right) d\gamma' + \delta (\gamma - \gamma_0). \]

In first approximation, for \( \gamma \neq \gamma_0 \), we have:
\[ |\alpha_{\gamma}|^2 = \frac{16m^2}{h^4(\gamma^2 - \gamma_0^2)^2} |k_{\gamma\gamma_0}|^2 \sin^2 \frac{\pi h(\gamma^2 - \gamma_0^2)t}{2m}. \]

Neglecting constant terms, for \( t \to \infty \) we get:
\[ |\alpha_{\gamma}|^2 = \frac{8\pi^2 m}{h^3} |k_{\gamma\gamma_0}|^2 t \delta (\gamma^2 - \gamma_0^2), \]

and the transition probability is:
\[ P_{\gamma_0\gamma} = \frac{8\pi^2 m}{h^3} |k_{\gamma\gamma_0}|^2 \delta (\gamma^2 - \gamma_0^2). \]

In second approximation:
\[ |\alpha_{\gamma}|^2 = \frac{16m^2}{h^4(\gamma^2 - \gamma_0^2)^2} |k_{\gamma\gamma_0}|^2 \sin^2 \frac{\pi h(\gamma^2 - \gamma_0^2)t}{2m} \]
\[ + \frac{32m^3}{h^6(\gamma^2 - \gamma_0^2)} \sin \frac{\pi h(\gamma^2 - \gamma_0^2)t}{2m} \int \left( k_{\gamma\gamma_0} k_{\gamma',\gamma_0} + k_{\gamma\gamma_0} \tilde{k}_{\gamma',\gamma_0} \right) \]
\[ \times \left( \frac{\sin \pi h(\gamma^2 - \gamma'^2)t/2m}{(\gamma^2 - \gamma_0^2)(\gamma'^2 - \gamma_0^2)} - \frac{\sin \pi h(\gamma^2 - \gamma_0^2)t/2m}{(\gamma^2 - \gamma_0^2)(\gamma'^2 - \gamma_0^2)} \right) d\gamma'. \]

### 6.3.1 Coulomb Field

For a Coulomb field:
\[ V = \frac{C}{r} = \int V_{\gamma} e^{2\pi i(\gamma x + \gamma y + \gamma z)} \, dx \, dy \, dz, \]
\[ V_\gamma = C \int \frac{e^{-2\pi i \gamma q}}{r} \, dq = C \int_0^\infty \frac{2}{\gamma} \sin 2\pi \gamma r \, dr = \frac{C}{\pi \gamma^2}; \]

\[ k_{\gamma'-\gamma''} = V_{\gamma'-\gamma''} = \frac{C}{\pi |\gamma' - \gamma''|^2}. \]

In first approximation:

\[ P_{\gamma_0\gamma} = \frac{8mc^2}{\hbar^3|\gamma - \gamma_0|^4} \delta (\gamma^2 - \gamma_0^2) = \frac{mc^2}{2\hbar^3\gamma_0^4 \sin^4 \theta / 2} \delta (\gamma^2 - \gamma_0^2). \]

In second approximation, for \( \gamma \neq \gamma_0 \):

\[ \alpha_\gamma = - \frac{2m}{h^2} \frac{C}{\gamma^2 - \gamma_0^2} \frac{\pi (\gamma - \gamma_0)^2}{C} \left( e^{2\pi i (h/2m)} (\gamma^2 - \gamma_0^2) t - 1 \right) \]

\[ + \frac{4m^2}{h^4} \int \frac{C^2}{\pi^2 |\gamma - \gamma'|^2 |\gamma' - \gamma_0|^2} \left( e^{2\pi i (h/2m)} (\gamma^2 - \gamma_0^2) t - 1 \right) \left( (\gamma^2 - \gamma_0^2) (\gamma^{2'} - \gamma_0^{2'}) - e^{2\pi i (h/2m)} (\gamma^2 - \gamma_0^2) t - 1 \right) \, d\gamma'. \]

### 6.4. THE BORN METHOD

The scattering from a given center was studied here by means of the Born method, and approximated expressions for the scattered partial waves were obtained.

\[ \nabla^2 \psi + k^2 \psi = F \psi. \]

\[ \psi = \psi_0 + \psi_1 + \psi_2 + \ldots, \]

\[ \nabla^2 \psi_0 + k^2 \psi_0 = 0, \]

\[ \nabla^2 \psi_1 + k^2 \psi_1 = F \psi_0, \]

\[ \nabla^2 \psi_2 + k^2 \psi_2 = F \psi_1, \]

\[ \ldots, \]

\[ \nabla^2 \psi_n + k^2 \psi_n = F \psi_{n-1}, \]

\[ \ldots. \]

\[ ^5 @ \text{Probably, the author started to evaluate the transition probability for Coulomb scattering in a second approximation, but succeeded only in obtaining an expression for the coefficient } \alpha_\gamma. \]
\[ \psi_n(q) = -\frac{1}{4\pi} \int \frac{e^{ik|q-q'|}}{|q-q'|} F(q') \psi_{n-1}(q') \, dq'. \]

\[ \psi_0(q) = e^{ik\mathbf{u}_0 \cdot \mathbf{q}}, \]

\[ \psi_1(q) = -\frac{1}{4\pi} \int \frac{e^{ik|q-q'|}}{|q-q'|} e^{ik\mathbf{u}_0 \cdot \mathbf{q}'} F(q') \, dq', \]

\[ \psi_2(q) = \frac{1}{16\pi^2} \int \frac{e^{ik|q-q'|}}{|q-q'|} \frac{e^{ik|q'-q''|}}{|q'-q''|} e^{ik\mathbf{u}_0 \cdot \mathbf{q}''} F(q') F(q'') \, dq' \, dq''. \]

\[ |u| = 1. \]

\[ |q| = r \to \infty: \]

\[ \psi_1(q) = -\frac{1}{4\pi r} \int e^{ik|q-q'|} e^{ik\mathbf{u}_0 \cdot \mathbf{q}'} F(q') \, dq'. \]

\[ |q| = r, \quad q = r \, u, \quad |u| = 1; \]

\[ |q'| = r', \quad q' = r' \, u', \quad |u'| = 1, \]

\[ r \to \infty: \]

\[ |q - q'| = r - r' \mathbf{u} \cdot \mathbf{u}', \]

\[ \psi_2(q) = \frac{e^{ikr}}{16\pi^2 r} \int \frac{e^{ik|q-q''|}}{|q'-q''|} \frac{e^{ikr''\mathbf{u}_0 \cdot \mathbf{u}''}}{|q'-q''|} e^{ikr'\mathbf{u}' \cdot \mathbf{u}} F(q') F(q'') \, dq' \, dq''. \]

\[ \mathbf{q}'' = \mathbf{q}' + \mathbf{\ell}; \]

\[ \psi_2(q) = \frac{e^{ikr}}{16\pi^2 r} \int e^{ik(\mathbf{u}_0 \cdot \mathbf{u}) \cdot \mathbf{q}'} F(q') \, dq' \int \frac{e^{ik|\ell|}}{|\ell|} e^{ik\mathbf{u}_0 \cdot \mathbf{\ell}} F(q' + \mathbf{\ell}) \, d\ell. \]

\[ F = \frac{1}{2\pi} \int \mathcal{F}_\gamma e^{i\gamma \cdot \mathbf{q}} \, d\gamma; \]

\[ \mathcal{F}_\gamma = \int F e^{-i\gamma \cdot \mathbf{q}} \, d\mathbf{q}, \]

\[ \int e^{-i\gamma \cdot \mathbf{q}} \frac{e^{ikr}}{r} \, d\mathbf{q} = \frac{4\pi}{\gamma^2 - k^2}. \]
\[ r \to \infty: \]
\[ \psi_1(q) = -\frac{e^{ikr}}{4\pi} F_{k(u-u_0)}, \]
\[ \psi_2(q) = \frac{e^{ikr}}{16\pi^2 r} \int e^{ik(u_0-u)\cdot q'} F(q') \, dq' \int \frac{2}{|ku_0 + \gamma|^2 - k^2} F_\gamma e^{i\gamma \cdot q'} \, d\gamma \]
\[ = \frac{e^{ikr}}{16\pi^2 r} \int \frac{2}{|ku_0 + \gamma|^2 - k^2} F_\gamma \, d\gamma \int e^{i(ku_0-ku+\gamma)\cdot q'} F(q') \, dq', \]
\[ \psi_2(q) = \frac{e^{ikr}}{8\pi^2 r} \int \frac{F_\gamma F_{k(u-u_0)-\gamma}}{|ku_0 + \gamma|^2 - k^2} \, d\gamma \]
\[ \psi_2(q) = \frac{e^{ikr}}{8\pi^2 r} \int \frac{F_\gamma - ku_0 F_{ku-\gamma}}{\gamma^2 - k^2} \, d\gamma. \]

### 6.5. COULOMB SCATTERING

The Schrödinger equation for the scattering of a wave from a Coulomb potential is solved and, in particular, the phase advancement is evaluated.

- \( Ze \) charge of the scatterer;
- \( Z'e \) charge of the incident particle;
- \( M \) mass of the incident particle.

We adopt units such that \( M = 1 \), \( ZZ'e^2 = 1 \), \( h/2\pi = 1 \). It follows that:
- the length unit is \( h^2/4\pi^2 MZZ'e^2 = (m/M)(1/ZZ')a_0 \); \(^6\)
- the energy unit is \( 4\pi^2 MZ^2Z'^2e^4/h^2 = 2(M/m)Z^2Z'^2\text{Rh} \); \(^7\)
- the velocity unit is \( 2\pi ZZ'e^2/h = ZZ'/137c \), where \( 1/137 = e^2/(1/2\pi)hc \).

The Schrödinger equation is:

\[ \nabla^2 \psi + 2 \left( E - \frac{1}{r} \right) \psi = 0. \]

\(^6\)Here \( m \) denotes the electron mass and \( a_0 \approx 0.529 \cdot 10^{-9} \) the Bohr radius.

\(^7\)1 Rh = 13.54 V [Remember that the symbol V used by Majorana should more appropriately understood as eV].
\( \psi = \sum_{\ell=0}^{n} \alpha_{\ell} \frac{X_{\ell}(r)}{r} P_{\ell}(\cos \theta), \)

\[ X''_{\ell} + \left( k^2 \frac{2}{r} - \frac{\ell(\ell + 1)}{r^2} \right) X_{\ell} = 0, \]

\( k^2 = 2E \) (the velocity of the ingoing particle in large units is \( v = (ZZ'/137)c/k \)).

\[ X_{\ell} = X_{\ell}^1 + X_{\ell}^2, \]

\[ X_{\ell}^1 = x^{\ell+1} e^{ikx} F \left( \ell + 1 + \frac{i}{k}, 2\ell + 2, -2ikx \right), \]

\[ X_{\ell}^2 = x^{\ell+1} e^{-ikx} F \left( \ell + 1 - \frac{i}{k}, 2\ell + 2, 2ikx \right), \]

\[ F(\alpha, \beta, x) = 1 + \frac{\alpha}{\beta} x + \frac{\alpha(\alpha + 1)}{2!\beta(\beta + 1)} x^2 + \frac{\alpha(\alpha + 1)(\alpha + 2)}{3!\beta(\beta + 1)(\beta + 2)} x^3 + \ldots \]

**Alternative solution**

\[ X'' + \left( k^2 \frac{2}{r} - \frac{\ell(\ell + 1)}{r^2} \right) X = 0, \]

\( \ell \) takes non-integer values greater than \(-1/2\),

\[ X = r^{\ell+1} u, \]

\[ u'' + 2 \frac{\ell + 1}{r} u' + \left( k^2 \frac{2}{r} \right) u = 0. \]

\[ u'' + \left( \delta_0 + \frac{\delta_1}{r} \right) u' + \left( \epsilon_0 + \frac{\epsilon_1}{r} \right) u = 0, \]

\( \delta_0 = 0, \delta_1 = 2(\ell + 1), \epsilon_0 = k^2, \epsilon_1 = -2; \)

\[ u \sim \int e^{iktr} (t - 1)^{\ell+i/k} (t + 1)^{\ell-i/k} dt. \]

\[ ^{\text{[@]} \text{This equation is a particular case of the more general one reported just after it, and is also considered by the author in another place; see Appendix 6.10.}} \]
\[ |\text{Im} \log(1 - t)| \leq \pi, \, |\text{Im} \log(1 + t)| \leq \pi: \]

\[
u = \int_{-1}^{1} e^{ikrt} (1 - t)^{\ell + i/k} (1 + t)^{\ell - i/k} dt. \]

For \( r = 0 \), on setting \( 1 - t = 2x \):

\[
u(0) = \int_{-1}^{1} (1 - t)^{\ell + i/k} (1 + t)^{\ell - i/k} dt = \int_{0}^{1} (2x)^{\ell - i/k} (2 - 2x)^{\ell + i/k} 2dx = 2^{2\ell+1} \int_{0}^{1} (x)^{\ell - i/k} (1 - x)^{\ell + i/k} dx, \]

\[
u(0) = 2^{2\ell+1} \frac{\Gamma(\ell + 1 - i/k) \Gamma(\ell + 1 + i/k)}{\Gamma(2\ell + 2)}. \quad (1)\]

For \( |r| > 0 \):

\[
u = \nu^1 + \nu^2, \]

\[
u^1 = e^{-i(\pi/2)(\ell+1+i/k)} e^{ikx} \int_{0}^{\infty} e^{-krp} p^{\ell+i/k} (2 + ip)^{\ell - ik} dp, \]

\[
u^2 = e^{i(\pi/2)(\ell+1-i/k)} e^{-ikx} \int_{0}^{\infty} e^{-krp} p^{\ell-i/k} (2 - ip)^{\ell + ik} dp. \]

For real \( r \) we have \( \nu^2 = \overline{\nu^1} \).

\[ \nu^1 = (kr)^{-(\ell+1)} e^{-i(\pi/2)(\ell+1+i/k)-(i/k)\log kr} e^{ikr} \]

\[ \times \int_{0}^{\infty} e^{-p} p^{\ell+i/k} (2 + ip/kr)^{\ell - ik} dp. \]

For \( r \to \infty \):

\[ \nu^1 = (kr)^{-(\ell+1)} e^{\pi/2k} e^{-i(\pi/2)(\ell+1)} e^{ikr-(i/k)\log kr} 2^{\ell-i/k} \Gamma(\ell + 1 + i/k) \]

\[ = 2^\ell (kr)^{-(\ell+1)} e^{\pi/2k} e^{-i(\pi/2)(\ell+1)} e^{ikr-(i/k)\log kr} 2kr \Gamma(\ell + 1 + i/k). \]

Now, replace \( \ell \) with \( \ell - \epsilon_\ell \); the phase advancement becomes then:

\[ k_\ell = \epsilon_\ell \frac{\pi}{2} - \arg \frac{\Gamma(\ell + 1 + i/k)}{\Gamma(\ell + 1 - \epsilon_\ell + i/k)}. \]

\[ @ \, The \, author \, then \, evaluates \, u' \, and \, u'' \, and \, verifies \, that \, the \, assumed \, form \, for \, u \, satisfies \, the previous \, differential \, equation. \]
\[
(\ell - \epsilon_\ell) (\ell + 1 - \epsilon_\ell) = \ell (\ell + 1) - a,
\]
\[
\left(\ell + \frac{1}{2} - \epsilon_\ell\right)^2 = \left(\ell + \frac{1}{2}\right)^2 - a,
\]
\[
\epsilon_\ell = \ell + \frac{1}{2} - \sqrt{\left(\ell + \frac{1}{2}\right)^2 - a}.
\]

6.6. QUASI COULOMBIAN SCATTERING OF PARTICLES

Let us assume a scattering potential of the form:
\[
k \frac{1}{\sqrt{r^2 + a^2}},
\]
\(a\) being the magnitude of the radius of the scatterer. By denoting with \(T\) the kinetic energy of the incident particles, let us define the minimum approach distance\(^{10}\) \(b\) in the limit Coulomb field \((a = 0)\) as:
\[
\frac{k}{b} = T; \quad b = \frac{k}{T}.
\]
The scattering intensity under an angle \(\theta\) will be obtained on multiplying that appearing in the Rutherford formula by a numerical factor depending on the mutual ratios of \(a, b, \lambda/2\pi\) (\(\lambda\) being the wavelength of the free particle) and \(\theta\). Let us set:
\[
i = f(\alpha, \beta, \theta) i_R,
\]
where \(i_R\) is the intensity calculated from the Rutherford formula \((a = 0)\) and
\[
\alpha = \frac{a}{\lambda/2\pi}, \quad \beta = \frac{b}{\lambda/2\pi}.
\]
Since for \(a = 0\) the Rutherford formula is exact, we have:
\[
f(0, \beta, \theta) = 1.
\]

\(^{10}@\) That is, the scattering parameter.
Let us now consider a fixed $\alpha$ and take the limit $\beta \to 0$. At zeroth order approximation, i.e., exactly for $\beta = 0$, we can use the Wentzel method. By choosing as mass unit $M$, wavelength unit $\lambda/2\pi$ and velocity unit $v$ for the incident particles, from $\lambda = h/Mv$ it follows that $h = 2\pi$ in our units. Moreover, the kinetic energy of the incident particle is $1/2$.

From Eqs. (4) and (2) it follows that $b = \beta$, $k = \beta/2$ and $a = \alpha$. By substituting these into Eq. (1), we get the expression for the potential energy, and the Schrödinger equation corresponding to the eigenvalue $1/2$ will be:

$$\nabla^2 \psi + \left(1 - \frac{\beta}{\sqrt{r^2 + \alpha^2}}\right) \psi = 0.$$  \hspace{1cm} (6)

Let us set:

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots,$$

where:

$$\nabla^2 \psi_n + \psi_n = \frac{\beta}{\sqrt{r^2 + \alpha^2}} \psi_{n-1}.$$ \hspace{1cm} (7)

In order to avoid convergence problems, instead of $\beta/\sqrt{r^2 + \alpha^2}$ let us consider the expression $\beta \left(1/\sqrt{r^2 + \alpha^2} - 1/R\right)$ for $r < R$ and 0 for $r > R$; in the final results we will take the limit $R \to \infty$. Eq. (7) is then replaced by:

$$\nabla^2 \psi_n + \psi_n = P \psi_{n-1};$$ \hspace{1cm} (8)

$$P = \begin{cases} 
\beta \left(\frac{1}{\sqrt{r^2 + \alpha^2}} - \frac{1}{R}\right), & \text{for } r < R; \\
0, & \text{for } r > R.
\end{cases}$$

Setting $\psi_0 = e^{iz}$, we have:

$$\nabla^2 \psi_1 + \psi_1 = P e^{iz}.$$ \hspace{1cm} (9)

For an univocal solution of Eq. (9) we will choose $\psi_1$ to represent a diverging wave. In this case Eq. (9) can be integrated and, putting

$$r_{12} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

@ In the original manuscript, the factor $\beta$ is lacking.
we get:

\[
\psi_1(x, y, z) = -\frac{1}{4\pi} \int P(x', y', z') \frac{e^{i(r_{12}+z')}}{r_{12}} \, dx' dy' dz'.
\]

(10)

Assuming the point \((x, y, z)\) to be far from the origin, we have \((r\) is the distance from the origin, \(\theta\) the angle between the vector radius and the z-axis):

\[
r \to \infty : \quad \psi_1(r, \theta) = -\frac{1}{4\pi r} \int P(x', y', z') e^{ir} \times e^{ir'(\cos \theta' - \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos \phi')} \, dx' dy' dz',
\]

\[
\cos \theta' (1 - \cos \theta) - \sin \theta \sin \theta' \cos \phi'
\]

\[
= 2 \sin \theta/2 \left[ \sin \theta/2 \cos \theta' - \cos \theta/2 \sin \theta' \cos \phi' \right]
\]

\[
= 2 \sin \theta/2 \left[ \cos (\pi/2 - \theta/2) \cos \theta' + \sin (\pi/2 - \theta/2) \sin \theta' \cos (\phi' - \pi) \right],
\]

\[
r \to \infty : \quad \psi_1(r, \theta) = -\frac{e^{ir}}{2r \sin \theta/2} \int_0^\infty r' P(r') \sin (2 \sin \theta/2 r') \, dr',
\]

(11)

whence we easily deduce:

\[
f(\alpha, 0, \theta) = \frac{2}{\beta} \sin \theta/2 \int_0^\infty r' P(r') \sin (2 \sin \theta/2 r') \, dr'.
\]

(12)

In we simply replace \(P\) with \(\beta/\sqrt{r^2 + \alpha^2}\), the integral in Eq. (12) does not converge; however, we can circumvent this difficulty by keeping indeterminate the upper integration limit and assuming, for the resulting integral, its mean value which for the upper limit tends to infinity. We thus find:

\[
f(\alpha, 0, \theta) = 2 \sin \theta/2 \int_0^\infty \frac{r}{\sqrt{r^2 + \alpha^2}} \sin (2 \sin \theta/2 r) \, dr
\]

\[
= \int_0^\infty \frac{x \sin x \, dx}{\sqrt{x^2 + 4\alpha^2 \sin^2 \theta/2}} = \varphi (\alpha \sin \theta/2).
\]

(13)
6.6.1 Method Of The Particular Solutions

\[ u''_\ell + \left( 1 - \frac{\beta}{\sqrt{r^2 + \alpha^2}} - \frac{\ell(\ell + 1)}{r^2} \right) u_\ell = 0. \] (14)

For the hydrogen atom we consider the values \( \beta = 0.4, 0.5, 0.6, 0.7 \) and \( \alpha = 0, 0.2, 0.4, 0.6, 0.8, 1 \). The solution of Eq. (14) is reported numerically in the following tables for \( \ell = 0 \) and \( \beta = 0.4 \).  

\[ \begin{array}{cccccc}
\hline
r & u & u' & u'' & u & u' & u'' \\
0 & 0 & 1 & 0.400 & 0 & 1 & 0 \\
0.1 & 0.1018 & 1.019 & 0.305 & & & \\
0.2 & 0.2067 & 1.049 & 0.207 & & & \\
0.3 & 0.3137 & 1.070 & 0.109 & & & \\
0.4 & 0.4217 & 1.080 & 0.000 & & & \\
0.5 & 0.5297 & 1.080 & -0.106 & & & \\
0.6 & 0.6366 & 1.069 & -0.212 & & & \\
0.7 & & & & & & \\
0.8 & & & & & & \\
0.9 & & & & & & \\
1.0 & & & & & & \\
1.1 & & & & & & \\
1.2 & & & & & & \\
1.3 & & & & & & \\
\hline
\end{array} \]

\[ \text{The author uses a numerical algorithm (unknown to us) in order to infer the solution } u(r) \text{ of Eq. (14) from its second (and first) derivative, and the first few results obtained are displayed in the tables.} \]
6.7. COULOMB SCATTERING: ANOTHER REGULARIZATION METHOD

Let us assume the potential to be as follows:

\[ V = \begin{cases} 
  V_0, & \text{for } r < R, \\
  k/r, & \text{for } r > R.
\end{cases} \]  

(1)
Denoting with $T$ the kinetic energy of the incident particles, the minimum approach distance in the Coulomb field will be:

$$ b = \frac{k}{T}. \quad (2) $$

The scattering intensity under an angle $\theta$ will be given by the product of the intensity scattering due to the Coulomb field, obtained from the Rutherford formula, times a numeric function depending on $\theta$, $R/\lambda$, $b/\lambda$, $V_0/T$:

$$ f\left(\frac{V_0}{T}, \frac{R}{\lambda/2\pi}, \frac{b}{\lambda/2\pi}, \theta\right), \quad (3) $$

where $\lambda$ is the wavelength of the free particle. Let us choose as mass unit $M$, velocity unit $v$ and length unit $\lambda/2\pi$ relative to the free particle. In such units, $h = \lambda M v$ is equal to $2\pi$, while $T$ is $1/2$. Moreover, let us set:

$$ A = \frac{V_0}{T}, \quad \alpha = \frac{R}{\lambda/2\pi}, \quad \beta = \frac{b}{\lambda/2\pi}, \quad (4) $$

so that:

$$ i_{i_R} = f\left(\frac{V_0}{T}, \frac{R}{\lambda/2\pi}, \frac{b}{\lambda/2\pi}, \theta\right) = f(A, \alpha, \beta, \theta). \quad (5) $$

In our units we have:

$$ V_0 = \frac{A}{2}, \quad R = \alpha, \quad b = \beta, \quad k = \frac{1}{2} \beta, \quad (6) $$

and the Schrödinger equation corresponding to the eigenvalue $1/2$ takes the form:

$$ \nabla^2 \psi + (1 - A) \psi = 0, \quad \text{for } r < R, \quad (7) $$

$$ \nabla^2 \psi + \left(1 - \frac{\beta}{r}\right) \psi = 0, \quad \text{for } r > R. $$

For the hydrogen we have:

$$ \beta = 0.4, \ 0.5, \ 0.6, \ 0.7; $$

$$ \alpha = 0.4, \ 0.5, \ 0.6, \ 0.7, \ 0.8; $$

$$ A = (2), \ (1.5), \ 1, \ 0.5, \ 0, \ -0.5, \ -1, \ -1.5, \ 2, \ -2.5, \ -3, $$

$$ -3.5, \ 4, \ -4.5, \ -5, \ -5.5, \ 6, \ -6.5, \ -7, \ -7.5, \ -8. \quad (8) $$

\footnote{In the original manuscript, the first dependent variable in Eq. (3) is $V_0/2T$ rather than $V_0/T$. However, in the following the author considered the latter parametrization.}

\footnote{In the original manuscript, the author wrote erroneously $h = \lambda/Mv$.}
6.8. TWO-ELECTRON SCATTERING

$v, v'$ be the velocities of the two beams;

$n_0, n'_0$ be the rest number densities of the two beams;

$n = n_0 / \sqrt{1 - v^2/c^2}$, $n' = n'_0 / \sqrt{1 - v'^2/c^2}$ be the number densities in the laboratory reference frame;

$v_r$ be the relative velocity according to the relativistic kinematics;

$S(v_r)$ be the cross section.

The number $N$ of collisions for unit volume and time can be written as:

$$N = a (v, v') n n' = S(v_r) n_0 n'_0 \frac{v_r}{\sqrt{1 - v_r^2/c^2}}.$$  

In terms of $a$ we thus have:

$$S = \frac{a}{v_r} \frac{\sqrt{1 - v_r^2/c^2}}{\sqrt{1 - u^2/c^2}} \sqrt{1 - v'_r^2/c^2}$$

(classically (that is: non relativistically), we have instead $S = a/|v - v'|$).

Without considering the resonance in the scattering cross section, let $u$ ($0 \leq u \leq v_r$) be the velocity of the first electron after the collision in its initial reference frame; we have:

$$dS = S(v_r, u) du.$$  

Let us now denote with $u_1$ the relative velocity between the frame of the first electron before (after) the collision and that of the second electron after (before) the collision. By taking into account the resonance between the two electrons, $u$ and $u_1$ are indistinguishable. Putting, conventionally, $u \leq u_1$, the maximum value of $u$ is given by \[^{15}\]:

$$u_{\text{max}} = u_{\text{min}} = c \sqrt{1 - \frac{4(1 - v_r^2/c^2)}{\left(1 + \sqrt{1 - v_r^2/c^2}\right)^2}} = \frac{y}{2} \sqrt{2 - \frac{y^2}{c^2}}$$

$$\left(y = c \sqrt{\frac{1 - \sqrt{1 - v_r^2/c^2}}{1 + \sqrt{1 + v_r^2/c^2}}}.\right)$$

The relation between $u$ and $u_1$ is the following:

$$\frac{1}{\sqrt{1 - u^2/c^2}} + \frac{1}{\sqrt{1 - u_1^2/c^2}} = 1 + \frac{1}{\sqrt{1 - v_r^2/c^2}}.$$  

\[^{15}\]In this case we have $u = u_1$
6.9. COMPTON EFFECT

\[ n = n_0 / \sqrt{1 - v^2 / c^2}, \]  
number of electrons per cm\(^3\);

\[ n_0, \text{ rest number densities of the electron beams}; \]

\[ N, \text{ number of photons}^{16} \text{ per cm}^3; \]

\[ N_0 = N \nu_0 / \nu, \text{ number of photons per cm}^3 \text{ in the electron frame}; \]

\[ h\nu, \text{ energy of one photon}; \]

\[ h\nu_0, \text{ energy of one photon in the electron frame (before or after the collision)}; \]

\[ u_1, \text{ relative velocity between the ingoing electron frame and the outgoing one (according to relativistic kinematics)}; \]

\[ S(\nu_0), \text{ cross section}. \]

The number of collisions for unit volume and time is thus:

\[ S(\nu_0) n_0 N_0 c = a n N, \]

so that, in terms of \( a \),

\[ S(\nu_0) = \frac{a}{c} \frac{1}{\sqrt{1 - v^2 / c^2}} \frac{\nu}{\nu_0}. \]

The differential cross section can be written as:

\[ dS = F(\nu_0, u) \, du, \]

so that

\[ S = \int_0^\infty F(\nu_0, u) \, du. \]

Classically, the cross section is given by:

\[ S_{\text{class}} = \frac{8\pi}{3} \frac{e^4}{m^2 c^4}. \]

\(^{16}\) For the sake of clarity, here and in the following we have translated with “photons” what was termed “quanta” in the original manuscript.
6.10. QUASI-STATIONARY STATES

The author considered the transition from a discrete (unperturbed) state $\psi_0$ with energy $E_0$ to a continuum (perturbed) state $\psi$, assuming that the unperturbed system has two continuum spectra $\phi_W$ and $\psi_W$ with energy $E_0 + W$. What here reported are the scratch calculations which prepared the Sect. 28 of Volumetto IV, to which we refer the reader for notations and further explanations of the arguments treated by the author. However, a further generalization is present here with respect to what considered after Eq. (4.499) of Volumetto IV.

$$\psi = \frac{1}{e^2/Q^2 + \pi^2 Q^2} \left( \frac{\epsilon I}{Q^2} \psi_0 + \frac{\epsilon^2 |I|^2}{Q^4} \psi_W - \frac{\epsilon |I|^2}{Q^2} \int \frac{\psi_{W'}}{W' - W} dW' \right)$$

$$+ \frac{\epsilon^2 I L}{Q^4} \phi_W - \frac{\epsilon I L}{Q^4} \int \frac{\phi_{W'}}{W' - W} dW'$$

$$+ I \int \frac{e^{-2\pi i(\epsilon' - \epsilon)t/h}}{(\epsilon^2/Q^2 + \pi^2 Q^2)(W' - W)} \psi_0$$

$$+ \frac{|I|^2}{Q^2} \int \frac{e^{-2\pi i(\epsilon' - \epsilon)t/h}}{(\epsilon^2/Q^2 + \pi^2 Q^2)(W' - W)} \psi_{W'}$$

$$- |I|^2 \int \frac{e^{-2\pi i(\epsilon' - \epsilon)t/h}}{(\epsilon^2/Q^2 + \pi^2 Q^2)(W' - W)} \int \psi_{W''} dW''$$

$$+ \frac{I L}{Q^2} \int \frac{e^{-2\pi i(\epsilon' - \epsilon)t/h}}{(\epsilon^2/Q^2 + \pi^2 Q^2)(W' - W)} \phi_{W'}$$

$$- I L \int \frac{e^{-2\pi i(\epsilon' - \epsilon)t/h}}{(\epsilon^2/Q^2 + \pi^2 Q^2)(W' - W)} \int \phi_{W''} dW''$$

$$+ \frac{|L|^2}{Q^2} \psi_W - \frac{I L}{Q^2} \phi_W.$$

$$\psi = \left( A \psi_0 + B \psi_W + C \phi_W + \int b \psi_{W'} dW' + \int c \phi_{W'} dW' \right) e^{-2\pi iEt/h},$$

Quantity $A$:

$$\frac{1}{(\epsilon^2/Q^2 + \pi^2 Q^2)(\epsilon' - \epsilon)} = \frac{Q^2}{(\epsilon' + i\pi Q^2)(\epsilon' - i\pi Q^2)(\epsilon' - \epsilon)},$$

$$R_1 = e^{2\pi iEt/h} e^{-2\pi^2 Q^2 t/h} \frac{1}{2\pi i(\epsilon + i\pi Q^2)}.$$
THE THEORY OF SCATTERING

\[ R^2 = \frac{1}{\epsilon^2/Q^2 + \pi^2 Q^2}, \]

\[ -2\pi i \left( R_1 + \frac{1}{2} R_2 \right) = \frac{1}{\epsilon^2/Q^2 + \pi^2 Q^2} \left[ e^{2\pi i \epsilon t} e^{-2\pi Q^2 t/h} \right. \]

\[ \times \left( -\epsilon/Q^2 + i\pi \right) - i\pi \right], \]

\[ A = \frac{1}{\epsilon^2/Q^2 + \pi^2 Q^2} \left[ \frac{\epsilon I}{Q^2} \left( 1 - e^{2\pi i \epsilon t/h} e^{-t/2T} \right) \right. \]

\[ -I\pi i \left( 1 - e^{(2\pi i/h)\epsilon t} e^{-t/2T} \right) \],

\[ A = \frac{I}{\epsilon + i\pi Q^2} \left( 1 - e^{2\pi i \epsilon t/h} e^{-t/2T} \right). \]

Quantity B:

\[ \frac{1}{\epsilon^2/Q^2 + \pi^2 Q^2} \frac{\epsilon^2 |I|^2}{Q^4} + \frac{\pi^2 |I|^2}{Q^4} + \frac{|L|^2}{Q^2} = \frac{1}{Q^2} \left( |I|^2 + |L|^2 \right) = 1, \]

\[ B = 1. \]

Quantity C:

\[ \frac{1}{\epsilon^2/Q^2 + \pi^2 Q^2} \frac{\epsilon^2 IL}{Q^4} + \frac{\pi^2 IL}{Q^4} - \frac{IL}{Q^2} = 0, \]

\[ C = 0. \]
Quantity $b$: \[ b = \int \frac{e^{-2\pi i (e'-e)t/h}de'}{(e'^2/Q^2 + \pi^2 Q^2)(e'-e)(e'-e'')} = \int \frac{Q^2 e^{-2\pi i (e'-e)t/h}de'}{(e'+i\pi Q^2)(e'-i\pi Q^2)(e'-e)(e'-e'')}, \]

\[ -2\pi i \left( R_0 + \frac{1}{2} R_1 + \frac{1}{2} R_2 \right), \]

\[ R_0 = e^{2\pi i et/h} e^{-2\pi^2 Q^2 t/h} \frac{1}{e'^2/Q^2 + \pi^2 Q^2} e^{-2\pi i (e'+i\pi Q^2)(e''+ipi Q^2)}, \]

\[ R_1 = \frac{1}{e'^2/Q^2 + \pi^2 Q^2} \frac{1}{e - e''}, \]

\[ R_2 = e^{2\pi i (e-e'')t/h} \frac{-1}{e''^2/Q^2 + \pi^2 Q^2} \frac{1}{e - e''}. \]

\[ -2\pi i \left( R_0 + \frac{1}{2} R_1 + \frac{1}{2} R_2 \right) = \frac{1}{e'^2/Q^2 + \pi^2 Q^2} \frac{1}{e''^2/Q^2 + \pi^2 Q^2} \]

\[ \times \left[ e^{2\pi i et/h} e^{-t/2T} \left( \frac{e}{Q^2} - i\pi \right) \left( \frac{e''}{Q^2} - i\pi \right) \right. \]

\[ - \left( \frac{e''^2}{Q^2} + \pi^2 Q^2 \right) \frac{\pi i}{e - e''} + \left( \frac{e^2}{Q^2} + \pi^2 Q^2 \right) \frac{\pi i e^{2\pi i (e-e'')t/h}}{e - e''} \right]. \]

\[ b = -\frac{|I|^2}{Q^2} \frac{1}{e - e'} \frac{1}{e'^2/Q^2 + \pi^2 Q^2} + \frac{|I|^2}{e'^2/Q^2 + \pi^2 Q^2} \frac{1}{e'' - e'} \frac{1}{e'^2/Q^2 + \pi^2 Q^2} \]

\[ + e^{2\pi i (e-e'')t/h} \frac{1}{e'^2/Q^2 + \pi^2 Q^2} \frac{1}{e'' - e'} \]

\[ \times \left[ e^{2\pi i et/h} e^{-t/2T} \left( \frac{e}{Q^2} - i\pi \right) \left( \frac{e'}{Q^2} - i\pi \right) \right. \]

\[ - \left( \frac{e'^2}{Q^2} + \pi^2 Q^2 \right) \frac{\pi i}{e - e'} + \left( \frac{e^2}{Q^2} + \pi^2 Q^2 \right) \frac{\pi i e^{2\pi i (e-e'')t/h}}{e - e'} \right] \]

\[ = \frac{|I|^2}{\epsilon + i\pi Q^2} \frac{1}{\epsilon - e'} - \frac{|I|^2}{e' + i\pi Q^2} \frac{1}{\epsilon - e'} e^{2\pi i (e-e'')t/h} \]

\[ + \frac{|I|^2}{e^{2\pi i et/h-t/2T}} e^{2\pi i (e-e'')t/h}, \]

\[ \text{Cf. the figure above.} \]
\[ b = \frac{|I|^2}{(\epsilon + i\pi Q^2)(\epsilon' + i\pi Q^2)} \left[ -1 + e^{2\pi i \epsilon t/h - t/2T} \right. \\
\left. \quad + \frac{\epsilon + i\pi Q^2}{\epsilon - \epsilon'} \left( 1 - e^{2\pi i (\epsilon - \epsilon') t/h} \right) \right], \]

\[ b = \frac{|I|^2}{(\epsilon + i\pi Q^2)(\epsilon' + i\pi Q^2)} \left[ -e^{2\pi i (\epsilon - \epsilon') t/h} + e^{2\pi i \epsilon t/h - t/2T} \right. \\
\left. \quad + \frac{\epsilon' + i\pi Q^2}{\epsilon - \epsilon'} \left( 1 - e^{2\pi i (\epsilon - \epsilon') t/h} \right) \right]. \]

Quantity \( c \):

\[ c = -\frac{\epsilon IL}{Q^2} \frac{1}{\epsilon - \epsilon'} \frac{1}{e^2/Q^2 + \pi^2 Q^2} + \frac{\epsilon' IL}{Q^2} \frac{1}{\epsilon' - e} \frac{e^{2\pi i (\epsilon - \epsilon') t/h}}{e^2/Q^2 + \pi^2 Q^2} \]
\[ + \frac{IL}{\epsilon^2/Q^2 + \pi^2 Q^2} \frac{1}{e^2/Q^2 + \pi^2 Q^2} \]
\[ \times \left[ e^{2\pi i \epsilon t/h} e^{-t/2T} \left( \frac{\epsilon}{Q^2} - i\pi \right) \left( \frac{\epsilon'}{Q^2} - i\pi \right) \right. \\
\left. - \left( \frac{\epsilon'^2}{Q^2} + \pi^2 Q^2 \right) \frac{\pi i}{\epsilon - \epsilon'} + \left( \frac{\epsilon^2}{Q^2} + \pi^2 Q^2 \right) \frac{\pi i e^{2\pi i (\epsilon - \epsilon') t/h}}{\epsilon - \epsilon'} \right] \\
= \frac{IL}{\epsilon + i\pi Q^2} \frac{1}{\epsilon - \epsilon'} - \frac{IL}{\epsilon' + i\pi Q^2} \frac{1}{\epsilon - \epsilon'} e^{2\pi i (\epsilon - \epsilon') t/h} \]
\[ + \frac{IL e^{2\pi i \epsilon t/h - t/2T}}{\epsilon + i\pi Q^2 (\epsilon' + i\pi Q^2)}. \]

\[ c = \frac{IL}{(\epsilon + i\pi Q^2)(\epsilon' + i\pi Q^2)} \left[ -1 + e^{2\pi i \epsilon t/h - t/2T} \right. \\
\left. \quad + \frac{\epsilon + i\pi Q^2}{\epsilon - \epsilon'} \left( 1 - e^{2\pi i (\epsilon - \epsilon') t/h} \right) \right]. \]
\[ \psi = e^{-2\pi i Et/h} \psi_W + \frac{I}{\epsilon + i\pi Q^2} \left( e^{-2\pi i Et/h} - e^{-2\pi i(E_0-k)t/h} e^{-t/2T} \right) \psi_0 \]

\[ - \frac{I}{\epsilon + i\pi Q^2} \int \frac{\overline{I}\psi_{W'} + \overline{L}\phi_{W'}}{\epsilon' + i\pi Q^2} e^{-2\pi i Et/h} \left( 1 - e^{2\pi i \epsilon t/h - t/2T} \right) d\epsilon' \]

\[ + I \int \frac{\overline{I}\psi_{W'} + \overline{L}\phi_{W'}}{\epsilon' + i\pi Q^2} e^{-2\pi i Et/h} \frac{1}{\epsilon' - \epsilon} \left( 1 - e^{2\pi i(\epsilon-\epsilon')t/h} \right) d\epsilon', \]

\[ \psi = e^{-2\pi i Et/h} \psi_W + a e^{-2\pi i E_0 t/h} \psi_0 \]

\[ + \int b_{W'} \left( \overline{I}\psi_{W'} + \overline{L}\phi_{W'} \right) dW' \cdot e^{-2\pi i E' t/h}, \]

\[ H\psi = E e^{-2\pi i Et/h} \psi_W + I_{W} e^{-2\pi i E_0 t/h} \psi_0 + a E_0 e^{-2\pi i E_0 t/h} \psi_0 \]

\[ + \int a e^{-2\pi i E_0 t/h} \overline{I}_W \psi_W dW' + \int a e^{-2\pi i E_0 t/h} \overline{L}_W \phi_W dW' \]

\[ + \int E' b_{W'} \left( \overline{I}_W \psi_{W'} + \overline{L}_W \phi_{W'} \right) e^{-2\pi i E' t/h} dW' \]

\[ + Q^2 \int b_{W'} dW' \cdot \psi_0 e^{-2\pi i E' t/h}, \]

\[ I_{W} = I:\]

\[ \dot{a} = -\frac{2\pi i}{h} \left( e^{-2\pi i W t/h} I + Q^2 \int b_{W'} e^{-2\pi i W' t/h} dW' \right), \]

\[ b_{W'} = -\frac{2\pi i}{h} e^{-2\pi i W' t/h} a. \]
\[ \psi = \psi' + \psi'', \]

\[ \psi' = e^{-2\pi i Et/h} \psi_W + \frac{I}{\epsilon + i\pi Q^2} e^{-2\pi i Et/h} \psi_0 \]
\[ - \frac{I}{\epsilon + i\pi Q^2} \int \frac{(I\psi_W' + \bar{L}\phi_W')}{e' - \epsilon} e^{-2\pi i Et/h} \left( 1 - e^{2\pi i (\epsilon - \epsilon') t/h} \right) d\epsilon', \]

\[ \psi'' = - \frac{I}{\epsilon + i\pi Q^2} e^{-2\pi i (E_0 - k)t/h} e^{-t/2T} \psi_0 \]
\[ - \frac{I}{\epsilon + i\pi Q^2} \int \frac{(I\psi_W' + \bar{L}\phi_W')}{e' + i\pi Q^2} e^{-2\pi i E't/h} \left( 1 - e^{2\pi i \epsilon't/h - t/2T} \right) d\epsilon'. \]

Appendix:
Transforming a differential equation

\[ u'' + \left( \delta_0 + \frac{\delta_1}{r} \right) u' + \left( \epsilon_0 + \frac{\epsilon_1}{r} \right) u = 0, \]

\[ \chi = r^k u. \]

\[ u = r^{-k} u, \]

\[ u' = u \left( \frac{\chi'}{\chi} - \frac{k}{r} \right), \]

\[ u'' = u \left( \frac{\chi''}{\chi} - \frac{k}{r} \right)^2 + u \left( \frac{\chi''}{\chi} - \frac{\chi'^2}{2} + \frac{k}{r^2} \right) \]

\[ = u \left( \frac{\chi''}{\chi} - 2 \frac{k}{r} \frac{\chi'}{\chi} + \frac{k(k+1)}{r^2} \right). \]

\[ \left( \frac{\chi''}{\chi} - 2 \frac{k}{r} \frac{\chi'}{\chi} + \frac{k(k+1)}{r^2} \right) + \delta_0 \frac{\chi'}{\chi} - \frac{\delta_0 k}{r} + \frac{\delta_1 \chi'}{r} - \frac{k\delta_1}{r^2} + \epsilon_0 + \frac{\epsilon_1}{r} = 0, \]

\[ \chi'' + \left( \delta_0 + \frac{\delta_1}{r} - 2 \frac{k}{r} \right) \chi' + \left( \epsilon_0 + \frac{\epsilon_1 - k\delta_0}{r} + \frac{k(k+1) - k\delta_1}{r^2} \right) \chi = 0. \]

\[ k = \frac{\delta_1}{2}; \quad \delta_1 = 2k, \]

\[ \chi'' + \delta_0 \chi' + \left( \epsilon_0 + \frac{\epsilon_1 - k\delta_0}{r} - \frac{k(k-1)}{r^2} \right) \chi = 0. \]
7

NUCLEAR PHYSICS

7.1. WAVE EQUATION FOR THE NEUTRON

Denoting with $\varepsilon$ the electric or diamagnetic susceptibility, the Lagrangian describing the electromagnetic field is:

$$-\frac{1}{2}\varepsilon(E^2 - H^2).$$

Using Dirac coordinates,

$$\left[\frac{W}{c} + \rho_1 \sigma \cdot p + \rho_3 mc + \frac{\varepsilon}{2c}\rho_3 (E^2 - H^2)\right]\psi = 0.$$

7.2. RADIOACTIVITY

In the following table the author referred to some radioactive nuclides grouped by their atomic number $Z$. The number following the (old) name of the given isotope is its mass number. Probably this table was aimed at cataloguing the isotopes existing at the time of Majorana according to $Z$ for further studies.

[1]

<table>
<thead>
<tr>
<th></th>
<th>Z = 90</th>
<th>U X</th>
<th>234</th>
<th>Z = 89</th>
<th>Ac</th>
<th>227</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U Y</td>
<td>231</td>
<td></td>
<td></td>
<td>Ms Th2</td>
<td>228</td>
</tr>
<tr>
<td></td>
<td>Io</td>
<td>230</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rd Ac</td>
<td>227</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Th</td>
<td>232</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rd Th</td>
<td>228</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[1] In the original manuscript, the unidentified Ref. 9.28 appears here.
7.3. NUCLEAR POTENTIAL

In the following pages, the author considered the problem of finding the nucleon potential inside a given nucleus. In particular, he focused on the interaction between neutrons and protons, assuming that the interaction between protons is approximatively given only by the usual electrostatic repulsion, while that between neutrons is negligible. Many of the results discussed apply to a general nucleus of atomic number \( Z \) and mass number \( A \), although particular attention was here given to \( \alpha \) particles. What reported in the following is, at the same time, a preliminary study and a generalization of what published by Majorana in Z. Phys. 82 (1933) 137, or in La Ricerca Scientifica 4 (1933) 559, on the nuclear exchange forces.

7.3.1 Mean Nucleon Potential

Some expressions for the matrix elements of the interaction potential between neutrons and protons in a given nucleus were defined in the following. The author considered the case of a nucleus composed of a number \( a \) of protons (whose wavefunctions, depending on the coordinates \( q \), were denoted with \( \psi \)) and \( A \) of neutrons (whose wavefunctions, depending on the coordinates \( Q \), were denoted with \( \varphi \)). The state function of the nucleons was written as a Slater determinant. With reference to the published papers quoted above, the given form of the matrix elements of the interaction potential (also considered in the
following subsections) in terms of Dirac $\delta$-functions corresponds to the hypothesis that the mean energy per nucleon cannot exceed a certain limit, whatever large be the nuclear density. It is also assumed that the density of neutrons is larger than that of protons.

In the second part, it seems that the author considered the particular case of a nucleus of helium (with only two protons and two neutrons), probably thought as composed of two deuterium nuclei (denoted, in the original manuscript, as $d$ and $D$, respectively). However, it is also possible that the author was initially studying the scattering of two nuclei with mass numbers $a$ and $A$, respectively, and that only later on he turned to the particular case cited above. The interaction potential between the nucleon $s$ in the first nucleus and the nucleon $S$ in the second one was denoted with $V^{sS}$.

\[
\psi_1, \psi_2, \ldots, \psi_a; \quad q_1, q_2, \ldots, q_a;
\]
\[
\varphi_1, \varphi_2, \ldots, \varphi_A; \quad Q_1, Q_2, \ldots, Q_A
\]

($A \geq a$).

\[
\psi = \frac{1}{\sqrt{a!}} \begin{vmatrix}
\psi_1(q_1) & \cdots & \psi_1(q_a) \\
\vdots & \ddots & \vdots \\
\psi_a(q_1) & \cdots & \psi_a(q_a)
\end{vmatrix},
\]
\[
\varphi = \frac{1}{\sqrt{A!}} \begin{vmatrix}
\varphi_1(Q_1) & \cdots & \varphi_1(Q_A) \\
\vdots & \ddots & \vdots \\
\varphi_A(Q_1) & \cdots & \varphi_A(A_A)
\end{vmatrix}.
\]

\[
\langle q', Q' | V | q'', Q'' \rangle = \delta(q'' - Q') \delta(Q'' - q') \, f |q' - Q'|.
\]

\[
\langle q'_s, Q'_s | V^{sS} | q''_s, Q''_S \rangle = \delta(q''_s - Q'_S) \delta(Q''_S - q'_s) \, f |q'_s - Q'_S|;
\]

\[
V^S = \sum_s V^{sS}, \quad V^s = \sum_S V^{sS};
\]

\[
\overline{V^{sS}} = \iint \int \int \overline{\psi}(q'_s) \overline{\varphi}(Q'_s) \delta(Q''_S - q'_s)
\]
\[
\times \delta(q''_s - Q'_S) \psi(q'_s) \psi(q''_s) \varphi(q'_s) \varphi(q''_s) dq'_s dQ'_s dq''_s dQ''_S;
\]

\[
f |q'_s - Q'_S| = \iint \int \int \psi(q'_s - q''_s) \overline{\psi}(q'_s) \overline{\varphi}(q''_s) \varphi(q'_s) \varphi(q''_s) dq'_s dq''_s dQ'_s;
\]

\[
\overline{V^s} = \iint \int \int \overline{\psi}(q'_s - q''_s) \overline{\psi}(q'_s) \left[ \sum_S \varphi_S(q'_s) \overline{\varphi}_S(q''_s) \right] dq'_s dq''_s.
\]
\[
\langle q' | V^s | q'' \rangle = f |q' - q''| \sum_S \varphi_S(q') \overline{\varphi}_S(q'').
\]

### 7.3.2 Computation Of The Interaction Potential between Nucleons

The following calculations seems to be aimed at obtaining an expression for the interaction potential between nucleons (the primed quantities probably refer to neutrons, while the unprimed ones to protons); see also the beginning of the next subsection.

\[
\int \int \frac{dq \, dq'}{|q - q'|^2} = \int dq \int \frac{dq'}{(q - q')^2}.
\]

\[
q' = (\rho, \vartheta, \varphi),
\]

\[
dq' = \rho^2 \sin \vartheta \, d\vartheta \, d\varphi \, d\rho,
\]

\[
|q - q'|^2 = |q|^2 + \rho^2 - 2 |q| \rho \cos \vartheta,
\]

\[
s = |q - q'|,
\]

\[
s^2 = |\varphi|^2 + |\rho|^2 - 2 |q| \rho \cos \vartheta,
\]

\[
2s \, ds = 2 |q| \rho \sin \vartheta \, d\vartheta.
\]

\[
R' > q:\]

\[
\int \frac{dq'}{|q - q'|^2} = \pi \left( 2R' + \frac{R'^2 - q^2}{q} \log \frac{R' + q}{R' - q} \right).
\]

\[
R' > q:\]

\[
|q| - \rho \leq s \leq |q| + \rho,
\]

\[
dq' = \frac{s \rho}{|q|} \, ds \, d\varphi \, d\rho = 2\pi \frac{s \rho}{|q|} \, ds \, d\rho.
\]

\[
\int \frac{dq'}{|q - q'|^2} = \frac{2\pi}{|q|} \int \rho d\rho \int \frac{ds}{s} \cdots = \frac{2\pi}{|q|} \int \rho d\rho \log \frac{|q| + \rho}{|q| - \rho} + \ldots.
\]

\[
\int \rho \, d\rho \, \log(q + \rho) = \frac{1}{2} \rho^2 \log(q + \rho) - \frac{1}{2} \int \frac{\rho^2}{q + \rho} \, d\rho
\]

\[
= \frac{1}{2} \rho^2 \log(q + \rho) - \frac{1}{4} (\rho - q)^2 - \frac{1}{2} q^2 \log(q + \rho).
\]
\[ \int \frac{dq'}{|q-q'|^2} = \frac{2\pi}{q} \int_0^q \rho \, d\rho \, \log \frac{q+\rho}{q-\rho} + \frac{2\pi}{q} \int_q^{R'} \rho \, d\rho \, \log \frac{\rho+q}{\rho-q} \]

\[ = \frac{2\pi}{q} \left\{ \frac{1}{2} R'^2 \log (R' + q) - \frac{1}{4} (R' - q)^2 
- \frac{1}{2} q^2 \log (R' + q) + \frac{1}{4} q^2 + \frac{1}{2} q^2 \log q 
- \left[ \frac{1}{2} R'^2 \log (R' - q) - \frac{1}{4} (R' + q)^2 
- \frac{1}{2} q^2 \log (R' - q) + \frac{1}{4} q^2 + \frac{1}{2} q^2 \log q \right] \right\}. \]

\[ dq' = dx' \, dy' \, dz'. \]

\[ F(q) = \int_{q<R'} \frac{dq'}{|q'-q|^2} = \begin{cases} 
\pi \left( 2R' + \frac{R'^2 - q^3}{q} \log \frac{R' + q}{R' - q} \right), & q < R'; \\
\pi \left( 2R' - \frac{q^2 - R'^2}{q} \log \frac{q + R'}{q - R'} \right), & q > R'. 
\end{cases} \]

\[ \frac{1}{\pi} F(0) = 4R', \quad \frac{1}{\pi} F(R') = 2R'. \]

\[ q > R': \]

\[ \log \frac{q+R'}{q-R'} = 2 \left( \frac{R'}{q} + \frac{1}{3} \frac{R'^3}{q^3} + \frac{1}{5} \frac{R'^5}{q^5} + \ldots \right); \]

\[ \frac{q^2 - R'^2}{q} \cdot 2 \left( \frac{R'}{q} + \frac{1}{3} \frac{R'^3}{q^3} + \ldots \right) = 2R' \left( 1 - \frac{2}{3} \frac{R'^2}{q^2} - \frac{2}{15} \frac{R'^4}{q^4} - \ldots \right). \]

\[ F(q) + F \left( \frac{R'^2}{q} \right) = 4\pi R'. \]

\[ (q > R'): \quad F(q) = \frac{4\pi R'^3}{q^2} \left( \frac{1}{3} + \frac{1}{15} \frac{R'^2}{q^2} + \frac{1}{35} \frac{R'^4}{q^4} + \ldots \right); \]

\[ (q < R'): \quad F(q) = 4\pi R' \left( 1 - \frac{1}{3} \frac{q^2}{R'^2} - \frac{1}{15} \frac{q^4}{R'^4} - \frac{1}{35} \frac{q^6}{R'^6} - \ldots \right). \]
\[ q < R < R', \ t < R < R': \]

\[
\int_{q<R} F(q) dq = 4\pi^2 \int_0^R \left\{ 2t^2 R' + (tR'^2 - t^3) \log \frac{R' + t}{R' - t} \right\} dt.
\]

\[ F(q) dq = 4\pi t^2 F(t) dt. \]

\[ F(t) = \pi \left( 2R' + \frac{R'^2 - t^2}{t} \log \frac{R' + t}{R' - t} \right). \]

\[
\int t \log(t + R') dt = \frac{1}{2} t^2 \log(t + R') - \frac{1}{2} \int \frac{t^2}{t + R'} dt
\]

\[ = \frac{1}{2} t^2 \log(t + R') - \frac{1}{4} (t - R')^2 - \frac{1}{2} R'^2 \log(t + R'). \]

\[
\int t^3 \log(t + R') dt = \frac{1}{4} \log(t + R') - \frac{1}{4} \int \frac{t^4}{t + R'} dt
\]

\[ = \frac{1}{4} t^4 \log(t + R') - \frac{1}{16} t^4 + \frac{1}{12} R't^3 - \frac{1}{8} R'^2 t^2
\]

\[ + \frac{1}{4} R'^3 t - \frac{1}{4} R'^4 \log(t + R'). \]

\[ R < R': \]

\[
\int F(q) dq = 4\pi^2 \left\{ \frac{2}{3} R^3 R' - \frac{1}{2} (R'^2 - R^2) R'^2 \log \frac{R' + R}{R' - R} + RR'^3
\]

\[ - \frac{1}{4} R'^4 \log \frac{R' + R}{R' - R} - \frac{1}{6} R'R^3 - \frac{1}{2} R'^3 R
\]

\[ + \frac{1}{4} R'^4 \log \frac{R' + R}{R' - R} \right\}

\[ = 4\pi^2 \left\{ \frac{1}{2} R^3 R' + \frac{1}{2} RR'^3 - \frac{1}{4} (R'^2 - R^2)^2 \log \frac{R' + R}{R' - R} \right\}. \]

\[ R' > R: \]

\[
\int_{q<R} \int_{q'<R'} \frac{dq \ dq'}{|q' - q|^2}
\]

\[ = \pi^2 \left\{ 2R^3 R' + 2RR'^3 - (R'^2 - R^2)^2 \log \frac{R' + R}{R' - R} \right\}. \]
### 7.3.3 Nucleon Density

In the following the author worked out some expressions for the nucleon density, starting from the potential and kinetic energy densities $V$ and $T$ of a system of nucleons (the proton and neutron density are denoted with $\rho(=\sum_1^Z \psi_i \bar{\psi}_1)$ and $\rho'(=\sum_1^Y \varphi_i \bar{\varphi}_1)$, respectively). Notice that the potential energy density $V$ is given, up to a factor $-\pi^2$, by the last formula in the previous subsection, with the replacements $R, R' \rightarrow \rho, \rho'$.

Potential energy per unit volume:

$$-V = 2\rho\rho' \frac{1}{3} + 2\rho'\rho \frac{1}{3} - (\rho' \rho \frac{2}{3} - \rho \rho \frac{2}{3})^2 \log \frac{\rho^\frac{1}{3} + \rho'^\frac{1}{3}}{|\rho^\frac{1}{3} - \rho'^\frac{1}{3}|}.$$

Kinetic energy per unit volume:

$$T = \frac{3}{5}(\rho^\frac{5}{3} + \rho'^\frac{5}{3}).$$

$\rho = \rho'$:\n
$$-V = 4\rho^\frac{4}{3} = \frac{2500}{81},$$

$$T = \frac{6}{5}\rho^\frac{5}{3} = \frac{1250}{81}.$$

$$T = -\frac{1}{2}V.$$

$$\frac{6}{5}\rho^\frac{5}{3} = \rho^\frac{4}{3}, \quad \rho^\frac{1}{3} = \frac{10}{6} = \frac{5}{3}, \quad \rho = \frac{125}{27};$$

$$\frac{-V}{\rho} = \frac{20}{3},$$

$$\frac{T}{2\rho} = \frac{5}{3}.$$

$$\frac{-\partial V}{\partial \rho} = -\frac{\partial V}{\partial \rho'} = \frac{8}{3}\rho^\frac{1}{3} = \frac{40}{9},$$

$$\frac{\partial T}{\partial \rho} = \frac{\partial T}{\partial \rho'} = \rho^\frac{2}{3} = \frac{25}{9};$$

---

2 The numerical values $2500/81$ and $1250/81$ seem to have been written by the author after he deduced the numerical value for $\rho$ (see below).
\[- \left( \frac{\partial V}{\partial \rho} + \frac{\partial T}{\partial \rho} \right) = \frac{5}{3}.\]

\[\rho \neq \rho' \ (\rho < \rho'):\]

\[- \frac{\partial V}{\partial \rho} = 2\rho'^{\frac{1}{3}} + \frac{2}{3}\rho^{-\frac{2}{3}}\rho' + \frac{4}{3}\rho^{-\frac{1}{3}} \left( \rho'^{\frac{2}{3}} - \rho^{\frac{2}{3}} \right) \log \frac{\rho'^{\frac{1}{3}} + \rho^{\frac{1}{3}}}{\rho'^{\frac{1}{3}} - \rho^{\frac{1}{3}}} \]

\[- \frac{1}{3}\rho^{-\frac{2}{3}} \left( \rho'^{\frac{2}{3}} - \rho^{\frac{2}{3}} \right) \left( \rho'^{\frac{1}{3}} + \rho^{\frac{1}{3}} \right) \cdot \log \left( \rho'^{\frac{1}{3}} + \rho^{\frac{1}{3}} \right),\]

\[- \frac{\partial V}{\partial \rho'} = 2\rho^{\frac{1}{3}} + \frac{2}{3}\rho'^{-\frac{2}{3}}\rho - \frac{4}{3}\rho'^{-\frac{1}{3}} \left( \rho^{\frac{2}{3}} - \rho'^{\frac{2}{3}} \right) \log \frac{\rho^{\frac{1}{3}} + \rho'^{\frac{1}{3}}}{\rho^{\frac{1}{3}} - \rho'^{\frac{1}{3}}} \]

\[+ \frac{1}{3}\rho'^{-\frac{2}{3}} \left( \rho^{\frac{2}{3}} - \rho'^{\frac{2}{3}} \right) \left( \rho^{\frac{1}{3}} + \rho'^{\frac{1}{3}} \right) \cdot \log \left( \rho^{\frac{1}{3}} + \rho'^{\frac{1}{3}} \right);\]

\[\frac{\partial T}{\partial \rho} = \rho^{\frac{2}{3}},\]

\[\frac{\partial T}{\partial \rho'} = \rho'^{\frac{2}{3}}.\]

\[T = -\frac{1}{2} V:\]

\[\frac{3}{5} \left( \rho^\frac{5}{3} + \rho'^\frac{5}{3} \right) = \rho\rho'^\frac{1}{3} + \rho\rho^{\frac{1}{3}} - \frac{1}{2}(\rho'^{\frac{2}{3}} - \rho^{\frac{2}{3}})^2 \log \frac{\rho'^{\frac{1}{3}} + \rho^{\frac{1}{3}}}{\rho'^{\frac{1}{3}} - \rho^{\frac{1}{3}}}.\]

\[\rho' = k\rho: \]

\[\frac{3}{5}\rho^{\frac{5}{3}} \left( 1 + k^{\frac{5}{3}} \right) = (k + k^{\frac{1}{3}})\rho^{\frac{4}{3}} - \frac{1}{2}\rho^{\frac{4}{3}}(k^{\frac{2}{3}} - 1)^2 \log \frac{k^{\frac{1}{3}} + 1}{k^{\frac{1}{3}} - 1},\]

\[\frac{3}{5}\rho^{\frac{1}{3}} \left( 1 + k^{\frac{5}{3}} \right) = (k + k^{\frac{1}{3}}) - \frac{1}{2}(k^{\frac{2}{3}} - 1)^2 \log \frac{k^{\frac{1}{3}} + 1}{k^{\frac{1}{3}} - 1}.\]

\[^{[3]}\]

\[\rho = \frac{125}{27} \left\{ \frac{k^{\frac{1}{3}} + k - \frac{1}{2}(k^{\frac{2}{3}} - 1)^2 \log \frac{k^{\frac{1}{3}} + 1}{k^{\frac{1}{3}} - 1}}{1 + k^{\frac{5}{3}}} \right\}^3.\]

\(^{[3]}\) In the original manuscript, the power 2 of the factor \((k^{\frac{2}{3}} - 1)\) in the following equation is missing.
7.3.4 Nucleon Interaction I

Explicit expressions for a particular form of the interaction potential between $Z$ protons and $Y$ neutrons are worked out. See also the paper published by Majorana in Z. Phys. 82 (1933) 137, or in La Ricerca Scientifica 4 (1933) 559.

Denote with $q$, $Q$ the center-of-mass coordinates.

\[
\langle q'Q' | V | q''Q'' \rangle = -\delta(q'' - Q') \delta(Q'' - q') \frac{\lambda e^2}{r}.
\]

\[N = Z + Y,\]

\[
\frac{1}{(Z/2)!} \sum \psi_1(q_1) \ldots \psi_{Z/2}(q_{Z/2}) \psi_1(q_{Z/2+1}) \ldots \psi_{Z/2}(q_Z),
\]

\[
\frac{1}{(Y/2)!} \sum \varphi_1(Q_1) \ldots \varphi_{Y/2}(q_{Y/2}) \varphi_1(q_{Y/2+1}) \ldots \varphi_{Y/2}(q_Y).
\]

\[U = -\sum_{i=1}^{Z} \sum_{\ell=1}^{Y} \int \overline{\psi}_i(q') \psi_i(q'') \varphi_\ell(q'') \varphi_\ell(q') \frac{\lambda e^2}{|q' - q''|} dq' dq''
\]

\[+ \sum_{i<k=2}^{Z} \int \overline{\psi}_i(q''') \psi_i(q'') \overline{\psi}_k(q'') \psi_k(q'') \frac{e^2}{|q' - q''|} dq' dq'' + \text{negligible exchange terms}.
\]

\[\rho = \sum_{i=1}^{Z} \psi_i \overline{\psi}_i, \quad \rho' = \sum_{i=1}^{Y} \varphi_i \overline{\varphi}_i.
\]

\[U = -\int \langle q''| \rho | q' \rangle \langle q'| \rho' | q'' \rangle \frac{\lambda e^2}{|q' - q''|} dq' dq''
\]

\[+ \frac{1}{2} \int \langle q'| \rho | q' \rangle \langle q'''| \rho | q'' \rangle \frac{e^2}{|q' - q'''|} dq' dq''.
\]

\[\langle q'| V_P | q'' \rangle = -\frac{\lambda e^2}{|q' - q''|} \langle q'| \rho' | q'' \rangle
\]

\[+ \delta(q' - q'') \int \langle q'''| \rho | q''' \rangle \frac{e^2}{|q' - q'''|} dq''',
\]

\[\langle Q'| V_N | Q'' \rangle = -\frac{\lambda e^2}{|Q' - Q''|} \langle Q'| \rho | Q'' \rangle.
\]

\[\text{Notice that here the author refers to the “ordinary” exchange energy depending on the electrostatic interaction among protons.}\]
\[ A_{q-\frac{v}{2}}q+\frac{v}{2} = \frac{1}{\hbar^3} \int e^{-2\pi i \frac{p \cdot v}{\hbar}} A(p, q) \, dp , \]
\[ A(p, q) = \int e^{2\pi i \frac{p \cdot v}{\hbar}} A_{q-\frac{v}{2},q+\frac{v}{2}} \, dv . \]

In Classical Mechanics:\footnote{\textsuperscript{5}}
\[ \rho = \begin{cases} 2, & p < P, \\ 0, & p > P; \end{cases} \quad \rho' = \begin{cases} 2, & p' < P', \\ 0, & p' > P'. \end{cases} \]

\[ V_P(p, q) = \frac{1}{\hbar^3} \int \frac{e^2}{|q' - q|} \rho(q', p') \, dq' dp' - \frac{1}{\hbar} \int \frac{\lambda e^2}{\pi |p - p'|^3} \rho'(Q, p') \, dp' , \]

\[ V_N = -\frac{1}{\hbar} \int \frac{\lambda e^2}{\pi |p - p'|^3} \rho(q, p) \, dp . \]

\[ P = P(q), \quad P' = P'(Q); \]
\[ \int_{p<P} dp = \frac{4}{3} \pi P^3, \quad \int_{p'<P'} dp' = \frac{4}{3} \pi P'^3 . \]

\[ V_P(p, q) = \int \frac{8\pi}{3} \frac{e^2}{|q' - q|} \frac{P^3}{\hbar^3} dq' \]
\[ - \frac{2\lambda e^2}{\hbar} \left( 2P' + \frac{P'^2 - p^2}{p} \log \frac{P' + p}{P' - p} \right) \quad p < P', \]
\[ V_P(p, q) = \int \frac{8\pi}{3} \frac{e^2}{|q' - q|} \frac{P^3(q')}{\hbar^3} dq' \]
\[ - \frac{2\lambda e^2}{\hbar} \left( 2P'(q) - \frac{p^2 - P'^2}{p} \log \frac{p + P'}{p - P'} \right) \quad p > P'. \]

\footnote{\textsuperscript{5}} In the following, the author deals with a semiclassical approach, which is valid when the number of particles is sufficiently large. The quantities \( V_P \) and \( V_N \) considered below are, then, the classical functions corresponding to the quantum matrix elements discussed before. See E. Majorana, Z. Phys. \textbf{82} (1933) 137 or La Ricerca Scientifica \textbf{4} (1933) 559.

\footnote{\textsuperscript{6}} In the following, the author postulates for simplicity that the one-particle states are either empty or doubly occupied with opposite spins. Moreover, by assuming that at a given position \( q \) (or \( Q \)) the protons (or neutrons) occupy the states with minimum kinetic energy, it follows that a maximum value \( P \) for the proton momentum (and, similarly, \( P' \) for neutrons) does exist. See the papers quoted in the previous footnote.
$V_N(p, q) = -\frac{2\lambda e^2}{h} \left( 2P + \frac{P^2 - p^2}{P} \log \frac{P + p}{P - p} \right), \quad p < P; $

$V_N(p, q) = -\frac{2\lambda e^2}{h} \left( 2P - \frac{p^2 - P^2}{P} \log \frac{p + P}{p - P} \right), \quad p > P. $

$C(q) = \int \frac{8\pi}{3} \frac{P^3(q')}{|q' - q|} \frac{e^2}{h^3} dq'. $

$V_P(P, q) = \begin{cases} 
C - \frac{2\lambda e^2}{h} \left( 2P' + \frac{P'^2 - P^2}{P} \log \frac{P' + P}{P' - P} \right), & P < P'; \\
C - \frac{2\lambda e^2}{h} \left( 2P' - \frac{P'^2 - P^2}{P} \log \frac{P + P'}{P - P'} \right), & P > P'; 
\end{cases} $

$V_N(P', q) = \begin{cases} 
-\frac{2\lambda e^2}{h} \left( 2P - \frac{P'^2 - P^2}{P'} \log \frac{P' + P}{P' - P} \right), & P < P'; \\
-\frac{2\lambda e^2}{h} \left( 2P - \frac{P^2 - P'^2}{P'} \log \frac{P + P'}{P - P'} \right), & P > P'. 
\end{cases} $

$T = \frac{P^2}{2M}. $

---

7\@ The original manuscript presents here an insert dealing with the following Fourier transforms:

$\varphi(\xi) = \int e^{-2\pi i \xi x} f(x) dx, \quad f(x) = \int e^{2\pi i \xi x} \varphi(\xi) d\xi,$

$\varphi'(\xi) = \int e^{-2\pi i \xi x} f'(x) dx, \quad f'(x) = \int e^{2\pi i \xi x} \varphi'(\xi) d\xi,$

where, in particular:

$\varphi'(\xi) = \frac{1}{\xi} \varphi(\xi), \quad f'(x) = \int \frac{1}{\pi(x - x_1)^2} f(x_1) dx_1.$
\[ V_P(P, q) + \frac{P^2}{2M} = -A_P, \]
\[ V_N(P', q) + \frac{P'^2}{2M} = -A_N. \]

By considering a statistical method:

\[ \bar{T}_P(q) = \frac{3P^2}{10M}; \]
\[ \nabla_P(q) = C - \frac{2\lambda e^2}{h} \left[ \frac{3}{2}P' + \frac{3}{2}P'^3 - \frac{3}{4} \frac{(P'^2 - P^2)^2}{P'^3} \log \frac{P' + P}{|P' - P|} \right]; \]
\[ \bar{T}_N(q) = \frac{3P'^2}{10M}, \]
\[ \nabla_N(q) = -\frac{2\lambda e^2}{h} \left[ \frac{3}{2}P + \frac{3}{2}P^3 - \frac{3}{4} \frac{(P'^2 - P^2)^2}{P^3} \log \frac{P' + P}{|P' - P|} \right]; \]

\[ P^3(\nabla_P - C) = P'^3 \nabla_N. \]

Limiting condition:

\[ -\left\{ \frac{P^3}{P^3 + P'^3} \left[ \nabla_P(Q) + \bar{T}_P(Q) \right] + \frac{P'^3}{P^3 + P'^3} \left[ \bar{T}_N(Q) \right] \right\} \]
\[ = \frac{P^3}{P^3 + P'^3} A_P + \frac{P'^3}{P^3 + P'^3} A_N. \]

---

8@ In the following, the author probably denotes with \( A_P \) (or \( A_N \)) the energy associated with the proton (or neutron) exchange interaction.

9@ An application of the theory of nuclear forces introduced above to heavy nuclei, composed of a large number of nucleons, is now apparently investigated, so that statistical methods may apply.
7.3.4.1 Zeroth approximation.

$C = 0; P = \text{constant}, P' = \text{constant},$

$k = P'/P$:

\[ V_P(P, q) = - \frac{2\lambda e^2}{h} P \left[ 2k + (k^2 - 1) \log \frac{k + 1}{|k - 1|} \right], \]

\[ V_N(P', q) = - \frac{2\lambda e^2}{h} P \left[ 2 - \frac{k^2 - 1}{k} \log \frac{k + 1}{|k - 1|} \right]. \]

\[ T_P = \frac{3P^2}{10M}, \]

\[ \bar{V}_N = - \frac{2\lambda e^2}{h} P \left[ \frac{3}{2} k + \frac{3}{2} k^3 - \frac{3}{4} (k^2 - 1)^2 \log \frac{k + 1}{|k - 1|} \right], \]

\[ T_N = \frac{3k^2 P^2}{10M}, \]

\[ \bar{V}_P = - \frac{2\lambda e^2}{h} P \left[ \frac{3}{2} + \frac{3}{2k^3} - \frac{3}{4} \frac{(k^2 - 1)^2}{k^3} \log \frac{k + 1}{|k - 1|} \right]. \]

Particular case: $k = 1$.

\[ V_P(P, q) = V_N(P', q) = -4 \frac{\lambda e^2}{h} P, \quad T = \frac{P^2}{2M}; \]

\[ \bar{V}_N(q) = -6 \frac{\lambda e^2}{h} P = \bar{V}_P(q), \quad T_N = \frac{3P^2}{10M}. \]

\[ -\frac{3\lambda e^2}{h} P + \frac{3P^2}{10M} = -\frac{4\lambda e^2}{h} P + \frac{P^2}{2M}, \]

\[ \frac{\lambda e^2}{h} P = \frac{P^2}{5M}, \]

\[ P = 5M \frac{\lambda e^2}{h}. \]

\[ V_P(P, q) = -20M \frac{\lambda^2 e^4}{h^2}, \quad T_{\text{nuc}} = \frac{25}{2} M \frac{\lambda^2 e^4}{h^2}; \]

\[ \bar{V} = -30M \frac{\lambda^2 e^4}{h^2}, \quad T = \frac{15}{2} M \frac{\lambda^2 e^4}{h^2}. \]
\[ A_P = A_N = \frac{15 \lambda^2 M e^4}{2 \hbar^2}. \]

7.3.5 Nucleon Interaction II

Explicit expressions for another particular form of the interaction potential between \( Z \) protons and \( Y \) neutrons are worked out.

\[ \langle q', Q' | V | q'', Q'' \rangle = -\delta(q'' - Q')\delta(Q'' - q') A e^{-|q' - q''|/\varepsilon}. \]

For protons: \( \rho = \sum_1^Z \psi_i \bar{\psi}_i \); for neutrons: \( \rho' = \sum_1^Y \varphi_i \bar{\varphi}_i \).

\[ \langle q' | V_P | q'' \rangle = -A e^{-|q' - q''|/\varepsilon} \langle q' | \rho' | q'' \rangle + \delta(q' - q'') \int \frac{e^2}{|q - q'|} \langle q | \rho | q \rangle \, dq, \]

\[ \langle q' | V_N | q'' \rangle = -A e^{-|q' - q''|/\varepsilon} \langle q' | \rho | q'' \rangle. \]

In Classical Mechanics\(^{11}\), assuming a degenerate gas of nucleons:

\[ \rho = \begin{cases} 2, & p < P, \\ 0, & p > P; \end{cases} \quad \rho' = \begin{cases} 2, & p < P', \\ 0, & p > P'. \end{cases} \]

\[ A e^{-|q' - q''|/\varepsilon} = A e^{-v/\varepsilon} = A e^{-(h/\varepsilon)(v/h)} = A e^{-k \, v/h}, \quad (k = h/\varepsilon). \]

\[ V_P(p, q) = \frac{1}{h^3} \int \frac{e^2}{|q - q'|} \rho(q', p') \, dq' \, dp' \]

\[ -A \int \frac{8\pi h/\varepsilon}{(h^2/\varepsilon^2 + 4\pi^2|p - p'|^2)^2} \rho'(q, p') \, dp', \]

\[ V_N(p, q) = -A \int \frac{8\pi h/\varepsilon}{(h^2/\varepsilon^2 + 4\pi^2|p - p'|^2)^2} \rho(q, p') \, dp'. \]

\(^{10} \@ \) In the original manuscript there appears also the following note:

\[ \frac{15 M e^4}{2 \hbar^2} = 9500 \text{ V} \]

(\( V \) stands for eV), where the nucleon mass value \( M \approx 938 \text{ MeV} \) had been used.

\(^{11} \@ \) See footnote 6.
Let us set:

\[
P_0 = \frac{\hbar}{2\pi \varepsilon}, \quad \frac{\hbar}{\varepsilon} = 2\pi P_0.
\]

\[\text{[12]}\]

\[
V_P(p, q) = \frac{1}{\hbar^3} \int \frac{e^2}{|q - q'|} \rho(q', p') \, dq' dp' - \frac{A}{\pi^2} \int \frac{P_0}{(P_0^2 + |p - p'|^2)^2} \rho'(q, p') \, dp',
\]

\[
V_N(p, q) = -\frac{A}{\pi^2} \int \frac{P_0}{(P_0^2 + |p - p'|^2)^2} \rho'(q, p') \, dp'.
\]

\[
P = P(q), \quad P' = P'(q), \quad \int_{p<P} dp = 4\pi P^3/3.
\]

For a degenerate gas of nucleons:

\[
V_P(p, q) = \frac{8\pi}{3} \frac{1}{\hbar^3} \int \frac{e^2}{|q - q'|} P^3(q') \, dq' - \frac{2A}{\pi^2} \int_{p<P} \frac{P_0}{(P_0^2 + |p - p'|^2)^2} \, dp',
\]

\[
V_N(p, q) = -\frac{2A}{\pi^2} \int_{p'<P} \frac{P_0}{(P_0^2 + |p - p'|^2)^2} \, dp'.
\]

\[
V_P(p, q) = \int \frac{8\pi}{3} \frac{e^2}{|q' - q|} \frac{P^3(q')}{\hbar^3} \, dq' - \frac{2A}{\pi} \left\{ \arctan \frac{P' + p}{P_0} \\
+ \arctan \frac{P' - p}{P_0} - \frac{1}{2} \frac{P_0}{p} \log \frac{P_0^2 + (P' + p)^2}{P_0^2 + (P' - p)^2} \right\},
\]

\[
V_N(p, q) = -\frac{2A}{\pi} \left\{ \arctan \frac{P + p}{P_0} + \arctan \frac{P - p}{P_0} \\
- \frac{1}{2} \frac{P_0}{p} \log \frac{P_0^2 + (P + p)^2}{P_0^2 + (P - p)^2} \right\}.
\]

\[\text{[12]}\] In the original manuscript, the unidentified Ref. 5.25 appears here.
\[ V_P(P, q) + \frac{P^2}{2M} = -A_P, \]
\[ V_N(P', q) + \frac{P^2}{2M} = -A_N. \]

\[ V_P(P, q) = \int \frac{8\pi e^2}{3} \frac{P^3(q')}{h^3} \frac{1}{|q' - q|} \text{d}q' - \frac{2A}{\pi} \left\{ \arctan \frac{P'}{P} + \frac{P'}{P_0} + \arctan \frac{P - P'}{P_0} - \frac{1}{2} \frac{P_0}{P} \log \frac{P_0^2 + (P + P')^2}{P_0^2 + (P - P')^2} \right\}, \]
\[ V_N(P', q) = -\frac{2A}{\pi} \left\{ \arctan \frac{P + P'}{P_0} + \arctan \frac{P - P'}{P_0} - \frac{1}{2} \frac{P_0}{P'} \log \frac{P_0^2 + (P + P')^2}{P_0^2 + (P - P')^2} \right\}. \]

Limiting conditions:

\[ - \left[ P^3 V_P(Q) + P^3 T_P(Q) + P'^3 T_N(Q) \right] = P^3 A_P + P'^3 A_N; \]

\[ P^3 V_P = P'^3 V_N + P^3 C, \]

\[ P^3 (V_P - C) = P'^3 V_N. \]

\[ C = C = \int \frac{8\pi e^2}{3} \frac{P^3(q')}{h^3} \frac{1}{|q - q'|} \text{d}q'. \]

\(^{13@}\) In the original manuscript, the unidentified Ref. 11.59 appears here.
\[
P^3 \nabla_N = \frac{-2A}{\pi} \left\{ P_0 P P' + (P^3 + P'^3) \arctan \frac{P + P'}{P_0} \right. \\
- (P^3 - P'^3) \arctan \frac{P' - P}{P_0} \\
- P_0 \frac{3(P^2 + P'^2)}{4} + P_0^2 \log \frac{P_0^2 + (P + P')^2}{P_0^2 + (P' - P)^2} \right\} \\
= P^3 (V_p - C).
\]

### 7.3.5.1 Evaluation of some integrals.

For \( p < P' \):

\[ \int_{p' < P'} \frac{P_0}{(P_0^2 + |p - p'|^2)^2} \, dp' = \]
\[ = \frac{2\pi P_0}{P} \left[ \int_0^p s \, ds \int_{p-s}^{p+s} \frac{t \, dt}{(P_0^2 + t^2)^2} + \int_p^{P'} s \, ds \int_{s-p}^{s+p} \frac{t \, dt}{(P_0^2 + t^2)^2} \right] \]
\[ = \frac{2\pi P_0}{P} \left[ \int_0^p s \, ds \cdot \frac{1}{2} \left\{ \frac{1}{P_0^2 + (p-s)^2} - \frac{1}{P_0^2 + (p+s)^2} \right\} \right. \]
\[ + \left. \int_p^{P'} s \, ds \cdot \frac{1}{2} \left\{ \frac{1}{P_0^2 + (s-p)^2} - \frac{1}{P_0^2 + (s+p)^2} \right\} \right] \]
\[ = \frac{2\pi P_0}{P} \int_0^{P'} s \, ds \cdot \frac{1}{2} \left\{ \frac{1}{P_0^2 + (p-s)^2} - \frac{1}{P_0^2 + (p+s)^2} \right\} \]

(the last expression holds also for \( p > P' \)).

\[ \int \frac{s \, ds}{P_0^2 + (p-s)^2} = \int \frac{(p-s) \, d(p-s)}{P_0^2 + (p-s)^2} + \int \frac{p \, ds}{P_0^2 + (p-s)^2} \]
\[ = \frac{1}{2} \log \left[ P_0^2 + (p-s)^2 \right] + \frac{p}{P_0} \arctan \frac{s-p}{P_0}, \]

\[^{14}\text{In the original manuscript, the unidentified Ref. 2.50 appears here.}\]
\[ \int \frac{s \, ds}{P_0^2 + (p + s)^2} = \frac{1}{2} \log \left[ \frac{P_0^2 + (p + s)^2}{P_0^2 + p^2} \right] - \frac{p}{P_0} \arctan \frac{p + s}{P_0}; \]

\[ \int_0^{P'} \frac{s \, ds}{P_0^2 + (p - s)^2} = \frac{1}{2} \log \frac{P_0^2 + (P' - p)^2}{P_0^2 + p^2} + \frac{p}{P_0} \left( \arctan \frac{P' - p}{P_0} + \arctan \frac{p}{P_0} \right); \]

\[ \int_0^{P'} \frac{s \, ds}{P_0^2 + (p + s)^2} = \frac{1}{2} \log \frac{P_0^2 + (P' + p)^2}{P_0^2 + p^2} - \frac{p}{P_0} \left( \arctan \frac{P' - p}{P_0} - \arctan \frac{p}{P_0} \right). \]

\[ \int^{P'}_{P' < P_0} \frac{P_0}{(P_0^2 + (p - p')^2)^2} \, dp' = \pi \left\{ -\frac{1}{2} \frac{P_0}{p} \log \frac{P_0^2 + (P' + p)^2}{P_0^2 + (P' - p)^2} + \arctan \frac{P' + p}{P_0} + \arctan \frac{P' - p}{P_0} \right\}. \]

\[ \int_{p < P} dp \int_{p' < p} \frac{P_0}{(P_0^2 + |p - p'|^2)^2} \, dp' = 4\pi^3 \left\{ \int_0^P p^2 \arctan \frac{P' + p}{P_0} \, dp - \frac{1}{2} P_0 \int_0^P p \log \frac{P_0^2 + (P' + p)^2}{P_0^2 + (P' - p)^2} \, dp \right\}. \]

\[ \int p^2 \arctan \frac{P' + p}{P_0} \, dp = \frac{1}{3} p^3 \arctan \frac{P' + p}{P_0} - \frac{1}{3} P_0 \int \frac{p^3}{P_0^2 + (P' + p)^2} \, dp \]

\[ = \frac{1}{3} p^3 \arctan \frac{P' + p}{P_0} - \frac{1}{3} P_0 \int \frac{(p + P')^3}{P_0^2 + (p + P')^2} \, dp \]
\[
= \frac{1}{3} p^3 \arctan \frac{p + P'}{P_0} - \frac{1}{3} P_0 \left[ \int (p + P') \, dp - \int 3P' \, dp \right] + \int \frac{(3P'^2 - P_0^2)(p + P')}{P_0^2 + (p + P')^2} \, dp - \int \frac{P'P'^2 - 3P_0^2}{P_0^2 + (p + P')^2} \, dp
\]

\[
= \frac{1}{3} p^3 \arctan \frac{p + P'}{P_0} - \frac{1}{3} P_0 \left[ \frac{1}{2} p^2 - 2P'p + \frac{3P'^2 - P_0^2}{2} \log \left\{ P_0^2 + (p + P')^2 \right\} - \frac{P'(P'^2 - 3P_0^2)}{P_0} \arctan \frac{p + P'}{P_0} \right].
\]

\[
\int p^2 \arctan \frac{P' - p}{P_0} = \frac{1}{3} p^3 \arctan \frac{P' - p}{P_0} + \frac{1}{3} P_0 \left[ \frac{1}{2} p^2 + 2P'p + \frac{3P'^2 - P_0^2}{2} \log \left\{ P_0^2 + (p - P')^2 \right\} - \frac{P'(P'^3 - 3P_0^2)^2}{P_0} \arctan \frac{P' - p}{P_0} \right].
\]

\[
\int p \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2} \, dp
\]

\[
= \frac{1}{2} p^2 \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2} - \int \frac{p^2(p + P')}{P_0^2 + (p + P')^2} \, dp
\]

\[
+ \int \frac{p^2(p - P')}{P_0^2 + (p - P')^2} \, dp
\]

\[
= \frac{1}{2} p^2 \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2}
\]

\[
- \int \frac{(p + P')^3 - 2P'(p + P')^2 + P'^2(p + P')}{P_0^2 + (p + P')^2} \, dp
\]

\[
+ \int \frac{(p - P')^3 + 2P'(p - P')^2 + P'^2(p - P')}{P_0^2 + (p - P')^2} \, dp
\]

\[
= \frac{1}{2} p^2 \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2} - \int (p + P') \, dp + \int 2P' \, dp
\]

\[
- \int \frac{(P'^2 - P_0^2)(p + P')}{P_0^2 + (p + P')^2} \, dp + \int \frac{2P'P_0^2}{P_1^2 + (p + P')^2} \, dp
\]

\[
+ \int (p - P') \, dp + \int 2P' \, dp + \int \frac{(P'^2 - P_0^2)(p - P')}{P_0^2 + (p - P')^2} \, dp
\]
\[ - \int \frac{2P'P_0^2}{P_0^2 + (p - P')^2} \, dp \]

\[ = \frac{1}{2} p^2 \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2} + 2P'p - \frac{P'^2 - P_0^2}{2} \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2} \]

\[ - 2P'P_0 \arctan \frac{P' + p}{P_0} + 2P'P_0 \arctan \frac{P' - p}{P_0}. \]

\[ \int p^2 \arctan \frac{P' + p}{P_0} \, dp + \int p^2 \arctan \frac{P' - p}{P_0} \, dp \]

\[ - \frac{1}{2} P_0 \int p \log \frac{P_0^2 + (p + P')^2}{P_0^2 + (p - P')^2} \, dp \]

\[ = \frac{1}{3} P_0 P' p + \frac{1}{3} (p^3 + P'^3) \arctan \frac{P' + p}{P_0} + \frac{1}{3} (p^3 - P'^3) \arctan \frac{P' - p}{P_0} \]

\[ - \frac{3P_0P'^2 + 3P_0P'^2 + P_0^3}{12} \log \frac{P_0^2 + (p + P)^2}{P_0^2 + (p - P)^2}. \]

\[ \int_{p<P} \int_{p'<p'} \frac{P_0 \, dp \, dp'}{(P^2 + \left| p - p' \right|^2)^2} \]

\[ = \frac{4}{3} \pi^2 \left\{ P_0 PP' + (P^3 + P'^3) \arctan \frac{P + P'}{P_0} \right\} \]

\[ - (P'^3 - P^3) \arctan \frac{P' - P}{P_0} \]

\[ - \frac{3P_0P'^2 + P_0^3 + 3P_0P'^2}{4} \log \frac{P_0^2 + (P + P')^2}{P_0^2 + (P' - P)^2} \right\}. \]

**7.3.5.2 Zeroth approximation.**

\( C = 0; \ P^3V_P = P'^3V_N; \)

\( P = \text{constant}, \ P' = \text{constant}. \)

\( k = P'/P, \ t = P_o/P: \)
\[ T_P(P, q) = \frac{P^2}{2M}, \]
\[ T_N(P', q) = \frac{P'^2}{2M} = \frac{k^2 P^2}{2M}. \]

\[ V_P(P, q) = -\frac{2A}{\pi} \left\{ \arctan \frac{1 + k}{t} + \arctan \frac{k - 1}{t} \right. \]
\[ -\frac{t}{2} \log \left( \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} \right), \]
\[ V_N(P', q) = -\frac{2A}{\pi} \left\{ \arctan \frac{1 + k}{t} - \arctan \frac{k - 1}{t} \right. \]
\[ -\frac{t}{2k} \log \left( \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} \right). \]

\[ P^3 V_P = -\frac{2A}{\pi} P^3 \left\{ kt + (1 + k^3) \arctan \frac{1 + k}{t} \right. \]
\[ -(k^3 - 1) \arctan \frac{k - 1}{t} \]
\[ -t \frac{3(1 + k^2) + t^2}{4} \log \left( \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} \right). \]

Particular case: \( k = 1, P = P' \).

\[ V_P(P, q) = V_N(P', q) = -\frac{2A}{\pi} \left\{ \arctan \frac{2}{t} - \frac{t}{2} \log \frac{4 + t^2}{t^2} \right\}; \]

\[ \overline{V}_P = -\frac{2A}{\pi} \left\{ t + 2 \arctan \frac{2}{t} - t \frac{6 + t^2}{4} \log \frac{4 + t^2}{t^2} \right\}. \]

\[ + \frac{2A}{\pi} \left\{ t + 2 \arctan \frac{2}{t} - t \frac{6 + t^2}{4} \log \frac{4 + t^2}{t^2} \right\} - \frac{3P^2}{5M} \]
\[ = \frac{2A}{\pi} \left\{ \arctan \frac{2}{t} - \frac{t}{2} \log \frac{4 + t^2}{t^2} \right\} - \frac{P^2}{M}. \]
\[
\frac{2P^2}{5M} = \frac{2A}{\pi} \left\{ t \frac{2 + t^2}{4} \log \frac{4 + t^2}{t^2} - t \right\}, \\
\frac{P^2}{2M} = \frac{5A}{2\pi} \left\{ t \frac{2 + t^2}{4} \log \frac{4 + t^2}{t^2} - t \right\}.
\]

\[
P^2 = Ay; \quad \frac{P_0^2}{2M} = Ax; \quad \frac{P_0}{P} = t = \sqrt{\frac{x}{y}};
\]

\[
y = \frac{5}{2\pi} \left\{ \sqrt{\frac{x}{y}} \left[ \frac{x + 2y}{4y} \log \frac{x + 4y}{x} - 1 \right] \right\}.
\]

\[
T = \frac{3}{5} T(P); \quad P_0 = \frac{h}{2\pi \varepsilon}.
\]

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<td>0.217</td>
</tr>
<tr>
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<td>0.600</td>
<td>0.159</td>
<td>0.159</td>
</tr>
<tr>
<td>0.6</td>
<td>0.134</td>
<td>0.498</td>
<td>0.115</td>
<td>0.115</td>
</tr>
<tr>
<td>0.7</td>
<td>0.127</td>
<td>0.417</td>
<td>0.081</td>
<td>0.0815</td>
</tr>
<tr>
<td>0.8</td>
<td>0.118</td>
<td>0.349</td>
<td>0.056</td>
<td>0.0565</td>
</tr>
</tbody>
</table>

\(^{16}\) In the original manuscript, the unidentified Ref. 1.03 appears here.

\(^{17}\) The numerical values in the following tables have been obtained by using the appropriate equations above, with a given value of the parameter \(t\). In particular, \(x\) has been calculated from \(x = t^2 y\). Note that, sometimes, the last digit in the numerical values appearing in the table is slightly erroneous.
For \( t = 0.6 \):

\[
A = 80 \cdot 10^6 \text{V}; \\
T(P_0) = 6.5 \cdot 10^6 \text{V}; \\
2\pi\varepsilon = 11 \cdot 10^{-13}, \quad \varepsilon = 1.75 \cdot 10^{-13}; \\
-V(P) = 27 \cdot 10^6 \text{V}; \\
T(P) = 18 \cdot 10^6 \text{V}; \\
-V(P) - T(P) = 9 \cdot 10^6 \text{V}; \\
-V = 40 \cdot 10^6 \text{V}; \\
\overline{T} = 11 \cdot 10^6 \text{V}; \\
-\overline{V}/2 - \overline{T} = 9 \cdot 10^6 \text{V}.
\]

\[
V(0, q) = \frac{2A}{\pi} \left( 2\arctan \frac{1}{t} - \frac{2t}{1 + t^2} \right).
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>(-V(0, q)/A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1.280</td>
</tr>
<tr>
<td>0.4</td>
<td>1.076</td>
</tr>
<tr>
<td>0.5</td>
<td>0.900</td>
</tr>
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<td>0.6</td>
<td>0.750</td>
</tr>
<tr>
<td>0.7</td>
<td>0.624</td>
</tr>
<tr>
<td>0.8</td>
<td>0.519</td>
</tr>
</tbody>
</table>

General case: \( k > 1, k = P'/P, t = P_0/P \).

\[
\overline{V}_P = k^3\overline{V}_N = -\frac{2A}{\pi} \left\{kt + (1 + k^3)\arctan \frac{1+k}{t} - (k^3 - 1)\arctan \frac{k-1}{t} - t \frac{3(1 + k^2) + t^2}{4} \log \frac{(k + 1)^2 + t^2}{(k-1)^2 + t^2} \right\}.
\]

\[
A_P = -V_P(P) - T_P(P) = \frac{2A}{\pi} \left\{ \arctan \frac{1+k}{t} + \arctan \frac{k-1}{t} - \frac{t}{2} \log \frac{(k+1)^2 + t^2}{(k-1)^2 + t^2} \right\} - \frac{P^2}{2M},
\]
\[ A_N = -V_N(P') - T_N(P') \]
\[ = \frac{2A}{\pi} \left\{ \arctan \frac{1 + k}{t} - \arctan \frac{k - 1}{t} - \frac{t}{2k} \log \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} \right\} - \frac{k^2 P^2}{2M}. \]

\[ T_P = \frac{3P^2}{10M}; \quad T_N = \frac{3k^2 P^2}{10M}. \]

\[ -V_P - T_P - k^3 T_N = A_P + k^3 A_N. \]

\[ \frac{P^2}{2M} + \frac{k^5 P^2}{2M} - \frac{3P^2}{10M} - \frac{3k^5 P^2}{10M} \]
\[ = \frac{2A}{\pi} \left\{ \frac{1 + k^2 + t^2}{4} \log \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} - kt \right\} = \frac{1 + k^5}{5M} P^2, \]

\[ \frac{P^2}{2M} = \frac{5}{1 + k^5 \pi} t \left\{ \frac{1 + k^2 + t^2}{4} \log \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} - k \right\}. \]

\[ y = \frac{1}{A} \frac{P^2}{2M}; \quad x = \frac{1}{A} \frac{P_0^2}{2M}; \]

\[ t = \frac{P_0}{P} = \sqrt{\frac{x}{y}} = \sqrt{\frac{T(P_0)}{T(P)}}, \quad T(P_0) = t^2 T(P). \]

\[ y = \frac{5}{1 + k^5 \pi} t \left\{ \frac{1 + k^2 + t^2}{4} \log \frac{(k + 1)^2 + t^2}{(k - 1)^2 + t^2} - k \right\}. \]

\[ y = T(P)/A: \]

<table>
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<tr>
<th></th>
<th>( k = 1 )</th>
<th>( k = 21/19 )</th>
<th>( k = 22/18 )</th>
<th>( k = 23/17 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0.5 )</td>
<td>0.235</td>
<td>0.204</td>
<td>0.157</td>
<td>0.109</td>
</tr>
<tr>
<td>0.6</td>
<td>0.225</td>
<td>0.196</td>
<td>0.154</td>
<td>0.111</td>
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<td>0.211</td>
<td>0.187</td>
<td>0.149</td>
<td>0.109</td>
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<td>0.8</td>
<td>0.195</td>
<td>0.174</td>
<td>0.142</td>
<td>0.106</td>
</tr>
<tr>
<td>0.9</td>
<td>0.179</td>
<td>0.162</td>
<td>0.133</td>
<td>0.101</td>
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<tr>
<td>1.0</td>
<td>0.165</td>
<td>0.194</td>
<td>0.124</td>
<td>0.096</td>
</tr>
</tbody>
</table>
\[ k^2 y = T(P')/A: \]

<table>
<thead>
<tr>
<th></th>
<th>(k = 1)</th>
<th>(k = 21/19)</th>
<th>(k = 22/18)</th>
<th>(k = 23/17)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t = 0.5)</td>
<td>0.236</td>
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<td>0.234</td>
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<tr>
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<td>0.225</td>
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<td>0.228</td>
<td>0.223</td>
<td>0.200</td>
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<tr>
<td>0.8</td>
<td>0.195</td>
<td>0.213</td>
<td>0.212</td>
<td>0.194</td>
</tr>
<tr>
<td>0.9</td>
<td>0.179</td>
<td>0.198</td>
<td>0.199</td>
<td>0.185</td>
</tr>
<tr>
<td>1.0</td>
<td>0.165</td>
<td>0.182</td>
<td>0.186</td>
<td>0.175</td>
</tr>
</tbody>
</table>

### 7.3.6 Simple Nuclei I

In the following pages the author considered the nucleon interaction discussed in Sect. 7.3.4.

\[
b_0 = \frac{\hbar^2}{4\pi^2 Me^2} = 2.9 \cdot 10^{-12} = a_0 \frac{m}{M},
\]

\[
S = \frac{2\pi^2 Me^4}{\hbar^2} = \frac{M}{m} \cdot 1 \text{ Rh} = 25000 \text{ V},
\]

\[
e^2/b_0 = 50000 \text{ V}.
\]

For deuterium \(^2\text{H}^\):

\[ q = q_1 - q_2, \quad \psi_0 = e^{-\lambda x/2b_0}, \quad E_0 = -\frac{\lambda^2}{2} S. \]

For \(Z + Y = N > 2\):

\[
\psi \sim \psi_1(q_1)\psi_2(q_2) \cdots \psi_n(q_n),
\]

with

\[
q_1 + q_2 + \ldots + q_n = 0.
\]

\[
Q = \frac{1}{n}(q_1 + q_2 + \ldots + q_n).
\]

\[
\psi = \psi(q_1 - Q, q_2 - Q, q_3 - Q, \ldots, q_n - Q),
\]

\[
q' = q_1 - Q, \quad q'_2 = q_2 - Q, \quad \ldots \quad q'_n = q_n - Q.
\]

\[
q'_1 + q'_2 + \ldots + q'_n = 0;
\]
\[ \psi = \psi(q'_1, q'_2, \ldots, q'_n). \]

\[ p'_i = \frac{h}{2\pi i} \frac{\partial}{\partial q'_i}; \quad p'_i = \frac{h}{2\pi i} \frac{\partial}{\partial q'_i}. \]

\[ p_1 = p'_1 - \frac{1}{n}(p'_1 + p'_2 + \ldots + p'_n), \]
\[ \ldots \]
\[ p_i = p'_i - \frac{1}{n}(p'_1 + p'_2 + \ldots + p'_n); \]
\[ p_i = p'_i - \frac{1}{n} \sum p_i. \]

\[ \sum p_i^2 = \sum p_i^2 - \frac{2}{n} \left( \sum p'_i \right)^2 + \frac{1}{n} \left( \sum p'_i \right)^2 = \sum p_i^2 - \frac{1}{n} \left( \sum p'_i \right)^2. \]

\[ T = \frac{1}{2M} \left\{ \sum p_i^2 - \frac{1}{n} \left( \sum p_i \right) \right\}. \]

For an α particle:

\[ \psi \sim e^{-s(r_1 + r_2 + r_3 + r_4)/b_0}, \]

\[ r_1 = |q_1|, \quad r_2 = |q_2|, \quad r_3 = |q_3|, \quad r_4 = |q_4|. \]

\[ \sum p_i^2 \psi = -\frac{\hbar^2}{4\pi^2} \left( \frac{4 s^2}{b_0^2} - \frac{2}{r_1 b_0} - \frac{2}{r_2 b_0} - \frac{2}{r_3 b_0} - \frac{2}{r_4 b_0} \right) \psi; \]
\[ p_i = (x^1_i, x^2_i, x^3_i), \]

\[ \sum p'_i \psi = -\frac{s}{b_0} \frac{4}{2\pi i} \left( \frac{x^k_1}{r_1} + \frac{x^k_2}{r_2} + \frac{x^k_3}{r_3} + \frac{x^k_4}{r_4} \right) \psi; \]
\[ \left( \sum p'_i \right)^2 \psi = \left\{ -\frac{\hbar^2}{4\pi^2} \frac{s^2}{b_0^2} \left( 4 + 2 \sum_{i<k} \frac{q_i \cdot q_k}{r_i r_k} \right) \right. \]
\[ + \left. \frac{s}{b_0} \frac{\hbar^2}{4\pi^2} \left( \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3} + \frac{2}{r_4} \right) \right\} \psi. \]
Since \( n = 4 \): \(^{18}\)

\[
H\psi = \left\{ -\frac{\hbar^2}{8\pi^2 M} \left[ \left( 3 - \frac{1}{2} \sum_{i<k} \frac{q_i \cdot q_k}{r_i r_k} \right) \frac{s^2}{i_0^2} - \frac{3}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) s b_0 \right] \right. \\
\left. -e^2 \left( \frac{\lambda}{r_{13}} + \frac{\lambda}{r_{14}} + \frac{\lambda}{r_{23}} + \frac{\lambda}{r_{24}} + \frac{1}{r_{12}} \right) \right\},
\]

where the indices 1,2 refer to the protons and 3,4 to the neutrons.

\[
E \sim -4s^2 S \sim -s^2 \cdot 100000 \text{ V.}
\]

Rough estimate:

\[
\frac{6}{b_0} \frac{\hbar^2}{8\pi^2 M} s \sim e^2 \left( \frac{5}{2} \lambda - \frac{5}{8} \right) \sim e^2 \frac{5}{2} \lambda;
\]

\[
s \sim \frac{5}{12} \lambda e^2 b_0 \frac{8\pi^2 M}{\hbar^2} \sim \frac{5}{6} \lambda;
\]

\[
E \sim \frac{25}{9} \lambda^2 S \sim -\lambda^2 \cdot 70000 \text{ V.}
\]

\(^{19}\)

### 7.3.7 Simple Nuclei II

In the following notes the author considered the nucleon interaction discussed in Sect. 7.3.5.

For deuterium \(^2\text{H}\) \((M = 1.65 \cdot 10^{-24},\ M' = M/2,\ \hbar^2/8\pi^2 M' = \hbar^2/4\pi^2 M)\): \(^{18}\)

\[
H\chi = E\chi, \quad \chi = \psi r.
\]

\(^{18}\) The following Hamiltonian was obtained by using the general expression for the kinetic energy \( T \) just reported above, specialized to the present case with 4 nucleons.

\(^{19}\) In the original manuscript there is also the following note:

\[
\frac{\hbar^2}{8\pi^2 M} \frac{1}{b_0^2} = \frac{2\pi^2 Me^4}{\hbar^2}.
\]
\[
H = -\left(\frac{\hbar^2}{4\pi^2 M} \frac{\partial}{\partial r^2} + A e^{-r/\varepsilon}\right).
\]
\[
\chi \sim r \left(1 + \frac{r}{\xi}\right) e^{-r/\eta}.
\]
\[
\int \chi^2 dr = \int r^2 e^{-2r/\eta} dr + \frac{2}{\xi} \int r^3 e^{-2r/\eta} dr + \frac{1}{\xi^2} \int r^4 e^{-2r/\eta} dr
= \frac{\eta^3}{4} + \frac{3\eta^4}{4\xi} + \frac{3\eta^5}{4\xi^2} = \frac{\eta^3}{4} \left(1 + 3\frac{\eta}{\xi} + \frac{\eta^2}{\xi^2}\right).
\]
\[
\frac{\partial \chi}{\partial r} = \left[1 + \left(\frac{2}{\xi} - \frac{1}{\eta}\right) r - \frac{1}{\xi \eta} r^2\right] e^{-r/\eta},
\]
\[
\frac{\partial^2 \chi}{\partial r^2} = \left[\left(\frac{2}{\xi} - \frac{2}{\eta}\right) - \left(\frac{4}{\xi \eta} - \frac{1}{\eta^2}\right) r + \frac{1}{\xi \eta^2} r^2\right] e^{-r/\eta}.
\]
\[
-H \chi = \frac{\hbar^2}{4\pi^2 M} \left[\left(\frac{2}{\xi} - \frac{2}{\eta}\right) - \left(\frac{4}{\xi \eta} - \frac{1}{\eta^2}\right) r + \frac{1}{\xi \eta^2} r^2\right] e^{-r/\eta}
+ A e^{-\left(1/\varepsilon + 1/\eta\right) r} \cdot r \left(1 + \frac{r}{\xi}\right).
\]
\[
-\chi H \chi = \frac{\hbar^2}{4\pi^2 M} \left[\left(\frac{2}{\xi} - \frac{2}{\eta}\right) r + \left(\frac{2}{\xi^2} - \frac{6}{\xi \eta} + \frac{1}{\eta^2}\right) r^2 \right.
+ \left(-\frac{4}{\xi^2 \eta} + \frac{2}{\xi \eta^2}\right) r^3 + \frac{1}{\xi^2 \eta^2} r^4\right] e^{-2r/\eta}
+ \left(r^2 + \frac{2}{\xi} r^3 + \frac{1}{\xi^2} r^4\right) A e^{-\left(1/\xi + 2/\eta\right) r}.
\]
\[
-\int \chi H \chi dr = \frac{\hbar^2}{4\pi^2 M} \left\{\frac{\eta^2}{2\xi} - \frac{\eta}{2} + \frac{\eta^3}{2\xi^2} - \frac{3\eta^2}{2\xi} + \frac{\eta}{4} - \frac{3\eta^3}{2\xi^2} + \frac{3\eta^2}{4\xi} + \frac{3\eta^3}{4\xi^2}\right\}
+ A \left\{\left(\frac{1}{\varepsilon} + \frac{2}{\eta}\right)^2 + \frac{12}{\xi \left(\frac{1}{\varepsilon} + \frac{2}{\eta}\right)^4} + \frac{24}{\xi^2 \left(\frac{1}{\varepsilon} + \frac{2}{\eta}\right)^5}\right\}.
\]
\[ B = \frac{h^2}{8\pi^2 M \varepsilon^2} = \frac{1}{2M} \left( \frac{h}{2\pi \varepsilon} \right)^2 = \frac{P_0^2}{2M} = T(P_0); \quad \frac{h^2}{4\pi^2 M} = 2B\varepsilon^2. \]

\[ k = \frac{\eta}{\xi}, \quad t = \frac{\eta}{\varepsilon}, \quad \eta = t \varepsilon, \quad \xi = \frac{1}{k} \eta = \frac{t}{k} \varepsilon. \]

\[ \int \chi^2 \, dr = \varepsilon^3 \frac{t^3}{4} (1 + 3k + 3k^2), \]

\[ -\int \chi H \chi \, dr = -B\varepsilon^3 \left\{ \frac{t}{2} + \frac{kt}{2} + \frac{k^2 t}{2} \right\} + A\varepsilon^3 \left\{ \frac{2t^3}{(2 + t)^3} + \frac{12kt^3}{(2 + t)^4} + \frac{24k^2 t^3}{(2 + t)^5} \right\}. \]

\[ -\overline{H} = A \frac{1}{(1 + \frac{t}{2})^3} + \frac{3k}{(1 + \frac{t}{2})^4} + \frac{3k^2}{(1 + \frac{t}{2})^5} \frac{1 + 3k + 3k^2}{1 + 3k + 3k^2} - B \cdot \frac{2}{t^2} \cdot \frac{1 + k + k^2}{1 + 3k + 3k^2}. \]

| \( k = 1 \) | \( t = 0.6 \) | \( 0.3303A - 2.381B \) |
| \( t = 0.7 \) | \( 0.2826A - 1.749B \) |
| \( t = 0.8 \) | \( 0.2432A - 1.339B \) |

7.3.7.1 Kinematics of two \( \alpha \) particles (statistics).

\( M_p \cong M_N \)

For one \( \alpha \) particle:

\[ \psi(q_1, q_2; Q_1, Q_2) = \psi(B) \varphi(q'_1, q'_2; Q'_1, Q'_2), \]

\[ q'_1 = q_1 - B, \quad q'_2 = q_2 - B, \quad Q'_2 = Q_1 - B, \quad Q'_2 = Q_2 = B; \]

\[ q'_1 + q'_2 + Q'_1 + Q'_2 = 0, \quad B = \frac{1}{4}(q_1 + q_2 + Q_1 + Q_2). \]

For two \( \alpha \) particles, without considering statistical effects (\( \psi \neq \psi_1 \)):

\[ \psi(q_1, q_2; Q_1, Q_2) \psi_1(q_3, q_4; Q_3, Q_4); \]

\[^{20\ast} \text{From the original manuscript it is evident that the author intended to obtain a similar table for the value } k = 0.8; \text{ however, no numerical value for } \overline{H} \text{ was reported.}\]
including statistical effects:

$$\psi = \frac{1}{6} \sum \pm \psi(q_{i_1}, q_{i_2}; Q_{k_1} Q_{k_2}) \psi_1(q_{i_3} q_{i_4}; Q_{k_3} Q_{k_4}),$$

with $i_1 < i_2$, $i_3 < i_4$, $k_1 < k_2$, $k_3 < k_4$.

7.4. THOMSON FORMULA FOR $\beta$ PARTICLES IN A MEDIUM

Majorana considered here the problem of the energy loss of $\beta$ particles in passing through a medium, as discussed in the articles by E.J. Williams, Proc. Roy. Soc. A130 (1930) 310, 328. By using the classical theorem of momentum $\int F dt = \int dp$, he first obtained an expression for the velocity $v'$ of $\beta$ particles and then, from their kinetic energy $T'$, the energy $Q$ acquired by atomic electrons during the collision. Here, quantity $a$ is the impact parameter and $\tau$ the Bohr’s time of collision. The classical number of collisions in which a certain $\beta$ particle looses energy between $Q$ and $Q + dQ$ in traversing the medium (assumed to be a gas of free electrons, initially at rest) is denoted by $\psi(Q) dQ$, while $J$ is the ionization potential.

---

21@ In the original manuscript, three handwritten lines appear in the table below, connecting the 1st with the 6th row, the 2nd with the 5th row, the 3rd with the 4th row, respectively, pointing out the possible proton+neutron states in the two $\alpha$ particles.

22@ In his notes the author quoted a paper by Williams and Terroux as present in the same issue of the above cited journal. However, no such a paper was published in that issue. Probably he referred to the important article of E.J. Williams and F.R. Terroux, Proc. Roy. Soc. A126 (1930) 289 which reported on some experimental observations.
\[ F = \frac{e^2}{r^2}, \quad F_n = \frac{e^2a}{r^3}, \]
\[ r = \sqrt{a^2 + x^2}. \]

\[
\int F_n dt = \int \frac{e^2a}{r^3} dt = \int \frac{e^2a}{(a^2 + x^2)^{3/2}} dt = \int \frac{e^2a}{(a^2 + x^2)^{1/2}} \frac{dx}{v}.
\]

\[
x = a \tan \varphi, \quad a^2 + x^2 = \frac{a^2}{\cos^2 \varphi}, \quad dx = \frac{a d \varphi}{\cos^2 \varphi}.
\]

\[
\int F_n dt = \int \frac{e^2a \cos^2 \varphi}{a^3} \frac{a d \varphi}{v \cos^2 \varphi} = \int_{-\pi/2}^{\pi/2} \frac{e^2 \cos \varphi}{av} d \varphi
\]
\[
= \frac{2e^2}{av} = \frac{2}{\tau} \frac{e^2}{a^2},
\]

\[
\tau = \frac{a}{v}, \quad v = \frac{a}{\tau}.
\]

\[
v' = \frac{2e^2}{a \sqrt{vm}},
\]

\[
T' = \frac{1}{2} mv'^2 = \frac{2e^4}{a^2 v^2 m},
\]

\[
T' = \frac{e^4}{a^2 T}.
\]

\[
Q = T' = \frac{2e^4}{a^2 mv^2} \left[ a^2 = \frac{2e^4}{Q mv^2} \right].
\]

For \( n \) electrons per unit volume.\textsuperscript{23}

\textsuperscript{23} In the original manuscript the typo “per centimeter” occurs.
\[ \psi(Q) dQ = -\pi n da^2 = \frac{2\pi e^4}{Q^2 mv^2} n dQ. \]

\[ \left[ \psi(Q) = \frac{2\pi e^4 n}{mv^2} \right]. \]

\[ \left[ 1 \approx \int_{J}^{\infty} \psi(Q) dQ = \frac{2\pi e^4 n}{mv^2} \frac{1}{J} \right], \]

that is, the Thomson formula.

### 7.5. SYSTEMS WITH TWO FERMIONS AND ONE BOSON

In the following the author seems to consider a system formed by one boson and two fermions, with momentum \( \gamma^0, \gamma', \gamma'' \), respectively. It is not clear to what he precisely referred himself; the topic was only sketched.

Let us consider three fields

\[ \psi(\gamma'), \varphi(\gamma''), \chi(\gamma^0), \]

with:

\[ \chi = (\chi_1, \chi_2), \quad \psi = (\psi_1, \psi_2), \quad \varphi = (\varphi_1, \varphi_2). \]

\[ \chi_i(\gamma)\overline{\chi}_i(\gamma') - \overline{\chi}_i(\gamma')\chi_i(\gamma) = \delta(\gamma - \gamma'), \]

\[ \psi_i(\gamma)\overline{\psi}_i(\gamma') + \overline{\psi}_i(\gamma')\psi_i(\gamma) = \delta(\gamma - \gamma'), \]

\[ \varphi(\gamma)\overline{\varphi}_i(\gamma') + \overline{\varphi}_i(\gamma')\varphi_i(\gamma) = \delta(\gamma - \gamma'). \]

\[ R = \int \overline{\chi} R' \chi d\gamma^0 + \int \overline{\psi} R \psi d\gamma' + \int \overline{\varphi} R'' \varphi d\gamma''. \]

### 7.6. SCALAR FIELD THEORY FOR NUCLEI?

In the following pages the author apparently elaborated a relativistic field theory for nuclei composed of scalar particles of two different kinds (one
with positive charge and the other with negative charge), described by
the complex scalar field $\psi$ and its conjugate $P$ (this is the continuation
of what reported in Sections 2.7 and 2.8). The total number of such
constituents is denoted with $N$, while $Z$ is the net charge; the num-
ber of “positive” particles is $L$, while that of the “negative” ones is $M$.
Explicit expressions of some operators and their matrix elements were
given. In particular, transitions between different nuclei were described
in the framework of the theory considered. For a more detailed discus-

$$
[\psi_0, P_0] = 1, \quad [\psi_0, \psi_1] = 0, \quad [P_0, P_1] = 0,
$$

$$
[\psi_1, P_1] = 1, \quad [\psi_0, P_1] = 0, \quad [\psi_1, P_0] = 0.
$$

$$
\psi = \frac{\psi_0 - i\psi_1}{\sqrt{2}}, \quad P = \frac{P_0 + iP_1}{\sqrt{2}}.
$$

$[24]$

$$
N = \int -\frac{2\pi i}{\hbar} (\psi P - \bar{\psi} \bar{P}) \, dV.
$$

$$
\bar{\psi}\psi = \frac{\psi_0^2 + \psi_1^2}{2}, \quad \bar{P}P = \frac{P_0^2 + P_1^2}{2}.
$$

$$
\psi P - \bar{\psi} \bar{P} = i(\psi_0 P_1 - \psi_1 P_0).
$$

$$
\psi_0 = \sum q_0^r u_r, \quad \psi_1 = \sum q_1^r u_r,
$$

$$
P_0 = \sum p_0^r u_r, \quad P_1 = \sum p_1^r u_r.
$$

$$
N = \frac{2\pi}{\hbar} \sum_r (q_0^r p_1^r - q_1^r p_0^r).
$$

$$
[\psi, \bar{\psi}] = \frac{1}{2}[\psi_0, \psi_0] + \frac{1}{2}[\psi_1, \psi_1] + \frac{i}{2}[\psi_0, \psi_1] - \frac{1}{2}[\psi_1, \psi_0],
$$

$$
[\psi, \psi] = \frac{1}{2}[\psi_0, \psi_0] - \frac{1}{2}[\psi_1, \psi_1] - \frac{i}{2}[\psi_0, \psi_1] + \frac{i}{2}[\psi_1, \psi_0].
$$

$^{[24]}$ Note that, in subsequent pages, the author denotes with $Z$ the following operator corre-
sponding, effectively, to the net charge rather than to the total number $N$ of particles.
\[
\n\nabla^2 u_r + k_r^2 u_r = 0, \quad \int u r^2 dV = 1.
\]

\[
\int \bar{PP} \ dV = \frac{1}{2} \sum (p_{0r}^2 + p_{1r}^2), \quad \int \bar{\psi} \psi \ dV = \frac{1}{2} \sum (q_{0r}^2 + q_{1r}^2),
\]

\[
\int \nabla \bar{\psi} \cdot \nabla \psi \ dV = \frac{1}{2} \sum k_r^2 (q_{0r}^2 + q_{1r}^2).
\]

The Hamiltonian \( H \) without external field is (we write \( q_0, q_1, p_0, p_1, k \) instead of \( q^0_r, q^1_r, p^0_r, p^1_r, k^r \)):

\[
H_0 = \sum_r \left\{ \frac{4\pi^2 m c^2}{\hbar^2} (p_0^2 + p_1^2) + \frac{\hbar^2}{16\pi^2 m} k^2 (q_0^2 + q_1^2) + \frac{1}{4} mc^2 (q_0^2 + q_1^2) \right\}
\]

\[
\nu^2 = \frac{c^2 k^2}{4\pi^2} + \frac{m^2 c^4}{\hbar^2}, \quad \hbar^2 \nu^2 = m^2 c^4 + \frac{c^2 \hbar^2 k^2}{4\pi^2},
\]

\[
\hbar \nu = \sqrt{m^2 c^4 + \frac{c^2 \hbar^2 k^2}{4\pi^2}} = \sqrt{m^2 c^4 + p^2 c^2} = c \sqrt{m^2 c^2 + p^2},
\]

\[
E = \sum E_r, \quad E_r = N_r \hbar \nu_r = N_r c \sqrt{m^2 c^2 + p^2}.
\]

\[
N_r = \frac{W^r - \hbar \nu_r}{\hbar \nu_r}.
\]

\[
N = \sum N_r, \quad Z = \sum Z_r,
\]

\[
N_r = 0, 1, 2, \ldots; \quad Z_r = N_r, N_r - 2, N_r - 4, \ldots, -N_r.
\]

\[
|Z_r| \leq N_r, \quad |Z| \leq N.
\]
With an external field endowed with vector potential $\mathbf{C} = 0$ and scalar potential $\varphi \neq 0$:

$$
\varphi = \sum \varphi_r u_r,
$$

$$
\varphi_r = \int \varphi u_r^2 dV, \quad \varphi_{rs} = \int u_r u_s \varphi dV,
$$

$$
H = H_0 - \frac{2\pi}{\hbar} e \sum_{rs} \varphi_{rs} (q_r^0 p_s^1 - q_1^r p_s^0),
$$

$$
\begin{array}{ccc}
N_r & Z_r \\
0 & 0 & 0 \\
1 & 0 & 1 \{10\} & 1, -1 \\
2 & 0 & 2 \{11, 20\} & 2, 0, -2 \\
3 & 0 & 3 \{12, 21, 30\} & 3, 1, -1, -3
\end{array}
$$

By using units such that $\hbar = 2\pi$, $\nu = 1/2\pi$, $\hbar \nu = 1$:

$$
\frac{W}{\hbar \nu} = \frac{1}{2} P_0^2 + \frac{1}{2} Q_0^2 + \frac{1}{2} P_1^2 + \frac{1}{2} Q_1^2,
$$

$$
N = \frac{1}{2} P_0^2 + \frac{1}{2} Q_0^2 + \frac{1}{2} P_1^2 + \frac{1}{2} Q_1^2 - 1,
$$

$$
Z = Q_0 P_1 - Q_1 P_0.
$$

$$
P_0 Q_0 - Q_0 P_0 = \frac{1}{i}, \quad P_1 Q_1 - Q_1 P_1 = \frac{1}{i},
$$

$$
P_0 P_1 - P_1 P_0 = 0, \text{ etc.}
$$

$$
N = 0, 1, 2, \ldots; \quad Z = N, N - 2, \ldots, -N.
$$
\[ NP_0 - P_0 N = iQ_0, \quad -(ZP_0 - P_0 Z) = -iP_1, \]
\[ NQ_0 - Q_0 N = -iP_0, \quad -(ZQ_0 - Q_0 Z) = -iQ_1, \]
\[ NP_1 - P_1 N = iQ_1, \quad -(ZP_1 - P_1 Z) = iP_0, \]
\[ NQ_1 - Q_1 N = -iP_1, \quad -(ZQ_1 - Q_1 Z) = iQ_0. \]

<table>
<thead>
<tr>
<th>(N, Z); ((N + 1, Z + 1))</th>
<th>(P_0)</th>
<th>(Q_0)</th>
<th>(P_1)</th>
<th>(Q_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f++((N, Z))</td>
<td>if++((N, Z))</td>
<td>+if++((N, Z))</td>
<td>-f++((N, Z))</td>
<td></td>
</tr>
<tr>
<td>f+−((N, Z))</td>
<td>if+−((N, Z))</td>
<td>-if+−((N, Z))</td>
<td>+f+−((N, Z))</td>
<td></td>
</tr>
<tr>
<td>f−−((N, Z))</td>
<td>−if−−((N, Z))</td>
<td>+if−−((N, Z))</td>
<td>-f−−((N, Z))</td>
<td></td>
</tr>
<tr>
<td>f−+((N, Z))</td>
<td>−if−+((N, Z))</td>
<td>-if−+((N, Z))</td>
<td>+f−+((N, Z))</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{1}{2} P_0^2 + \frac{1}{2} Q_0^2 + \frac{1}{2} P_1^2 + \frac{1}{2} Q_1^2 - 1
\]

| \(N, Z\); \((N + 2, Z + 2)\) | 0 | 0 | 2f++\((N, Z)\) \cdot f+−\((N + 1, Z + 1)\) | -2f+−\((N, Z)\) \cdot f++\((N + 1, Z + 1)\) |
| \(N, Z\); \((N + 2, Z)\) | 0 | 2f++\((N, Z)\) \cdot f+−\((N + 1, Z + 1)\) | -2f+−\((N, Z)\) \cdot f++\((N + 1, Z + 1)\) |
| \(N, Z\); \((N + 2, Z − 2)\) | 2|f++\((N, Z)\)| \cdot 2|f+−\((N, Z)\)| + 2|f−−\((N, Z)\)| \cdot 2|f−+\((N, Z)\)| + 2|f−+\((N, Z)\)| \cdot 2|f−−\((N, Z)\)| |
| \(N, Z\); \((N, Z)\) | \frac{2}{2}|f++\((N, Z)\)| \cdot \frac{2}{2}|f+−\((N, Z)\)| + \frac{2}{2}|f−−\((N, Z)\)| \cdot \frac{2}{2}|f−+\((N, Z)\)| + \frac{2}{2}|f−+\((N, Z)\)| \cdot \frac{2}{2}|f−−\((N, Z)\)| |
| \(N, Z\); \((N, Z − 2)\) | 0 | | |
| \(N, Z\); \((N − 2, Z + 2)\) | 0 | 0 | 0 | 0 |
| \(N, Z\); \((N − 2, Z)\) | 0 | 0 | 0 | 0 |
| \(N, Z\); \((N − 2, Z − 2)\) | 0 | 0 | 0 | 0 |
\[ f_{++}(N, Z) = \bar{f}_{--}(N + 1, Z + 1), \]
\[ f_{+-}(N, Z) = \bar{f}_{-+}(N + 1, Z - 1). \]
\[ |f_{++}(N, Z)|^2 + |f_{--}(N, Z)|^2 = \frac{N + Z + 1}{4}, \]
\[ |f_{+-}(N, Z)|^2 + |f_{-+}(N, Z)|^2 = \frac{N - Z + 1}{4}. \]
\[ f_{--}(N, Z) = \bar{f}_{++}(N - 1, Z - 1), \]
\[ f_{-+}(N, Z) = \bar{f}_{+-}(N - 1, Z + 1). \]
\[ |f_{++}(N, Z)|^2 + |f_{++}(N - 1, Z - 1)|^2 = \frac{N + Z + 1}{4}, \]
\[ |f_{+-}(N, Z)|^2 + |f_{+-}(N - 1, Z + 1)|^2 = \frac{N - Z + 1}{4}. \]
\[ \sqrt{(N + Z + 2)(N - Z + 2)} - \sqrt{(N - Z + 2)(N + Z + 2)} = 0. \]
\[ |f_{++}(N, Z)|^2 = \frac{N + Z + 2}{8}, \]
\[ |f_{+-}(N, Z)|^2 = \frac{N - Z + 2}{8}. \]
\[ f_{++} = \sqrt{\frac{N + Z + 2}{8}}, \]
\[ f_{+-} = \sqrt{\frac{N - Z + 2}{8}}, \]
\[ f_{--} = \sqrt{\frac{N - Z}{8}}, \]
\[ f_{-+} = \sqrt{\frac{N + Z}{8}}. \]
\[ P_0(N, Z; N', Z') = \sqrt{\frac{N + Z + 2}{8}} \delta_{N+1,N'} \delta_{Z+1,Z'} \]
\[ + \sqrt{\frac{N - Z + 2}{8}} \delta_{N+1,N'} \delta_{Z-1,Z'} \]
\[ + \sqrt{\frac{N - Z}{8}} \delta_{N-1,N'} \delta_{Z+1,Z'} \]
\[ + \sqrt{\frac{N + Z}{8}} \delta_{N-1,N'} \delta_{Z-1,Z'} \]
\[ = a + b + c + d, \]
\[ Q_0(N, Z; N', Z') = ia + ib - ic - id, \]
\[ P_1(N, Z; N', Z') = ia - ib + ic - id, \]
\[ Q_1(N, Z; N', Z') = -a + b + c - d. \]

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 2 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 2 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 2 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & -1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 2 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & -2 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

\[25\]

The columns and rows of the following matrix are ordered for \( N, Z \) equal to 0,0; 1,1; 1,-1; 2,2; 2,0; 2,-2; 3,3; 3,1; 3,-1; 3,-3; \ldots , \) respectively.
\[ P_0 = \]

\[
\begin{array}{c|ccc|ccc|c}
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \ldots \\
\hline
\frac{1}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \ldots \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \ldots \\
\hline
0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & \ldots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \ldots \\
0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & \ldots \\
\hline
0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

\[ Z = -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ N = 0 \]
\[ N + Z = 2L, \quad N - Z = 2M, \]

\[ L = 0, 1, 2, \ldots; \quad M = 0, 1, 2, \ldots. \]

*L numbers the* particles with positive *charge, while M numbers the* particles with negative *charge.*

<table>
<thead>
<tr>
<th>( N )</th>
<th>( Z )</th>
<th>( L )</th>
<th>( M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ N = L + M, \quad Z = L - M. \]
\[ P_0(L, M; L', M') = \frac{\sqrt{L+1}}{2} \delta_{L+1,L'} \delta_{M M'} + \frac{\sqrt{L}}{2} \delta_{L-1,L'} \delta_{M M'} \]
\[ + \frac{\sqrt{M+1}}{2} \delta_{L L'} \delta_{M+1,M'} + \frac{\sqrt{M}}{2} \delta_{L L'} \delta_{M-1,M'}. \]

\[
\begin{align*}
\sqrt{2} P_0 &= P_0^L + P_0^M = P_L + P_M, \\
\sqrt{2} Q_0 &= Q_0^L + Q_0^M = Q_L + Q_M, \\
\sqrt{2} P_1 &= Q_0^L - Q_0^M = Q_L - Q_M, \\
\sqrt{2} Q_1 &= -P_0^L + P_0^M = -P_L + P_M.
\end{align*}
\]

\[
\frac{P_0^L}{\sqrt{2}} = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\
-\frac{1}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \ldots \\
0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & \ldots \\
0 & 0 & \frac{\sqrt{3}}{2} & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{bmatrix},
\]
\[
\begin{pmatrix}
0 & \frac{i}{2} & 0 & 0 & 0 & \ldots \\
-\frac{i}{2} & 0 & \frac{i\sqrt{2}}{2} & 0 & 0 & \ldots \\
0 & -i\frac{\sqrt{2}}{2} & 0 & \frac{i\sqrt{3}}{2} & 0 & \ldots \\
0 & 0 & -i\frac{\sqrt{3}}{2} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots 
\end{pmatrix}
\]

\[Q_0^L = \sqrt{2} P_0^L,\]
\[P_0^L Q_0^L - Q_0^L P_0^L = -i.\]

\[
\sqrt{2} P_L = P_0 - Q_1, \\
\sqrt{2} Q_L = Q_0 + P_1, \\
\sqrt{2} P_M = P_0 + Q_1, \\
\sqrt{2} Q_M = Q_0 - P_1.
\]

For \(h = 2\pi, \nu = 1/2\pi:\)

\[
W_{\hbar\nu} = \frac{1}{2} P_L^2 + \frac{1}{2} Q_L^2 + \frac{1}{2} P_M^2 + \frac{1}{2} Q_M^2,
\]

\[
N = L + M = \frac{1}{2} P_L^2 + \frac{1}{2} Q_L^2 - \frac{1}{2} P_M^2 + \frac{1}{2} Q_M^2 - \frac{1}{2},
\]

\[
L = \frac{1}{2} P_L^2 + \frac{1}{2} Q_L^2 - \frac{1}{2}, \quad M = \frac{1}{2} P_M^2 + \frac{1}{2} Q_M^2 - \frac{1}{2},
\]

\[
Z = L - M = Q_0 P_1 - Q_1 P_0 = \frac{1}{2} P_L^2 + \frac{1}{2} Q_L^2 - \frac{1}{2} P_M^2 - \frac{1}{2} Q_M^2.
\]

\[^{26}\textit{Notice that, by using the matrices given above, the following relation is not actually satisfied.}\]
\[
\psi P = \frac{1}{4} \left\{ \psi_L P_L + \psi_M P_M + \psi_L P_M + \psi M P_L \\
- P_L \psi L - P_M \psi M + P_L \psi_M + P_M \psi L \\
+ i \left( \psi_L^2 + P_L^2 - \psi_M^2 - P_M^2 \\
- \psi_L \psi_M + \psi_M \psi L + P_L P_M - P_M P_L \right) \right\}.
\]

Versuchsweise: \(^{27}\)

\[P_M = \psi_M = 0\]

\((mc^2 = 1, \ h = 2\pi).\)

\[ [\psi, P] = \frac{1}{2}, \ [\psi, \bar{\psi}] = -i, \ [P, \bar{P}] = \frac{i}{4}. \]

We have, thus, the classical theory! \(^{28}\)

\[
\bar{\psi} \psi = \frac{\psi_L^2 + P_L^2}{2}, \\
\bar{P} P = \frac{\psi_L^2 + P_L^2}{8} = \frac{1}{4} \bar{\psi} \psi, \\
\psi P = \frac{i}{4} (\psi_L^2 + P_L^2). 
\]

\(^{27}\) This German word means “tentatively”, and refers to the successive assumptions. Note, however, that in the original paper the cited word is written as “versucherweiser”.

\(^{28}\) That is, a theory with only positively charged particle, without antiparticles.
PART IV
8

CLASSICAL PHYSICS

8.1. SURFACE WAVES IN A LIQUID

The author studied the propagation of surface waves in liquids under the action of the gravitational potential $U$ and the liquid pressure $P$. Some particular cases were considered in detail.

$$\mu \alpha = \mu F - \nabla p.$$

$F = \nabla U$:

$$\alpha = \nabla U - \frac{1}{\mu} \nabla p.$$

$\mu = \mu(p)$;

$$\int P \frac{dp}{\mu}, \quad \nabla P = \frac{1}{\mu} \nabla p.$$

$$\alpha = \nabla (U - P).$$

$$v = \nabla \phi,$$

$$\alpha = \nabla \frac{\partial \phi}{\partial t} + v_x \nabla \frac{\partial \phi}{\partial x} + v_y \nabla \frac{\partial \phi}{\partial y} + v_z \nabla \frac{\partial y}{\partial r}$$

$$= \nabla \frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla V^2.$$

$$\nabla \frac{\partial \phi}{\partial t} + \nabla \frac{1}{2} V^2 - \nabla U + \nabla P = 0,$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} V^2 - U + P = 0.$$

For a liquid:
\[ U = g z, \quad P = \frac{p}{\mu}. \]

\[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} V^2 - g z + \frac{p}{\mu} = 0, \]

\[ \nabla^2 \varphi = 0. \]

\[ \varphi = A e^{\omega i(t-x/v)} e^{kz}. \]

\[ \nabla^2 \varphi = -\varphi \left( \frac{\omega^2}{v^2} + k^2 \right). \]

Since \( \nabla^2 \varphi = 0 \), we have:

\[ k = \pm \frac{\omega}{v} i, \]

\[ \varphi = e^{\omega i(t-x/v)} \left( A e^{\omega z v} + B e^{-\omega z v} \right). \]

For small amplitudes:

\[ \frac{\partial \varphi}{\partial t} - g z + \frac{p}{\mu} = 0. \]

For \( z = 0 \), \( \frac{dp}{dt} = 0 \):

\[ \frac{\partial^2 \varphi}{\partial t^2} - g \frac{\partial \varphi}{\partial z} = 0. \]

For \( z = \ell \):

\[ \frac{\partial \varphi}{\partial z} = 0. \]

\[ -\omega^2 e^{\omega i(t-x/v)} (A + B) = \frac{\omega}{v} (A - B) e^{\omega i(t-x/v)}, \]

\[ \frac{g}{v} (A - B) = -\omega (A + B). \]

\[ Ae^{\omega \ell/v} - Be^{-\omega \ell/v} = 0, \]

\[ B = Ae^{2\omega \ell/v}. \]
\[
\frac{B + A}{B - A} = \frac{g}{\omega v}.
\]
\[
\frac{g}{\omega v} = \frac{e^{\omega \ell v} + e^{-\omega \ell v}}{e^{\omega \ell v} - e^{-\omega \ell v}}.
\]
\[
\lambda = v \sqrt{\frac{\omega}{2\pi}} = \frac{2\pi v}{\omega}, \quad \frac{v}{\omega} = \frac{\lambda}{2\pi},
\]
\[
v = \omega \frac{\lambda}{2\pi}, \quad \omega = v \frac{2\pi}{\lambda}.
\]
\[
\frac{\lambda}{2\pi} \frac{g}{v^2} = \frac{e^{2\pi \ell/\lambda} + e^{-2\pi \ell/\lambda}}{e^{2\pi \ell/\lambda} - e^{-2\pi \ell/\lambda}},
\]
\[
\frac{2\pi}{\lambda} \frac{v^2}{g} = \frac{e^{2\pi \ell/\lambda} - e^{-2\pi \ell/\lambda}}{e^{2\pi \ell/\lambda} + e^{-2\pi \ell/\lambda}} = \tanh \frac{2\pi \ell}{\lambda},
\]
\[
v^2 = g \frac{\lambda}{2\pi} \tanh \frac{2\pi \ell}{\lambda}, \quad v = \sqrt{g \frac{\lambda}{2\pi} \tanh \frac{2\pi \ell}{\lambda}}.
\]

For \( \ell \ll \frac{\lambda}{2\pi} \):
\[
v = \sqrt{g \ell}.
\]

For \( \ell \gg \frac{\lambda}{2\pi} \):
\[
v = \sqrt{g \frac{\lambda}{2\pi}}.
\]

8.2. THOMSON’S METHOD FOR THE DETERMINATION OF \( e/m \)

The equations of motion for the electron moving in the Thomson apparatus, aimed at the determination of the charge to mass ratio, \( e/m \), are studied by the author in these pages.
For photoelectric electrons:

\[ m\ddot{x} = E e + H e \dot{y}, \]
\[ m\ddot{y} = -H e x. \]

\[ m\dddot{x} = H e \ddot{y} = -\frac{H^2 e^2}{m} \dot{x}, \]
\[ \dddot{x} = H e \ddot{y} = -\frac{H^2 e^2}{m^2} \dot{x}. \]
\[ \dot{x} = c \sin \frac{H e}{m} t. \]

By the substitution above, the constant \( c \) is determined as follows:

\[ c \frac{H e}{m} = \frac{E e}{m}, \quad c H = E, \quad c = E \frac{H}{H}. \]
\[ \dot{x} = \frac{E H}{E m} \sin \frac{H e}{m} t. \]
\[ \frac{E m}{H^2 e} \left( 1 - \cos \frac{H e}{m} t \right), \]
\[ x_0 = \frac{2E m}{H^2 e}. \]

8.3. **WIEN’S METHOD FOR THE DETERMINATION OF \( e/m \) (POSITIVE CHARGES)**

The equations of motion for positively charged particles moving in the Wien apparatus, aimed at the determination of the charge to mass ratio,
e/m, are solved and compared with the experimental results by Thomson.

\[
\begin{align*}
\dot{y} &= \dot{H} e x, \\
\ddot{y} &= \int H e \, dx, \\
m \frac{dy}{dt} &= \int H e \, dx, \\
m v \, dy &= dx \int H e \, dx, \\
m v \, y &= \int dx \int H e \, dx = e A. \\
y &= A \frac{e}{m v}, \\
m \frac{d^2 z}{dt^2} &= Z e, \\
m v^2 \frac{d^2 z}{dx^2} &= Z e, \\
m v^2 z &= B e.
\end{align*}
\]

Thomson has repeated the experiment by Wien, obtaining, as a result, the parabola:

\[
\frac{y^2}{z} = \frac{A^2}{B} \frac{e}{m}.
\]
\[
m v_{\text{max}}^2 = 2V e, \\
z_{\text{min}} = \frac{B}{2V}.
\]

8.4. **DETERMINATION OF THE ELECTRON CHARGE**

In the following, the author studied several electrical effects in gases, with particular reference to the Townsend effect, that is, the increase of the photoelectric saturation current from an electrode as a function of the distance \(d\) between plane parallel electrodes for high values of the electric field (whose strength was denoted with \(X\)). The quantity \(n\) gives the number of electric charges (electrons) per unit volume, while the Townsend coefficient \(\alpha\) is the number of new ion pairs produced per centimeter of path in the gas by electron impacts. The gas is at the pressure \(p\) and temperature \(T\), while \(D\) is a diffusion coefficient.

This study was aimed to obtain determinations of the electron charge \(e\) (with different experimental methods).

8.4.1 **Townsend Effect**

\[
\frac{dn}{dt} = \frac{dm}{dt} = q - \alpha m n. \tag{1}
\]

8.4.1.1 **Ion recombination.**
\[ n = m: \]
\[
\frac{dn}{dt} = q - \alpha n^2. \tag{2}
\]
\[
n - \alpha n^2 = 0; \quad n_0 = \sqrt{\frac{q}{\alpha}}. \tag{3}
\]
\[
\frac{dn}{q - \alpha n^2} = dt,
\]
\[
\frac{dn}{2\sqrt{q}} \left( \frac{1}{\sqrt{q} + n\sqrt{\alpha}} + \frac{1}{\sqrt{q} - n\sqrt{\alpha}} \right) = dt,
\]
\[
\frac{1}{2\sqrt{qa}} \log \frac{\sqrt{q} + n\sqrt{\alpha}}{\sqrt{q} - n\sqrt{\alpha}} = t,
\]
\[
\frac{\sqrt{q} + n\sqrt{\alpha}}{\sqrt{q} - n\sqrt{\alpha}} = e^{2t\sqrt{qa}} = \frac{e^{2t\sqrt{qa}} - 1}{e^{2t\sqrt{qa}} + 1},
\]
\[
n\sqrt{\frac{\alpha}{q}} = \frac{e^{2t\sqrt{qa}} - 1}{e^{2t\sqrt{qa}} + 1},
\]
\[
n = \sqrt{\frac{q}{\alpha}} \frac{e^{2t\sqrt{qa}} - 1}{e^{2t\sqrt{qa}} + 1},
\]
\[
n = n_0 \frac{e^{3\alpha q t} - 1}{e^{3\alpha q t} + 1},
\]
\[
n = n_0 \frac{e^{2n_0\alpha t} - 1}{e^{2n_0\alpha t} + 1}. \tag{4}
\]

(formula applying to a source active for a time \( t \)).

\[
\frac{dn}{dt} = -\alpha n^2,
\]
\[
\frac{dn}{n^2} = -\alpha dt,
\]
\[
\frac{1}{n_0} - \frac{1}{n} = -\alpha t,
\]
\[
\frac{1}{n} = \frac{1}{n_0} + \alpha t,
\]
\[
n = \frac{1}{\frac{1}{n_0} + \alpha t} = \frac{n_0}{1 + n_0\alpha t} \tag{5}
\]
For the determination of \( \alpha \) we can use the following setup, where \( i_A, i_B \) are the saturation currents measured by setting alternately the electrical tension in \( A \) and \( B \), respectively, with \( i_B = \frac{1}{2} i_A \).

\[
V = \sigma v, \quad T = \frac{d}{v}.
\]

\[
n_B = \frac{n_A}{1 + n_A \alpha T},
\]

and since \( n_B = \frac{1}{2} n_A \),

\[
n_A \alpha T = 1.
\]

\[
i_A = n_A V e, \quad i_A \alpha T = V e.
\]

\[
\alpha = \frac{V e}{i_A T}.
\]

For air we have \( \alpha = 1.65 \cdot 10^{-6} = 3480e \) (Townsend).

### 8.4.1.2 Ion diffusion.

\[
\frac{dn}{dt} = q - \alpha n^2 + D \nabla^2 n.
\]

\[
\frac{dn}{dt} = q - \alpha n^2 + D \frac{d^2 n}{dx^2}.
\]

For \( \frac{dn}{dt} = 0 \) and neglecting \( \alpha \),

\[
D \frac{d^2 n}{dx^2} - q = 0,
\]

\[
\frac{d^2 n}{dx^2} = -\frac{q}{D},
\]

\[
n = \frac{q}{2D} (e^2 - x^2).
\]
\[
\int n \, dx = \frac{q \ell^3}{D} - \frac{1}{3} \frac{q \ell^3}{D} = \frac{2}{3} \frac{q}{D} \ell^3.
\]

\[
Q = \frac{2}{3} \frac{q}{D} \ell^3 e.
\]

\[
Q = 2q \ell t, \quad \frac{1}{3} \frac{\ell^2}{D} e = t.
\]

<table>
<thead>
<tr>
<th>(D) coefficients (Townsend)</th>
<th>+ ions</th>
<th>- ions</th>
</tr>
</thead>
<tbody>
<tr>
<td>dry air</td>
<td>0.028</td>
<td>0.043</td>
</tr>
<tr>
<td>wet air</td>
<td>0.032</td>
<td>0.026</td>
</tr>
<tr>
<td>dry CO(_2)</td>
<td>0.023</td>
<td>0.026</td>
</tr>
<tr>
<td>dry H(_2)</td>
<td>0.123</td>
<td>0.190</td>
</tr>
</tbody>
</table>

8.4.1.3 Velocity in the electric field.

\[
N_1 = D \frac{dn}{dx}, \quad N_1 = V n.
\]

\[
V n = D \frac{dn}{dx}, \quad V = D \frac{1}{n} \frac{dn}{dx}, \quad V = D \frac{1}{p} \frac{dp}{dx},
\]

\[
V = \frac{D}{p} n e X.
\]

\[
p = n kT, \quad \frac{n}{p} = \frac{1}{kT} = \frac{N}{\pi},
\]

\[\text{[1]}\]

\[
V = D \frac{N}{\pi} e X = \frac{D}{kT} eX,
\]

The relation utilized by Townsend relation is for \(X = 1\):

\[
V = D \frac{N}{\pi} e = \frac{D}{kT} e.
\]

8.4.1.4 Charge of an ion.

\[
n = D \frac{N}{\pi} e,
\]

\[
N e = \frac{\pi n}{D}.
\]

\[\text{[1]}\] \(N\) is the total number of charged particles, while \(\pi\) is the atmospheric pressure (see below).
where \( \pi \) is the atmospheric pressure. Townsend has found:

\[
Ne' = \frac{96540 \cdot 3 \times 10^9}{22400} = 1.3 \cdot 10^{10},
\]

\[
\frac{e}{e'} = 1.04.
\]

8.4.2 Method of the Electrolysis (Townsend)

The oxygen and hydrogen which are formed at the electrode are strongly electrified, positively or negatively depending on the kind of electrolysis. From the Stokes law:

\[
v = ka^2.
\]

\[
n = q \sqrt[4]{\frac{4}{3} \pi a^3}.
\]

\[
e = \frac{Q}{n},
\]

where \( q \) is evaluated thermodynamically.

8.4.3 Zaliny’s Method For The Ratio Of The Mobility Coefficients

\[
V - ku = 0,
\]

\[
V - k_1 v = 0,
\]

\[
\frac{u}{v} = \frac{k_1}{k} = \frac{1}{1.24}.
\]
8.4.4 Thomson’s Method
8.4.5 Wilson’s Method

It is as the Thomson’s method, with the addition of an electric field to the gravity. The charge \( e \) is obtained from the ratio between the fall velocities with and without the field:

\[
\frac{v_1}{v} = \frac{4}{3} \pi \rho g a^3 + Xe \quad \frac{4}{3} \pi \rho g a^3.
\]

By determining \( a \) from the Stokes formula (see below), we can obtain the value of \( e \).

8.4.6 Millikan’s Method

The Stokes law:

\[
v = \frac{2}{9} \frac{g a^2}{\mu} (\sigma - \rho)
\]

has been corrected by Cunningham for droplets with small radius:

\[
v = \frac{2}{9} \frac{g a^2}{\mu} (\sigma - \rho) \left( 1 + A \frac{\ell}{a} \right)
\]

where \( A \) is a numerical constant and \( \ell \) is the mean free path. By setting \( B = A\ell \) we have:

\[
v = \frac{2}{9} \frac{g a^2}{\mu} (\sigma - \rho) \left( 1 + \frac{B}{a} \right)
\]
8.5. ELECTROMAGNETIC AND ELECTROSTATIC MASS OF THE ELECTRON

The expressions for the electromagnetic and the electrostatic mass of the electron are derived, by evaluating the magnetic energy $W$ and the analogous electrostatic energy $W/e^2$.

\[
H = \frac{e u \sin \theta}{r^2},
\]
\[
H^2 = \frac{e^2 u^2 \sin^2 \theta}{r^4},
\]
\[
\frac{H^2}{8\pi} = \frac{e^2 u^2 \sin^2 \theta}{8\pi r^4}.
\]

\[
4\pi r^2 \frac{H^2}{8\pi} dr = \frac{e^2 u^2}{3r^2} dr.
\]

\[
\int_a^{\infty} dr \frac{dr}{r^2} = \frac{1}{a}.
\]

\[
W = \frac{e^2 u^2}{3a} = \frac{1}{2} m u^2.
\]

\[
m = \frac{2}{3} \frac{e^2}{a} \quad \text{(electromagnetic)},
\]
\[
m = \frac{2}{3} \frac{e^2}{a e^2} \quad \text{(electrostatic)}.
\]

8.6. THERMIonic EFFECT

In the following the author studied electron emission induced by thermionic effect, obtaining the Richardson formula for the electron current. Moreover, he subsequently considered also the Langmuir effect (for low voltage) induced by the cloud of (slowly moving) electrons (space charge) around the cathode, which limits the electron emission from the cathode.
Let $V_e$ be the extraction work; in order that an electron comes out of the metal, the following relation must hold:

$$V_e \leq \frac{1}{2} m u^2.$$ 

The Maxwell distribution gives:

$$d n = C e^{-m u^2/kT} du,$$

$$d n = n \sqrt{\frac{h m}{\pi}} e^{-h m u^2} du,$$

$$h = 1/2kT.$$ 

$$V_e = \frac{1}{2} m u_0^2, \quad u_0 = \sqrt{\frac{2V_e}{m}}.$$ 

The number of electrons emitted is then given by:

$$\int_{\sqrt{2V_e/m}}^{\infty} 2d n = 2n \sqrt{\frac{h m}{\pi}} \int_{\sqrt{2V_e/m}}^{\infty} e^{-h m u^2} du$$

$$= n \sqrt{\frac{h m}{\pi}} \frac{1}{hm \sqrt{2Ve/m}} e^{-b/T}$$

$$= n \sqrt{\frac{1}{2Ve h \pi}} e^{-b/T}$$

$$= n \sqrt{\frac{kT}{\pi V e}} e^{-b/T}.$$ 

From this, the Richardson formula for the electron current $i$ follows (Richardson effect):

$$i = a T^{1/2} e^{-b/T}.$$ 

Instead, with the photoelectric theory, it has been found that:

$$i = a T^2 e^{-b/T}.$$ 

Electron emission starts around 1000°C; for several elements (sodium) it starts around 200°C. If $T$ is small, the value for the saturation current is reached very quickly.
\[ V e = \frac{1}{2} m u^2. \]

\[ V = u/300: \]

\[ \frac{u}{300} e = \frac{1}{2} m u^2, \]

\[ u = \sqrt{u} \sqrt{\frac{2e}{300m}}. \]

\[ e = 4.77 \cdot 10^{-10}, \ m = 0.9 \cdot 10^{-27}, \]

\[ \frac{2e}{300m} = 5.53 \cdot 10^{15}, \]

\[ u = \sqrt{u} \cdot 594 \text{ km/s}. \]

### 8.6.1 Langmuir Experiment on the Effect of the Electron Cloud

At low values of the potential, the electron current does not change with varying \( T \).

\[ \frac{d^2 V}{dx^2} = -4\pi \rho. \]

\[ i = \rho v = \text{const.} \]

\[ v = k \sqrt{V}, \quad -4\pi \rho = \frac{c}{\sqrt{V}}. \]

\[ \frac{d^2 V}{dx^2} = \frac{c}{\sqrt{V}}, \]

\[ \frac{d^2 V}{dx^2} = \frac{d}{dx} \frac{dV}{dx}; \]

\[ \frac{d}{dx} \frac{dV}{dx} = \frac{c}{\sqrt{V}}, \]

\[ \frac{dV}{dx} \frac{dV}{dx} = \frac{c}{\sqrt{V}}. \]

\[ \left( \frac{dV}{dx} \right)^2 = c \sqrt{V} + \text{const.} \]
\[ V(0) = 0, \quad V(\ell) = V_1. \]
\[ v = v_0 \sqrt{V}, \quad \rho = \frac{i}{v_0 \sqrt{V}}. \]

\[ i = k \frac{V^{3/2}}{x^2}. \]

Effects that are an obstacle to the reaching of the value of the saturation current are the following.
1) the cloud of slowly moving electrons around the cathode (Langmuir effect):
\[ i_{\text{max}} = k V^{3/2}; \]
2) the magnetic field produced by the filament (a voltage of the order of 1 volt is required):
\[
\begin{align*}
    m \ddot{x} & = E e - H \dot{z}, \\
    m \ddot{z} & = H \dot{x},
\end{align*}
\]
\[
E = \frac{A}{x}, \quad H = \frac{B}{x},
\]
\[
\begin{align*}
    m \ddot{x} & = \frac{A}{x} e - \frac{B}{x} \dot{z}, \\
    m x \ddot{x} & = A e - B \dot{z}, \\
    m \ddot{z} & = \frac{B}{x} \dot{x};
\end{align*}
\]
3) a non-vanishing gradient of the voltage along the filament (of the order of 1 volt/cm).

\[ \text{It is not clear how the author solved the differential equation for } V, \text{ thus obtaining the expression for } \rho \text{ and, finally, the following expression for the current } i. \text{ Nevertheless, the expression for } i \text{ is correct, choosing in a given way the integration constant in the differential equation above.} \]
If the effects 1), 2) and 3) are removed in some way, the saturation of the current is reached at a very lower voltage. This has been verified experimentally by Schottky.\textsuperscript{3} The effect 3) is removed by switching off the voltage and measuring $i$ at the same time instant.

\textsuperscript{3}In the original manuscript, the author writes this name (between brackets) as “Sciochi”.
MATHEMATICAL PHYSICS

In the following six Sections, the author studied a number of topics dealing with tensor calculus, following closely the text T. Levi-Civita, Lezioni di calcolo differenziale assoluto (Stock, Rome, 1925), which was present in the Majorana personal library. For the notations used and further comments on the topics treated, we refer the reader to this book (we denote it as Levi-Civita I) or to its English translation (denoted as Levi-Civita E) in T. Levi-Civita, The Absolute Differential Calculus – Calculus of Tensors (Blackie & Son, London, 1926). Some explicit references to chapters (III and IV) or pages (pp. 48, 60, 123, 137, 140, 141, 143, 160, 173, 174, 178, 197 of Levi-Civita I or pp. 36, 47, 107, 119, 121, 123, 131, 140, 152, 153, 156, 172 of Levi-Civita E) of this book are reported throughout the manuscript. A few results, on the contrary, do not appear in the mentioned book; they were obtained by Majorana, or he simply reported what was expounded in the university course taught by Levi-Civita at the University of Rome and followed by Majorana himself.

9.1. LINEAR PARTIAL DIFFERENTIAL EQUATIONS. COMPLETE SYSTEMS

\[ X_1, \ldots, X_n: \]
\[ \sum X_i \, dx_i = 0. \]
\[ y(x_1, \ldots, x_n) = C, \]
\[ dy = \sum \frac{\partial y}{\partial x_i} \, dx_i. \]
\[ \frac{\partial y}{\partial x_i} = pX_i, \quad p = p(x_1 \ldots x_n). \]
\[ \text{d} y = \sum p X_i \text{d} x_i = \sum A_i \text{d} x_i. \]

\[ \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = 0. \]

\[ p \left( \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) + X_i \frac{\partial p}{\partial x_j} - X_j \frac{\partial p}{\partial x_i} = 0, \]

\[ p \left( \frac{\partial X_j}{\partial x_k} - \frac{\partial X_k}{\partial x_j} \right) + X_j \frac{\partial p}{\partial x_k} - X_k \frac{\partial p}{\partial x_j} = 0, \]

\[ p \left( \frac{\partial X_k}{\partial x_i} - \frac{\partial X_i}{\partial x_k} \right) + X_k \frac{\partial p}{\partial x_i} - X_i \frac{\partial p}{\partial x_k} = 0; \]

\[ X_k \left( \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) + X_i \left( \frac{\partial X_j}{\partial x_k} - \frac{\partial X_k}{\partial x_j} \right) + X_j \left( \frac{\partial X_k}{\partial x_i} - \frac{\partial X_i}{\partial x_k} \right) = 0. \]

### 9.1.1 Linear Operators

\[ Auv = vAu + uAv = (-Au)v + uAv. \]

\[ A = \sum_{r=1}^{N} a_r \frac{\partial}{\partial x_r} , \quad B = \sum_{r=1}^{N} b_r \frac{\partial}{\partial x_r} . \]

\[ AB = \sum_{r,s=1}^{N} a_r \frac{\partial}{\partial x_r} \left( b_s \frac{\partial}{\partial x_s} \right) = \sum_{r,s=1}^{N} a_r b_s \frac{\partial^2}{\partial x_r \partial x_s} + \sum_{r,s=1}^{N} a_r \frac{\partial b_s}{\partial x_r} \frac{\partial}{\partial x_s}, \]

\[ BA = \sum_{r,s=1}^{N} a_r b_s \frac{\partial^2}{\partial x_r \partial x_s} + \sum_{r,s=1}^{N} b_r \frac{\partial a_s}{\partial x_r} \frac{\partial}{\partial x_s}, \]

\[ AB - BA = (A, B) = \sum_{r,s=1}^{N} \left( a_r \frac{\partial b_s}{\partial x_r} - b_r \frac{\partial a_s}{\partial x_r} \right) \frac{\partial}{\partial x_s}. \]
\[ AB = \sum_{rs} a_r b_s \frac{\partial^2}{\partial x_r \partial x_s} + \sum_s (Ab_s) \frac{\partial}{\partial x_s}, \]

\[ BA = \sum_{rs} a_r b_s \frac{\partial^2}{\partial x_r \partial x_s} + \sum_s (Ba_s) \frac{\partial}{\partial x_s}, \]

\[ AB - BA = (A, B) = \sum_{s=1}^{N} (Ab_s - Ba_s) \frac{\partial}{\partial x_s}. \]

\[ A_1, \ldots, A_n: \]

\[ B = \sum_1^n \lambda_i A_i, \quad C = \sum_1^n \mu_i A_i. \]

\[ BC = \sum_{i,k=1}^{n} \lambda_i A_i \mu_k A_k = \sum_{i,k} \lambda_i \mu_k A_i A_k + \sum \lambda_i (A_i \mu_k) A_k, \]

\[ CB = \sum_{i,k} \lambda_i \mu_k A_k A_i - \sum_{i,k} \mu_i (A_i \lambda_k) A_k, \]

\[ (B.C) = BC - CB = \sum_{i,k} \lambda_i \mu_k (A_i, A_k) + \sum_k \left( \sum_i (\lambda_i A_i \mu_k - \mu_i A_i \lambda_k) \right) A_k. \]

### 9.1.2 Integrals Of An Ordinary Differential System And The Partial Differential Equation Which Determines Them

\[ x_1, \ldots, x_n: \]

\[ \frac{dx_i}{dt} = X_i(x|t). \quad (1) \]

\[ f(x|t) = \text{constant}: \]

\[ \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = 0, \]

\[ \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial x_i} X_i = 0. \]
\[ A = \frac{\partial}{\partial t} + \sum X_i \frac{\partial}{\partial x_i}, \]

\[ Af = 0. \]

\( f(x|t) \) constant for any value of \( t \) implies \( Af = 0 \).

Conversely, \( Af = 0 \) implies \( f(x|t) \) constant for any value of \( t \).


\[ du_\alpha = \sum_{i=1}^{n} X_{\alpha i} dx_i, \quad \alpha = 1, \ldots, m. \]

\( f(x|u) = \text{constant} \):

\[ df = \sum \frac{\partial f}{\partial x_i} dx_i + \sum \frac{\partial f}{\partial u_\alpha} du_\alpha \]

\[ = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} + \sum_{\alpha=1}^{m} \frac{\partial f}{\partial u_\alpha} X_{\alpha i} \right) dx_i. \]

\[ \frac{\partial f}{\partial x_i} + \sum_{\alpha=1}^{m} \frac{\partial f}{\partial u_\alpha} X_{\alpha i} = 0 \quad (i = 1, 2, \ldots, n). \]

\[ \Omega_i = \sum_{\alpha=1}^{m} X_{\alpha i} \frac{\partial}{\partial u_\alpha}. \]

\[ B_i = \frac{\partial}{\partial x_i} + \Omega_i, \quad (i = 1, 2, \ldots, n). \]

\[ B_i f = 0, \quad (i = 1, 2, \ldots n). \]

Complete systems:

\[ A_k f = 0, \]
\[ A_k = \sum_{1}^{N} a_{k\nu} \frac{\partial}{\partial x_\nu} \quad (k = 1, 2, \ldots, n); \]

\[ (A_i, A_k) = \sum_{1}^{n} p_{i kl} A_l, \quad p_{i kl} = -p_{k il}. \]

**Jacobian systems:**

\[ (A_i, A_k) = 0. \]

**Reduction of a complete system to a Jacobian one:**

\[ B_i f = \sum_{k=1}^{n} c_{ik} A_k f, \quad \|c_{ik}\| \neq 0. \]

\( N - n = m; \ x_{n+1} = u_1, \ x_{n+2} = u_2, \ldots \ x_N = u_m; \)

\[ A_k f = \sum_{i=1}^{N} a_{ki} \frac{\partial f}{\partial x_i} = \sum_{i=1}^{m} a_{ki} \frac{\partial f}{\partial x_i} + U_k f = 0, \]

\[ U_k = \sum_{r=1}^{m} a_{k,n+r} \frac{\partial}{\partial u_r}. \]

\[ \sum_{i=1}^{n} a_{ki} \frac{\partial f}{\partial x_i} + U_k f = 0, \quad k = 1, 2, \ldots, n, \quad \|a_{ki}\| \neq 0; \]

\[ \sum_{i=1}^{n} a_{ki} \frac{\partial f}{\partial x_i} = -U_k f. \]

\[ \sum_{i=1}^{n} \alpha_{kr} a_{ki} \frac{\partial f}{\partial x_i} = -\alpha_{kr} U_k f, \]

where \( \alpha_{ri} = \frac{A_{ri}}{A} \) is the reciprocal element of \( a_{ri}; \)

\[ \sum_{i} \alpha_{ri} a_{ki} = \delta_{ik}, \quad \sum_{i} \alpha_{ki} a_{kr} = \delta_{ir}. \]

\[ \sum_{i=1}^{n} \sum_{r=1}^{n} \alpha_{kr} a_{ki} \frac{\partial f}{\partial x_i} = \sum_{i=1}^{n} \delta_{ir} \frac{\partial f}{\partial x_i} \]

\[ = \frac{\partial f}{\partial x_r} = -\sum_{r=1}^{n} \alpha_{kr} U_k f, \]
that is a Jacobian system. 
Conversely, let us start from a Jacobian system:
\[ \frac{\partial f}{\partial x_i} + \Omega_i f = 0, \quad i = 1, 2, \ldots, n, \]
where \( \Omega_i \) are linear operators depending only on \( u_1, \ldots, u_m \),
\[ \Omega_i = \sum_{\alpha=1}^{m} X_{i\alpha} \frac{\partial}{\partial u_\alpha}. \]
By setting:
\[ B_i = \frac{\partial}{\partial x_i} + \Omega_i, \]
we have:
\[ B_i f = 0, \quad i = 1, 2, \ldots, n, \]
\[ B_i = \sum_{k} \alpha_{ki} A_k. \]
The Poisson brackets of the \( B \) operators are linear combinations of the Poisson brackets of the \( A \) operators and of the \( A \) themselves, and since the \( A \) operators define a complete system and, in turn, are combinations of the \( B \) operators, we have:
\[ (B_i, B_k) = \sum_{\ell} q_{ik\ell} B_{\ell}. \]
\[ B_i = \frac{\partial}{\partial x_i} + \Omega_i, \quad B_k = \frac{\partial}{\partial x_k} + \Omega_k. \]
\[ B_i B_k = \frac{\partial^2}{\partial x_i \partial x_k} + \Omega_i \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_i} \Omega_k + \Omega_i \Omega_k, \]
\[ B_k B_i = \frac{\partial^2}{\partial x_i \partial x_k} + \Omega_k \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_k} \Omega_i + \Omega_k \Omega_i, \]
\[ (B_i, B_k) = \left( \Omega_i \frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_k} \Omega_i \right) - \left( \Omega_k \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} \Omega_i \right) + \Omega_i \Omega_k - \Omega_k \Omega_i \]
\[ = \Omega_{ik} = 0. \]
9.2. **ALGEBRAIC FOUNDATIONS OF THE TENSOR CALCULUS**

9.2.1 **Covariant And Contravariant Vectors**

Covariant:

\[ u'_i = u_k \frac{\partial x'_k}{\partial x_i}, \quad u''_i u'_k \frac{\partial x'_k}{\partial x''_i} = u_r \frac{\partial x'_r}{\partial x'_i} \frac{\partial x'_i}{\partial x''_r} = u_r \frac{\partial x'_r}{\partial x''_i} = u_k \frac{\partial x'_k}{\partial x''_i} \]

Contravariant:

\[ u''_i = u^k \frac{\partial x'_i}{\partial x^k}, \quad u''_i u^k \frac{\partial x''_i}{\partial x^k} = u^r \frac{\partial x''_r}{\partial x^i} \frac{\partial x^i}{\partial x''_r} = u^r \frac{\partial x''_r}{\partial x^i} = u^k \frac{\partial x''_i}{\partial x^k}. \]

9.3. **GEOMETRICAL INTRODUCTION TO THE THEORY OF DIFFERENTIAL QUADRATIC FORMS I**

9.3.1 **The Symbolic Equation Of Parallelism**

\[ dR \cdot \delta P = 0 \]

(\(\delta P\) taken on the surface);

\[ \sum_{\nu=1}^{3} dY_\nu \delta y_\nu = 0 \]

(\(\delta y_\nu\) being the most general ones).

9.3.2 **Intrinsic Equations Of Parallelism**

Deduction of the intrinsic equations:

\[ \delta y_\nu = \sum_{k=1}^{2} \frac{\partial y_\nu}{\partial x_k} \frac{\partial x_k}{\partial x}, \]

\[ Y_\nu = \sum_{i=1}^{2} \frac{\partial y_\nu}{\partial x_i} R^i \]
\( R^2 = \sum a_{ik} R^i R^k. \)

\[
\sum_{\nu=1}^{2} \frac{dy_\nu}{\partial x_i} \delta y_\nu = \sum_{\nu=1}^{3} \sum_{i=1}^{2} \left( \frac{\partial y_\nu}{\partial x_i} R^i \right) \sum_{k=1}^{2} \frac{\partial y_\nu}{\partial x_k} \delta x_k
\]

\[= \sum_k \tau_k \delta r_k = 0,\]

\[\tau_k = \sum_{\nu=1}^{3} \sum_{i=1}^{2} \frac{\partial y_\nu}{\partial x_k} \left( \frac{\partial y_\nu}{\partial x_i} R^i \right).\]

\[\tau = 0.\]

\[
\sum_{\nu=1}^{3} \frac{\partial y_\nu}{\partial x_k} \frac{\partial^2 y_\nu}{\partial x_i \partial x_j} = \sum_\nu \frac{\partial}{\partial x_j} \frac{\partial y_\nu}{\partial x_k} \frac{\partial y_\nu}{\partial x_i} - \sum_\nu \frac{\partial y_\nu}{\partial x_i} \frac{\partial^2 y_\nu}{\partial x_k \partial x_j}.
\]

\[
\begin{bmatrix}
i & j \\
k & \end{bmatrix} = \frac{\partial}{\partial x_j} a_{ik} - \begin{bmatrix}
k & j \\
i & \end{bmatrix},
\]

\[
\begin{bmatrix}
i & j \\
k & \end{bmatrix} + \begin{bmatrix}
j & k \\
i & \end{bmatrix} = \frac{\partial}{\partial x_j} a_{ik},
\]

\[
\begin{bmatrix}
j & k \\
i & \end{bmatrix} + \begin{bmatrix}
k & i \\
j & \end{bmatrix} = \frac{\partial}{\partial x_k} a_{ji},
\]

\[
\begin{bmatrix}
k & i \\
j & \end{bmatrix} + \begin{bmatrix}
i & j \\
k & \end{bmatrix} = \frac{\partial}{\partial x_i} a_{kj}.
\]
\[
\begin{bmatrix}
  i & j \\
  k &
\end{bmatrix} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} a_{ki} + \frac{\partial}{\partial x_j} a_{kj} - \frac{\partial}{\partial x_k} a_{ij} \right).
\]

\[dR \cdot \delta P = \sum \tau_k \delta x_k, \quad \tau_k = 0,
\]

\[\tau_k = \sum_{i=1}^{2} a_{ik} dR^i + \sum_{i,j=1}^{2} \begin{bmatrix}
  i & j \\
  k &
\end{bmatrix} R^i dx_j = 0.
\]

\(\tau_k\) is a covariant vector; in fact, \(\sum \tau_k \delta x_k = \text{invariant}.
\]

\[\tau^\ell = \sum_k a^{\ell k} \tau_k,
\]

\(\tau^\ell\) is a contravariant vector.

\[\tau^\ell = dR^\ell + \sum_{i,j=1}^{2} \begin{bmatrix}
  i & j \\
  \ell &
\end{bmatrix} R^i dx_j = 0.
\]

\[dR^\ell = - \sum_{i,j=1}^{2} \begin{bmatrix}
  i & j \\
  \ell &
\end{bmatrix} R^i dx_j
\]

(which is the equation of the parallelism).

\[9.3.3 \quad \text{Christoffel’s Symbols}
\]

\[
\begin{bmatrix}
  j & \ell \\
  k &
\end{bmatrix} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} a_{\ell k} + \frac{\partial}{\partial x_\ell} a_{kj} - \frac{\partial}{\partial x_k} a_{ij} \right),
\]

\[
\begin{bmatrix}
  j & \ell \\
  i &
\end{bmatrix} = \sum_k a^{\ell k} \begin{bmatrix}
  j & \ell \\
  k &
\end{bmatrix},
\]

\[
\begin{bmatrix}
  j & \ell \\
  k &
\end{bmatrix} = \begin{bmatrix}
  \ell & j \\
  k &
\end{bmatrix}, \quad \begin{bmatrix}
  j & \ell \\
  i &
\end{bmatrix} = \begin{bmatrix}
  \ell & j \\
  i &
\end{bmatrix} = \begin{bmatrix}
  \ell & j \\
  i &
\end{bmatrix};
\]

\[
\frac{\partial a_{ik}}{\partial x_j} = \begin{bmatrix}
  j & k \\
  i &
\end{bmatrix} + \begin{bmatrix}
  j & i \\
  k &
\end{bmatrix},
\]
\[ \begin{bmatrix} j & \ell \\ k \end{bmatrix} = \sum a_{ik} \begin{bmatrix} j \\ i \end{bmatrix}. \]

\[ a = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}. \]

\[ \frac{\partial a}{\partial x_i} = \sum_{r,s} \frac{\partial a_{rs}}{\partial x_i} a^{rs}, \]

\[ \frac{\partial \log a}{\partial x_i} = \sum_r \left\{ \frac{i}{r} \right\} + \sum_s \left\{ \frac{i}{s} \right\} = 2 \sum_r \left\{ \frac{i}{r} \right\}, \]

\[ \frac{\partial \log \sqrt{a}}{\partial x_i} = \sum_r \left\{ \frac{i}{r} \right\}. \]

### 9.3.4 Equations Of Parallelism In Terms Of Covariant Components

\[ dR^\ell = - \sum_{ij} \left\{ \begin{array}{c} i \\ j \\ \ell \end{array} \right\} R^i dx_j \quad \text{(contravariant components)}, \]

\[ R_s = \sum a_{s\ell} R_\ell, \]

\[ dR_s = \sum_\ell a_{s\ell} dR_\ell + \sum_\ell R_\ell da_{s\ell}, \]

\[ da_{s\ell} = \sum_\ell \frac{\partial a_{s\ell}}{\partial x_\ell} dx_\ell = \sum_\ell \left( \begin{bmatrix} t & s \\ \ell & s \end{bmatrix} + \begin{bmatrix} t & \ell \\ s \end{bmatrix} \right) R_\ell dx_\ell, \]
\[ dR_s = \sum_{\ell} a_{sl} dR^\ell + \sum_{\ell,t} R^\ell \left( \begin{bmatrix} t & s \\ \ell & \ell \end{bmatrix} + \begin{bmatrix} t & \ell \\ s & s \end{bmatrix} \right) dx_t \]

\[ = - \sum_{i,j} \begin{bmatrix} i & j \\ s & s \end{bmatrix} R^i dx_j + \sum_{\ell,t} R^\ell \left( \begin{bmatrix} t & s \\ \ell & \ell \end{bmatrix} + \begin{bmatrix} t & \ell \\ s & s \end{bmatrix} \right) dx_t \]

\[ = \sum_{\ell,t} R^\ell \begin{bmatrix} t & s \\ \ell & \ell \end{bmatrix} dx_t. \]

\[ \begin{bmatrix} t & s \\ \ell & \ell \end{bmatrix} = \sum_r a_{\ell r} \begin{bmatrix} t & s \\ r & r \end{bmatrix}, \]

\[ dR_s = \sum_{\ell,t,r} a_{\ell r} R^\ell dx_t \begin{bmatrix} t & s \\ r & r \end{bmatrix} = \sum_{\ell,r} R_r dx_t \begin{bmatrix} t & s \\ r & r \end{bmatrix}. \]

**Equations of the parallelism**

Contravariant components: \( dR^i = - \sum_{\ell,k} \begin{bmatrix} \ell & k \\ i & i \end{bmatrix} R^\ell dx_k \)

Covariant components: \( dR_i = \sum_{\ell,k} \begin{bmatrix} i & k \\ \ell & \ell \end{bmatrix} R_\ell dx_k \)

**9.3.5 Some Analytical Verifications**

\[ x_i = x_i(s), \quad i = 1, 2, \]

\[ \dot{R}^i = - \sum_{\ell=1}^{2} \begin{bmatrix} \ell & k \\ i & i \end{bmatrix} R^\ell \dot{x}_k \]

\[ \dot{V}^i = - \sum_{\ell,k} \begin{bmatrix} \ell & k \\ i & i \end{bmatrix} V^\ell \dot{x}_k; \]
\[
\sum_i \dot{R}^i V_i = - \sum_{i,\ell,k} \begin{bmatrix} \ell & k \\ i \end{bmatrix} R^\ell V_i \dot{x}_k;
\]

\[
\dot{R}^i = \sum_{\ell,k} \begin{bmatrix} i & k \\ \ell \end{bmatrix} R_\ell \dot{x}_k,
\]

\[
\dot{V}^i = \sum_{\ell,k} \begin{bmatrix} i & k \\ \ell \end{bmatrix} V_\ell \dot{x}_k;
\]

\[
\sum_i \dot{R}^i V_i = - \sum_{i,\ell,k} \begin{bmatrix} \ell & k \\ i \end{bmatrix} R^\ell V_i \dot{x}_k,
\]

\[
\sum_i R^i \dot{V}^i = \sum_{i,\ell,k} \begin{bmatrix} i & k \\ \ell \end{bmatrix} R^i V_\ell \dot{x}_k = \sum_{i,\ell,k} \begin{bmatrix} \ell & k \\ i \end{bmatrix} R^\ell V_i \dot{x}_k,
\]

\[
\frac{d}{ds} (R \cdot V) = \frac{d}{ds} \sum_i R^i V_i = \sum_i \dot{R}^i V_i + \sum_i R^i \dot{V}_i = 0.
\]

### 9.3.6 Permutability

\[
d\delta x_i = - \sum_{k,\ell} \begin{bmatrix} k & \ell \\ i \end{bmatrix} \delta x_k dx_\ell,
\]

\[
\delta x_i = \sum_{k,\ell} \begin{bmatrix} k & \ell \\ i \end{bmatrix} dx_k \delta x_\ell,
\]

\[
d\delta x_i = \delta dx_i.
\]

\[
x_i + dx_i + \delta x_i + d\delta x_i = x_i + \delta x_i + dx_i + \delta dx_i.
\]

### 9.3.7 Line Elements

\[
ds^2 = \sum_{i,k=1}^n a_{ik} dx_i dx_k.
\]

\[
\lambda^i = \frac{dx_i}{ds}, \quad \lambda_i = \sum_{k=1}^n a_{ik} \lambda^k, \quad \lambda^i = \sum a^{ik} \lambda_k,
\]
\[
\sum_{i,k=1}^{n} a_{ik} \lambda^i \lambda^k = \sum_{i=1}^{n} \lambda^i \lambda_i = \sum_{i,k=1}^{n} a^{ik} \lambda_i \lambda_k = 1,
\]

\[R^i = R \lambda^i, \quad R_i = R \lambda_i,\]

\[R^2 = \sum_{i,k=1}^{n} a_{ik} R^i R^k = \sum_{i=1}^{n} R^i R_i = \sum_{i,k=1}^{n} a^{ik} R_i R_k.\]

\[\cos \theta = \sum_{i,k=1}^{n} a_{ik} \lambda^i \mu^k = \sum_{i=1}^{n} \lambda^i \mu_i = \sum_{k=1}^{n} \lambda_k \mu_k = \sum_{i,k=1}^{n} a^{ik} \lambda_i \mu_k,\]

\[R \cdot V = \sum_{i=1}^{n} R^i V_i.\]

9.3.8 Euclidean Manifolds. Any \( V_n \) Can Always Be Considered As Immersed In A Euclidean Space

\( W_p \) (immersed in \( V_n \)):

\[x_i = f_i(u_i, \ldots, u_p) \quad (i = 1, 2, \ldots, n; \ p < n).\]

\[ds^2 = \sum_{i,k=1}^{n} a_{ik} dx_i dx_k = \sum_{i,k=1}^{n} \sum_{r,s=1}^{p} a_{ik} \frac{\partial x_i}{\partial u_r} \frac{\partial x_k}{\partial u_s} du_r du_s\]

\[= \sum_{r,s=1}^{p} b_{rs} du_r du_s,\]

\[b_{rs} = \sum_{i,k=1}^{n} a_{ik} \frac{\partial x_i}{\partial u_r} \frac{\partial x_k}{\partial u_s} .\]

An arbitrary \( V_n \) can always be considered as immersed in a Euclidean space. \( V_n \) immersed in \( S_N, \ N > n \).
$y_1(x), y_2(x), \ldots, y_N(x)$.

$$\sum_{i,k=1}^{n} a_{ik} dx_i dx_k = \sum_{\nu=1}^{N} dy_\nu^2,$$

$$dy_\nu = \sum_{i=1}^{n} \frac{\partial y_\nu}{\partial x_i} dx_i = \sum_{k=1}^{n} \frac{\partial y_\nu}{\partial x_k} dx_k,$$

$$dy_\nu^2 = \sum_{i,k=1}^{n} \frac{\partial y_\nu}{\partial x_i} \frac{\partial y_\nu}{\partial x_k} dx_i dx_k.$$

$$\sum_{i,k=1}^{n} a_{ik} dx_i dx_k = \sum_{\nu=1}^{N} \sum_{i,k=1}^{n} \frac{\partial y_\nu}{\partial x_i} \frac{\partial y_\nu}{\partial x_k} dx_i dx_k,$$

$$a_{ik} = \sum_{\nu=1}^{N} \frac{\partial y_\nu}{\partial x_i} \frac{\partial y_\nu}{\partial x_k}$$  \hspace{1cm} (i, k = 1, 2, \ldots, n).

If $N = \frac{n(n+1)}{2}$, the problem has a solution.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$N = n(n+1)/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

$C = \min(N - n)$,

$$\min N \leq \frac{n(n+1)}{2},$$

$$C \leq \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}.$$

$$R^2 = \sum a_{ik} R^i R^k, \quad V^2 = \sum a_{ik} V^i V^k,$$
\[ \sum a_{ik}(R^i + V^i)(R^k + V^k) = R^2 + V^2 + 2 \sum a_{ik}R^iV^k, \]
\[ R \cdot V = \sum_{i,k} a_{ik} R^i V^k = \sum_i R^i V_i = \sum_i R_i V^i = \sum_{i,k} a_{ik} R_i V_k. \]

For a definite form \( a \), and taking \( x_i \) and \( y_i \) not proportional, it follows:
\[ \left| \sum a_{ik} x_i y_k \right|^2 < \sum a_{ik} x_i x_k \cdot \sum a_{ik} y_i y_k. \]
\[ z_i = \lambda x_i + \mu y_i. \]
\[ \sum a_{ik} z_i z_k > 0, \]
\[ \lambda^2 \sum a_{ik} x_i x_k + 2\lambda \mu \sum a_{ik} x_i y_k + \mu^2 \sum a_{ik} y_i y_k > 0, \]
\[ \left( \sum a_{ik} x_i y_k \right)^2 < \sum a_{ik} x_i x_k \cdot \sum a_{ik} y_i y_k. \]
\[ \cos \theta = \sum_{i,k=1}^n a_{ik} \lambda^i \mu^k = \sum_{i=1}^n \lambda^i \mu_i = \sum_{i=1}^n \lambda_i \mu^i = \sum_{i,k=1}^n a^{ik} \lambda_i \mu_k, \]
\[ R \cdot V = \sum a_{ik} R^i V^k = \sum R^i V_i = \sum R_i V^i = \sum a^{ik} R_i V_k. \]

9.3.10 Coordinate Lines

For the coordinate line \( i \) (\( a_j = \text{constant for } j \neq i \)), the parameters \( \lambda^i \) are:
\[ \chi^j = \frac{dx_j}{ds} = \begin{cases} 0 & (j \neq i), \\ 1/\sqrt{a_{ii}} & (j = i). \end{cases} \]

The moments of the normal to the surface \( x_i = \text{constant} \) are:
\[ \mu_j = 0 \quad \text{for } j \neq i, \quad \mu_i = \frac{1}{\sqrt{a_{ii}}}. \]

The angle between the coordinate lines \( i \) and \( k \) is given by:
\[ \cos \theta = \sum a_{rs} \lambda^r \lambda^s = \frac{a_{ik}}{\sqrt{a_{ii}a_{kk}}}. \]
The angle between the hypersurfaces $x_i = \text{constant}$ and $x_k = \text{constant}$ is given by:

$$\cos \theta = \frac{a^{ik}}{\sqrt{a^{ii}a^{kk}}},$$

Let $\mathbf{s}_i$ be a unitary vector along the line $i$ (the parameters are then equal to the contravariant components: $\lambda^j = 0$ for $j \neq i$, $\lambda^i = 1/\sqrt{a_{ii}}$).

Let $\mathbf{n}_i$ be a unitary vector normal to the hypersurface $x_i = \text{constant}$ (the moments are then equal to the covariant components: $\mu_j = 0$ for $j \neq i$, $\mu_i = 1/\sqrt{a_{ii}}$).

$$R \cdot \mathbf{s}_i = \sum R_j (s_i)^j = \frac{R_i}{\sqrt{a_{ii}}},$$

$$R \cdot \mathbf{n}_i = \sum R^j n_{i,j} = \frac{R^i}{\sqrt{a_{ii}}};$$

$$R_i = \sqrt{a_{ii}} R \cdot \mathbf{s}_i, \quad R^i = \sqrt{a_{ii}} R \cdot \mathbf{n}_i.$$

**9.3.11 Differential Equations Of Geodesics**

$x_i = x_i(t)$:

$$s = \int \sqrt{a_{ik}dx_i dx_k} = \int ds.$$  

$$\mathcal{I} = \int_A^B ds = \int_A^B \sqrt{a_{ik}dx_i dx_k},$$

$$\delta \mathcal{I} = \int_A^B \delta ds.$$

$$ds^2 = \sum a_{ik}dx_i dx_k,$$

$$ds \delta ds = \sum a_{ik}dx_i d\delta x_k + \frac{1}{2} \sum \delta a_{ik} \cdot dx_i dx_k,$$

$$\delta a_{ik} = \sum \frac{\delta a_{ik}}{\delta x_j} \delta x_j,$$

$$\delta ds = \sum a_{ik} \dot{x}_i \delta x_k + \frac{1}{2} \sum_{i,k,j} \delta a_{ik} \frac{\delta x_j}{\delta x_j} \dot{x}_i \dot{x}_k ds.$$
\[ \delta I = \int \sum a_{ik} \dot{x}_i \delta x_k + \frac{1}{2} \int \sum \frac{\delta a_{ik}}{\delta x_j} \delta x_j \dot{x}_i \dot{x}_k ds. \]

\[ \int \sum a_{ik} \dot{x}_i d\delta x_k = \sum a_{ik} \dot{x}_i \delta x_k \bigg|_B^A - \int_A^B \sum a_{ik} (\dot{a}_{ik} \dot{x}_i + a_{ik} \ddot{x}_i) dt. \]

\[ \delta I = \int_A^B \sum_k \delta x_k \cdot \left( \frac{1}{2} \sum_{i,j} \frac{\partial a_{ij}}{\partial x_k} \dot{x}_i \dot{x}_j - \sum_i \dot{a}_{ik} \dot{x}_i - \sum_i a_{ik} \ddot{x}_i \right) ds. \]

\[ \dot{a}_{ik} = \sum_j \frac{\partial a_{ik}}{\partial x_j} \dot{x}_j, \]

\[ dI = \int_A^B \sum_k \delta x_k \left( \frac{1}{2} \sum_{i,j} \frac{\partial a_{ij}}{\partial x_k} \dot{x}_i \dot{x}_j - \sum_{i,j} \frac{\partial a_{ik}}{\partial x_j} \dot{x}_i \dot{x}_j - \sum_i a_{ik} \ddot{x}_i \right) ds. \]

\[ dI = -\int p_k \delta x_k ds, \quad \delta I + \int_A^B p_k \delta x_k ds = 0, \]

\[ p_k = \sum_{i,j} \begin{bmatrix} i \ j \ k \end{bmatrix} \dot{x}_i \dot{x}_j + \sum_i a_{ik} \ddot{x}_i. \]

\[ \sum_i a_{ik} \ddot{x}_i + \sum_{i,j} \begin{bmatrix} i \ j \ k \end{bmatrix} \ddot{x}_i \dot{x}_j = 0 \quad (k = 1, 2, \ldots, n). \]

\[ p^i = \sum_k a^{ik} p_k, \]

\[ p^k = \sum_{i,j} \begin{bmatrix} i \ j \ k \end{bmatrix} \ddot{x}_i \dot{x}_j + \ddot{x}_k. \]
Equations of the geodesic lines

\[ dI = - \int_{AB} \sum p_k \delta x_k ds, \quad p_k = \sum_{ij} \begin{bmatrix} i & j \\ k \end{bmatrix} \dot{x}_i \dot{x}_j + \sum_i a_{ij} \ddot{x}_i \]

\[ p^i = \sum a^{ik} x_k, \quad p^k = \sum_{ij} \begin{bmatrix} i & j \\ k \end{bmatrix} \dot{x}_i \dot{x}_j + \ddot{a}_k \]

\[ p_k = 0, \quad \text{that is:} \quad \sum_{i,j=1}^{n} \begin{bmatrix} i & j \\ k \end{bmatrix} \dot{x}_i \dot{x}_j + \sum_{i=1}^{n} a_{ik} \ddot{x}_i = 0 \]

\[ (k = 1, 2, \ldots, n), \]

or

\[ p^k = 0, \quad \text{that is:} \quad \ddot{x}_k + \sum_{i,j=1}^{n} \begin{bmatrix} i & j \\ k \end{bmatrix} \dot{x}_i \dot{x}_j = 0 \]

\[ (k = 1, 2, \ldots, n). \]

### 9.3.12 Application

\[ ds^2 = dx_1^2 + r^2 dx_2^2, \]

\[ a_{11} = 1, \quad a_{22} = r^2, \quad a_{12} = 0; \]

\[ a^{11} = 1, \quad a^{22} = \frac{1}{r^2}, \quad a^{12} = 0. \]

\[ \frac{\partial a_{11}}{\partial x_1} = \frac{\partial a_{11}}{\partial x_2} = 0, \quad \frac{\partial a_{22}}{\partial x_1} = 2rr', \quad \frac{\partial a_{22}}{\partial x_2} = 0, \quad \frac{\partial a_{12}}{\partial x_1} = \frac{\partial a_{12}}{\partial x_2} = 0. \]

\[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \left( \frac{\partial a_{11}}{\partial x_1} + \frac{\partial a_{11}}{\partial x_1} - \frac{\partial a_{11}}{\partial x_1} \right) = 0, \]

\[ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \left( \frac{\partial a_{11}}{\partial x_2} + \frac{\partial a_{12}}{\partial x_1} - \frac{\partial a_{12}}{\partial x_1} \right) = 0, \]
\[
\begin{bmatrix}
2 & 1 \\
0 & 1
\end{bmatrix} = -rr', \\
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix} = 0,
\]
\[
\begin{bmatrix}
1 & 2 \\
2 & 0
\end{bmatrix} = rr', \\
\begin{bmatrix}
2 & 0 \\
0 & 0
\end{bmatrix} = 0.
\]

\[
\left\{ \begin{array}{c}
1 & 1 \\
0 & 1
\end{array} \right\} = 0, \\
\left\{ \begin{array}{c}
1 & 2 \\
0 & 1
\end{array} \right\} = 0,
\]
\[
\left\{ \begin{array}{c}
2 & 2 \\
0 & 1
\end{array} \right\} = -rr', \\
\left\{ \begin{array}{c}
1 & 1 \\
0 & 2
\end{array} \right\} = 0,
\]
\[
\left\{ \begin{array}{c}
1 & 2 \\
2 & 0
\end{array} \right\} = \frac{r'}{r}, \\
\left\{ \begin{array}{c}
2 & 0 \\
0 & 0
\end{array} \right\} = 0.
\]

\[\ddot{x}_1 - r \frac{dr}{dx_1} \dot{x}_2^2 = 0, \quad r^2 \ddot{x}_2 + 2r \frac{dr}{dx_1} \dot{x}_1 \dot{x}_2 = 0,\]

or

\[\ddot{x}_1 - r \frac{dr}{dx_1} \dot{x}_2^2 = 0, \quad \ddot{x}_2 + \frac{2}{r} \frac{dr}{dx_1} \dot{x}_1 \dot{x}_2 = 0.\]

\[\sin \alpha = r \dot{x}_2, \quad r \sin \alpha = r^2 \dot{x}_2,\]

\[\frac{d}{ds} (r \sin \alpha) = 2r \frac{dr}{dx_1} \dot{x}_1 \dot{x}_2 + r^2 \ddot{x}_2 = 0,\]

\[r^2 \dot{x}_2 = \text{constant}.\]

\[
\left\{ \begin{array}{c}
\ddot{x}_1 = r \frac{dr}{dx_1} \dot{x}_2^2, \\
\ddot{x}_1 = r \frac{dr}{dx_1} \frac{c^2}{r^4}, \\
\ddot{x}_2 = -\frac{2}{r} \frac{dr}{dx_1} \dot{x}_2 \dot{x}_2, \\
\ddot{x}_2 = \frac{c^2}{r^4},
\end{array} \right\}
\]
9.4. GEOMETRICAL INTRODUCTION TO THE THEORY OF DIFFERENTIAL QUADRATIC FORMS II

9.4.1 Geodesic Curvature

\[ x_i = x_i(s); \ x_1, x_2, \ldots, x_n \] are the coordinates in the space \( V_n \).

\[
dI = -\int_{AB} \sum_k p_k \delta x_k ds
= -\int_{AB} \sum_k \bar{p}_k \delta \bar{x}_k ds;
\]

\[
\int \sum_k (p_k dx_k - \bar{p}_k \delta \bar{x}_k) ds = 0,
\]

\[
\sum_k p_k \delta x_k = \sum_k \bar{p}_k \delta \bar{x}_k.
\]

\[
p_k = \sum_i a_{ki} \ddot{x}_i + \sum_{i,j} \left[ \begin{array}{c} i \\ j \\ k \end{array} \right] \dot{x}_i \dot{x}_j, \quad \text{covariant components};
\]

\[
p^k = \ddot{x}_i + \sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} \dot{x}_i \dot{x}_j, \quad \text{contravariant components}.
\]

9.4.2 Vector Displacement

\[
\begin{array}{c|ccc}
\text{parallel displacement} & s & s + ds \\
\hline
\dot{x}^i & \dot{x}_i - \sum_{i,k} \left\{ \begin{array}{c} l \\ k \\ i \end{array} \right\} \ddot{x}_l \dot{x}_k ds = u_i
\end{array}
\]

\[
\text{line displacement} & \dot{x}_i & \dot{x}_i + \ddot{x}_i ds = v_i
\]

\[ @ In the original manuscript, a reference appears (p. 154) of a unspecified text. \]
\[ v_i - u_i = \left[ \sum_{\ell,k} \left\{ \begin{array}{cc} \ell & k \\ i & \end{array} \right\} \dot{x}_l \dot{x}_k \right] ds = p^i ds. \]

\[ u_i, \quad u_i + p^i ds; \]
\[ \dot{x}_i, \quad \dot{x}_i + p^i ds. \]

\[ \sum a_{ik} \dot{x}_i \dot{x}_k = 1. \]

\[ \sum a_{ik} (\dot{a}_{ik} (\dot{a}_i + p^i ds) (\dot{x}_k + p^k ds) = 1 + 2 \sum a_{ik} \dot{x}_i p^k ds + \sum a_{ik} p^k p^k ds^2, \]

\[ \sum_{i,k} a_{ik} \dot{x}_i p^k = \sum_{i,k} a_{ik} \dot{x}_i \dot{x}_k + \sum_{i,k,\ell,m} a_{ik} \left\{ \begin{array}{cc} \ell & m \\ k & \end{array} \right\} \dot{x}_\ell \dot{x}_m \dot{x}_i \]

\[ = \sum_{i,k} a_{ik} \dot{x}_i \dot{x}_k + \sum_{i,\ell,m} \left[ \begin{array}{cc} \ell & m \\ i & \end{array} \right] \dot{x}_\ell \dot{x}_m \dot{x}_i \]

\[ = \frac{1}{2} \left[ \sum_{i,k} a_{ik} \frac{d}{ds} (\dot{x}_i \dot{x}_k) \right. \]

\[ + \sum_{i,k,\ell} \dot{x}_i \dot{x}_k \left( \frac{\partial a_{\ell k}}{\partial x_k} + \frac{\partial a_{k \ell}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_\ell} \right) \dot{x}_\ell \]

\[ = \frac{1}{2} \left[ \sum_{i,k} a_{ik} \frac{d}{ds} (\dot{x}_i \dot{x}_k) \right. \]

\[ + \sum_{i,k} \dot{x}_i \dot{x}_k \left( - \frac{\partial a_{ik}}{\partial s} + \frac{\partial a_{ik}}{\partial s} + \frac{\partial a_{ik}}{\partial s} \right) \]

\[ = \frac{1}{2} \frac{d}{ds} \sum a_{ik} \dot{x}_i \dot{x}_k = 0. \]

\[ \sum_{i,k} a_{ik} p^i = \rho^2. \]
\[ t \cdot t = 1, \quad t \cdot (t + \rho \, ds) = 1, \]
\[ (t + \rho \, ds)(t + \rho \, ds) = 1 + \rho^2 ds^2; \]
\[ \cos(t, t + \rho \, ds) = \frac{1}{1 + (1/2)\rho^2 ds} = 1 - \frac{1}{2} \rho^2 ds, \]
\[ \sin(t, t + \rho \, ds) = \rho \, ds. \]

9.4.3 Autoparallelism Of Geodesics

\[ p_k = \sum a_{ik} \ddot{x}_i + \sum_{i,j} \left[ \begin{array}{c} i \\ j \\ k \end{array} \right] \dot{x}_i \dot{x}_j = 0, \]
\[ p^k = \ddot{x}_k + \sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} \dot{x}_i \dot{x}_j = 0. \]

\[ \lambda^i = \dot{x}_i, \]
\[ d\lambda^k = \ddot{x} \, ds = -\sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} \lambda^i dx_j \quad \text{(antiparallelism)} \]

9.4.4 Associated Vectors

\[ V^k = \frac{dR^k}{ds} + \sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} R^i \dot{x}_j = \frac{\tau^k}{ds} \]
\[ \tau^k = 0: \quad dR^k + \sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} R^i dx_j = 0 \quad \text{(parallelism)}; \]
for \( R^k = \dot{x}_k : \)
\[ V^k = p^k = \ddot{x}_k + \sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} \dot{x}_i \dot{x}_j \quad \text{(geodesic curvature)}; \]
\[ p_k = \ddot{x}_k + \sum_{i,j} \left\{ \begin{array}{c} i \\ j \\ k \end{array} \right\} \dot{x}_i \dot{x}_j = 0 \quad \text{(equation of the geodesic lines)}. \]
9.4.5 Remarks On The Case Of An Indefinite $ds^2$

\[ ds^2 = \sum a_{ik} dx_i dx_k, \quad \|a_{ik}\| \neq 0. \]

- time directions: $ds^2 > 0 (\infty^{n-1})$;
- space directions: $ds^2 < 0 (\infty^{n-1})$;
- null interval directions: $ds^2 = 0 (\infty^{n-1})$.

9.5. COVARIANT DIFFERENTIATION. INVARIANTS AND DIFFERENTIAL PARAMETERS. LOCALLY GEODESIC COORDINATES

9.5.1 Geodesic Coordinates

$x_i = x_i(x_1, x_2, \ldots, x_n)$ ($i = 1, 2, \ldots, n$).

\[ \partial a_{ik} \over \partial x_j = 0 \quad (i, k, j = 1, 2, \ldots, n). \]

\[ P = P_0(x_1^0, x_2^0, \ldots, x_n^0) = \overline{P}_0(\overline{x}_1^0, \overline{x}_2^0, \ldots, \overline{x}_n^0) \]

\[ \overline{a}_{ik} = \sum_{r,s} a_{rs} \frac{\partial x_r}{\partial \overline{x}_i} \frac{\partial x_s}{\partial \overline{x}_k}, \]

\[ \begin{aligned}
\frac{\partial \overline{a}_{ik}}{\partial \overline{a}_{jl}} & = \sum_{r,s,t} a_{rs} \frac{\partial x_r}{\partial x_t} \frac{\partial x_s}{\partial \overline{x}_i} \frac{\partial x_t}{\partial \overline{x}_k} + \sum_{r,s} a_{rs} \frac{\partial^2 x_r}{\partial x_i \partial x_j} \frac{\partial x_s}{\partial \overline{x}_k} \\
& \quad + \sum_{r,s} a_{rs} \frac{\partial x_r}{\partial \overline{x}_i} \frac{\partial^2 x_s}{\partial x_k \partial \overline{x}_j}. 
\end{aligned} \]

\[ \frac{\partial x_i}{\partial \overline{x}_k} = a_{ik}, \quad dx_i = \sum a_{ik} d\overline{x}_k, \]

\[ dx = Sd\overline{x}. \]

\[ \overline{x} = Ux', \quad d\overline{x} = Ud\overline{x}'. \]
\( \text{d}x = S U \text{d}x \).

\[
\begin{array}{c|c}
P & P_0 \\
\hline
x_r = \bar{x}_r + \frac{1}{2} \sum_{i,k} q_{ik}^r \bar{x}_i \bar{x}_k, & q_{ik}^r = q_{ki}^r, \\
\frac{\partial x_r}{\partial \bar{x}_j} = \delta_{rj} + \sum_k q_{jk}^r \bar{x}_k, & (\frac{\partial x_r}{\partial \bar{x}_j})_0 = 1, \\
\frac{\partial^2 x_r}{\partial \bar{x}_j \partial \bar{x}_\ell} = q_{j\ell}^r, & (\frac{\partial^2 x_r}{\partial \bar{x}_j \partial \bar{x}_\ell})_0 = q_{j\ell}^r.
\end{array}
\]

\[
\left( \frac{\partial a_{ik}}{\partial \bar{x}_j} \right)_0 = \left( \frac{\partial a_{ik}}{\partial x_j} \right)_0 + \sum_r a_{rk} \left( \frac{\partial^2 x_r}{\partial \bar{x}_i \partial \bar{x}_j} \right)_0 + \sum_s a_{is} \left( \frac{\partial^2 x_s}{\partial \bar{x}_k \partial \bar{x}_j} \right)_0
\]

\[
\sum_r a_{ir} q_{kj}^r + \sum_r a_{kr} q_{ij}^r = - \left( \frac{\partial a_{ik}}{\partial x_j} \right)_0,
\]

\[
\sum_r a_{kr} q_{ji}^r + \sum_r a_{jr} q_{ki}^r = - \left( \frac{\partial a_{kj}}{\partial x_i} \right)_0,
\]

\[
\sum_r a_{jr} q_{ik}^r + \sum_r a_{ir} q_{jk}^r = - \left( \frac{\partial a_{ji}}{\partial x_k} \right)_0.
\]

\[
\sum_r a_{ir} q_{kj}^r = \frac{1}{2} \left\{ \left( \frac{\partial a_{ik}}{\partial x_j} \right)_0 + \left( \frac{\partial a_{ji}}{\partial x_k} \right)_0 - \left( \frac{\partial a_{kj}}{\partial x_i} \right)_0 \right\} = \left[ \begin{array}{c} k \\ j \end{array} \right]_0.
\]

\[\sum_i a^{si} a_{ir} = \delta_{rs}.\]

\[
q^s_{kj} = \left\{ \begin{array}{c} k \\ j \end{array} \right\}_0 = \left( \frac{\partial^r x_s}{\partial \bar{x}_k \partial \bar{x}_j} \right)_0.
\]
geodesic coordinates $\bar{x}_i$ for the point $x_i = \bar{x}_i = 0$

$$d\bar{x}_i = dx_i + \frac{1}{2} \sum_{k,j} \begin{pmatrix} k & j \end{pmatrix}_i dx_k dx_j,$$

$$dx_i = d\bar{x}_i - \frac{1}{2} \sum_{k,j} \begin{pmatrix} k & j \end{pmatrix}_i d\bar{x}_k d\bar{x}_j + \ldots,$$

$$\frac{\partial^2 x_i}{\partial \bar{x}_k \partial \bar{x}_j} = - \begin{pmatrix} k & j \end{pmatrix}_{i}, \quad \frac{\partial^2 \bar{x}_i}{\partial x_k \partial x_j} = \begin{pmatrix} k & j \end{pmatrix}_i.$$  

9.5.1.1 Applications. 1° parallelism: $(dR = 0)$, $\left(\bar{R}_0^i\right) = (R^i)_0$, $(d\bar{x}_i)_0 = (dx_i)_0$.

$$R^i = \sum_k R^k_i \frac{\partial x_i}{\partial \bar{x}_k},$$

$$dR^i = - \sum_{k,j} \begin{pmatrix} k & j \end{pmatrix}_i R^k dx_j,$$  

covariant components.

$$R_i = \sum_k R_k \frac{\partial \bar{x}_k}{\partial x_i}, \quad (R_i)_0 = (\bar{R}_i)_0,$$

$$dR_i = \sum_{k,j} \begin{pmatrix} i & j \end{pmatrix}_k R_k dx_j,$$  

covariant components.
$2^\circ$ geodesic lines:

\[
\left( \frac{dx_i}{ds} \right)_0 = \left( \frac{d\pi_i}{ds} \right)_0.
\]

\[
\frac{dx_i}{ds} = \sum_k \frac{\partial x_i}{\partial \pi_k} \frac{d\pi_k}{ds},
\]

\[
\frac{d^2x_i}{ds^2} = \sum_k \frac{\partial x_i}{\partial \pi_k} \frac{d^2\pi_k}{ds^2} + \sum_{k,j} \frac{\partial^2 x_i}{\partial \pi_k \partial \pi_j} \frac{d\pi_k}{ds} \frac{d\pi_j}{ds}.
\]

\[
\frac{d^2\pi_k}{ds} = 0.
\]

\[
\ddot{x}_i + \sum_{k,j} \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \dot{x}_k \dot{x}_j = 0.
\]

\[
\sum a_{ir} \ddot{x}_r + \sum_{k,j} \left[ \begin{array}{c} k \\ j \\ r \end{array} \right] \dot{x}_k \dot{x}_j = 0.
\]

$3^\circ$ geodesic curvature:

\[
p^k = \frac{d^2\pi_k}{ds^2} = \ddot{x}_k + \sum_{k,j} \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \dot{x}_k \dot{x}_j.
\]

$4^\circ$ Associated vectors:

\[
V^i = \frac{d\bar{R}^i}{ds}, \quad \bar{R}^i = \sum_k \frac{R^k d\pi_i}{dx_k}, \quad \frac{d\bar{R}^i}{ds} = \frac{dR^i}{ds} + \sum_{k,j} \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} R^k \dot{x}_j.
\]

$5^\circ$ Covariant differentiation:

\[
A^{i_1...i_\mu}_{k_1...k_m | r} = \frac{\partial}{\partial \pi_r} \bar{A}^{i_1...i_\mu}_{k_1...k_m}.
\]

\[
\bar{A}^{i_1...i_\mu}_{k_1...k_m} = \sum_{p,q} A^{p_1...p_\mu}_{q_1...q_m} \frac{\partial x_{i_1}}{\partial \pi_{p_1}} \ldots \frac{\partial x_{i_\mu}}{\partial \pi_{p_\mu}} \frac{\partial x_{q_1}}{\partial \pi_{k_1}} \ldots \frac{\partial x_{q_\mu}}{\partial \pi_{k_\mu}}.
\]
\[ A^{i_1 \ldots i_\mu}_{k_1 \ldots k_m | r} = \frac{\partial}{\partial x_r} A^{i_1 \ldots i_\mu}_{k_1 \ldots k_m} + \sum_{p} A^{i_2 \ldots i_\mu}_{k_1 \ldots k_m} \left\{ \begin{array}{c} p \\ r \end{array} \right\} + \ldots \]

\[ - \sum_{p} A^{i_1 \ldots i_\mu}_{p, k_2 \ldots k_m} \left\{ \begin{array}{c} k_1 \\ r \end{array} \right\} + \ldots \]

\[ A^{k_1 \ldots k_\mu}_{i_1 \ldots i_m | l} = \frac{\partial}{\partial x_l} A^{k_1 \ldots k_\mu}_{i_1 \ldots i_m} \]

\[ = \frac{\partial A^{k_1 \ldots k_\mu}_{i_1 \ldots i_m}}{\partial x_l} + \sum_{j} A^{k_1 \ldots k_{r-1} j k_{r+1} \ldots k_\mu}_{i_1 \ldots i_m} \left\{ \begin{array}{c} j \\ \ell \end{array} \right\} \]

\[ - \sum_{j} A^{k_1 \ldots k_\mu}_{i_1 \ldots i_{\rho-1} j i_{\rho+1} \ldots i_m} \left\{ \begin{array}{c} i_\rho \\ \ell \end{array} \right\} \]

\[ A^{i_1 \ldots i_\mu}_{k_1 \ldots k_m | l} = \frac{\partial}{\partial x_l} A^{i_1 \ldots i_\mu}_{k_1 \ldots k_m} \]

\[ = \frac{\partial A^{i_1 \ldots i_\mu}_{k_1 \ldots k_m}}{\partial x_l} + \sum_{j} A^{i_1 \ldots i_m}_{k_1 \ldots k_{r-1} j k_{r+1} \ldots k_\mu} \left\{ \begin{array}{c} j \\ \ell \end{array} \right\} \]

\[ - \sum_{j} A^{i_1 \ldots i_m}_{i_1 \ldots i_{\rho-1} j i_{\rho+1} \ldots i_\mu} \left\{ \begin{array}{c} i_\rho \\ j \end{array} \right\} \]

9.5.2 Particular Cases

1) \[ A_{i | k} = \frac{\partial A_i}{\partial x_k} - \sum_{p=1}^{n} \left\{ \begin{array}{c} i \\ p \end{array} \right\} A_p, \]

\[ A_{i | k} - A_{k | i} = \frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i}. \]

2) \[ A^{i}_{i | k} = \frac{\partial A^i}{\partial x_k} + \sum_{p=1}^{n} A^p \left\{ \begin{array}{c} p \\ i \end{array} \right\}. \]

3) \[ f_{i | i} = \frac{\partial f}{\partial x_i} = f_i. \]

\[ f_{i | k} = f_{i k} = f_{i i | k} = \frac{\partial^2 f}{\partial x_i \partial x_k} - \sum_{p} f_p \left\{ \begin{array}{c} i \\ p \end{array} \right\}. \]

\[ f_{i k} = f_{k i}. \]
4) 
\[ A_{ik|j} = \frac{\partial A_{ik}}{\partial x_j} - \sum_{p=1}^{n} A_{pk} \begin{bmatrix} i \\ p \end{bmatrix} - \sum_{p=1}^{n} A_{ip} \begin{bmatrix} k \\ p \end{bmatrix}. \]

5) 
\[ A_{ik|j} = \frac{\partial A_{ik}}{\partial x_j} + \sum_{p=1}^{n} A_{pk} \begin{bmatrix} p \\ i \end{bmatrix} + \sum_{p=1}^{n} A_{ip} \begin{bmatrix} p \\ k \end{bmatrix}. \]

6) 
\[ a_{ik|j} = \frac{\partial a_{ik}}{\partial x_j} - \sum_{p=1}^{n} a_{pk} \begin{bmatrix} i \\ p \end{bmatrix} - \sum_{p=1}^{n} a_{ip} \begin{bmatrix} k \\ p \end{bmatrix} = \frac{\partial a_{ik}}{\partial x_j} - \begin{bmatrix} i \\ j \\ k \end{bmatrix} - \begin{bmatrix} k \\ j \\ i \end{bmatrix} = 0 \quad \text{(Ricci lemma).} \]

9.5.3 Applications

\[ V^i = \sum a^{ik} V_k, \quad V_i = \sum a_{ik} V^k; \]
\[ V^{|i} = \sum a^{ik} V_{k|j}, \quad V_{i|j} = \sum a_{ik} V^{k}_{|j}. \]

Covariant derivative of the scalar product:
\[ \chi = U \cdot V = \sum U^i V_i = \sum U_i V^i = \sum a_{ik} U^i V^k = \sum a_{ik} U_i V_k. \]
\[ \chi_j = \sum_i (U^i_{|j} V_i + U^i V_{i|j}), \]
\[ \sum_i U^i_{|j} V_i = \sum_{k=1}^{n} a^{ik} U_{k|j} V_i = \sum_{i=1}^{n} U_{i|j} V^i, \]
\[ \chi_j = \sum_i (U_{i|j} V^i + U^i V_{i|j}). \]

\[ U = V: \]
\[ \chi_j = 2 \sum U_{i|j} U^i. \]
## 9.5.4 Divergence Of A Vector

\( \Theta = \sum_{i,j=1}^{n} a^{ij} X_{i|j} = \sum_{i} X_{i|i} \),

\( X_{i|j} = \sum_{k} a_{ik} X_{k|j} ^{k} \),

\( \Theta = \sum_{i,j,k}^{n} a^{ij} a_{ik} X_{j} ^{k} = \sum_{j,k=1}^{n} \delta_{jk} X_{j} ^{k} = \sum_{k=1}^{n} X_{k} |_{k} \).

\( X_{i|i} = \frac{\partial X_{i}}{\partial x_{i}} + \sum_{p} X_{p} \left\{ \begin{array}{c} p \\ i \end{array} \right\} \),

\( \Theta = \sum_{i} X_{i|i} = \sum_{i=1}^{n} \frac{\partial X_{i}}{\partial x_{i}} + \sum_{i,p=1}^{n} X_{p} \left\{ \begin{array}{c} p \\ i \end{array} \right\} \).

\( \frac{1}{a} da = \sum a^{ki} da_{ik} \).

dx \rightarrow da:

\( \frac{1}{a} \frac{\partial a}{\partial x_{r}} = \sum_{k,i} a^{ki} \frac{\partial a_{ik}}{\partial x_{r}} = \sum_{i,k} a^{ki} \left[ \begin{array}{c} i \\ r \end{array} \right] + \sum_{i,k} a^{ki} \left[ \begin{array}{c} k \\ r \end{array} \right] \)

\( \frac{1}{a} \frac{\partial a}{\partial x_{r}} = 2 \sum_{i,k} a_{ik} \left[ \begin{array}{c} i \\ r \end{array} \right] \).

\( \frac{\partial \log \sqrt{a}}{\partial x_{r}} = \sum_{i=1}^{n} \frac{\partial a_{ik}}{\partial x_{r}} \left[ \begin{array}{c} i \\ k \end{array} \right] = \sum_{i=1}^{n} \left\{ \begin{array}{c} i \\ i \end{array} \right\} \).

\( \sum_{i,p=1}^{n} \left\{ \begin{array}{c} p \\ i \end{array} \right\} X_{p} = \sum_{p=1}^{n} \frac{\partial \log \sqrt{a}}{\partial x_{p}} X_{p} \).

\( \Theta = \sum_{i=1}^{n} \left( \frac{\partial X_{i}}{\partial x_{i}} + \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial x_{i}} X_{i} \right) \),

\( \Theta = \frac{1}{\sqrt{a}} \sum_{i=1}^{n} \frac{\partial \sqrt{a} X_{i}}{\partial x_{i}} \).
Special case:

\[ X = \nabla u, \quad X_i = \frac{\partial u}{\partial x_i}. \]

\[ \nabla \cdot X = \nabla^2 u. \]

\[ \nabla^2 u = \sum_{i,k=1}^{n} a^{ik} u_{ik} = \sum_{i=1}^{n} u_{i|i} = \frac{1}{\sqrt{a}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sqrt{a} u^i, \]

where

\[ u_i = \sum_{k=1}^{n} a^{ik} u_k = \sum_{k=1}^{n} a^{ik} \frac{\partial u}{\partial x_k}. \]

### 9.5.5 Divergence Of A Double (Contravariant) Tensor

Given \( X^{ik} \):

\[ Y^i = \sum_{k=1}^{n} X^{ik} |^k \]

(which, in general, is different from \( \sum_{k=1}^{n} X^{ki} |^k \)),

\[ Y_i = \sum_{k,\ell=1}^{n} a^{k\ell} X_{ik}|^\ell. \]

\[ Y_i = \sum_{r} a_{ir} Y^r = \sum_{r,k=1}^{n} a_{ir} X^{rk} |^k = \sum_{k=1}^{n} X^{k} |^k = \sum_{\ell,k} a^{k\ell} X_{i\ell}|^k \]

\[ = \sum_{\ell,k} a^{k\ell} X_{ik}|^\ell. \]

Coming back to

\[ Y^i = \sum_{k=1}^{n} X^{ik} |^k, \]

let us suppose \( X \) to be antisymmetric:

\[ X^{ik} + X^{ki} = 0. \]

\[ X^{ik}|^j = \frac{\partial X^{ik}}{\partial x_j} + \sum_{p} \left\{ \begin{array}{c} p \cr i \end{array} \right\} X^{pk} + \sum_{p} \left\{ \begin{array}{c} p \cr k \end{array} \right\} X^{ip}, \]
\[ X^i_k |_k = \frac{\partial X^i_k}{\partial x_k} + \sum_p \begin{bmatrix} p & k \\ i & \end{bmatrix} X^{pk} + \sum_p \begin{bmatrix} p & k \\ k & \end{bmatrix} X^{ip}. \]

\[ \sum_{p,k} \begin{bmatrix} p & k \\ i & \end{bmatrix} X^{pk} = 0 \]

(if \( X \) is antisymmetric).

\[ Y^i = \sum_k \frac{\partial X^i_k}{\partial x_k} + \sum_{p,k} \begin{bmatrix} p & k \\ k & \end{bmatrix} X^{ip}. \]

\[ \sum_k \begin{bmatrix} p & k \\ k & \end{bmatrix} = \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial x_p}. \]

\[ Y^i = \sum_k \left( \frac{\partial X^i_k}{\partial x_k} + \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial x_k} X^i_k \right) \]

\[ = \frac{1}{\sqrt{a}} \sum_k \left( \sqrt{a} \frac{\partial X^i_k}{\partial x_k} + X^i_k \frac{\partial \sqrt{a}}{\partial x_k} \right) \]

\[ = \frac{1}{\sqrt{a}} \sum_{k=1}^{n} \frac{\partial (\sqrt{a}X^i_k)}{\partial x_k}. \]

### 9.5.6 Some Laws Of Transformation

For \( n \) covariant systems \( \lambda_{a|i} \) (\( i \) is the covariance index; \( \alpha \) is the ordering number of the system):

\[ \nabla = |\lambda_{a|i}|, \quad \nabla = |\bar{\lambda}_{a|i}|, \]

\[ \nabla = \nabla D, \quad D = \begin{pmatrix} x_1 \ldots x_n \\ \bar{x}_1 \ldots \bar{x}_n \end{pmatrix}, \]

\[ d\bar{x} = Sdx. \]

\[ \bar{\lambda}_\alpha = S^{-1}\lambda_{a}. \]

\[ P = \|p_{i\alpha}\|, \quad p_{i\alpha} = \lambda_{a|i}, \]

\[ \overline{P} = S^{-1}P, \quad x = S^{-1}\bar{x}. \]
\[
\sum a_{ik} \, dx_i \, dx_k = \sum \overline{a}_{ik} \, d\overline{x}_i \, d\overline{x}_k,
\]
\[
dx^* \, a \, dx = d\overline{x}^* \, \overline{a} \, d\overline{x},
\]
\[
dx^* \, a \, dx = (S \, dx)^* \, a \, S \, dx = dx^* \, S^* \overline{a} \, S \, dx.
\]
\[
a = S^* \overline{a} \, S, \quad \overline{a} = S^{-1} \, a \, S^{-1}.
\]
\[
\overline{a} = aD^2, \quad \nabla = \nabla D, \quad \frac{\nabla}{\pm \sqrt{a}} = \pm \frac{\nabla}{\sqrt{a}}.
\]

### 9.5.7 \(\varepsilon\) Systems

**Contravariant \(\varepsilon\) system:**
\[
\frac{1}{\sqrt{a}} \sum_{i_1 \ldots i_n=1}^{n} \lambda_{1|i_1} \lambda_{2|i_2} \ldots \lambda_{n|i_n} = \sum_{i_1 \ldots i_n=1}^{n} \varepsilon^{i_1,i_2,\ldots,i_n} \lambda_{1|i_1} \lambda_{2|i_2} \ldots \lambda_{n|i_n} = \text{invariant},
\]
\(\varepsilon^{i_1 \ldots i_n}\) is an antisymmetric contravariant tensor:
\[
\varepsilon^{i_1 \ldots i_n} = \begin{cases} 
0 & \text{if } i_n \text{ are not all different each other}, \\
\frac{1}{\sqrt{a}} & \text{if } i_n \text{ form an even permutation of } 1, 2, \ldots, n, \\
-\frac{1}{\sqrt{a}} & \text{if } i_n \text{ form an odd permutation of } 1, 2, \ldots, n.
\end{cases}
\]

**Covariant \(\varepsilon\) system (it is the reciprocal of the previous one):**
\[
\varepsilon_{i_1 \ldots i_n} = \begin{cases} 
0 & \ldots \\
\sqrt{a} & \ldots \\
-\sqrt{a} & \ldots
\end{cases}
\]
\[
\varepsilon^{i_1 \ldots i_n} = \sum_{k_1 \ldots k_n} a_{i_1 k_1} a_{i_2 k_2} \ldots a_{i_n k_n} \varepsilon_{k_1 \ldots k_n} = a \varepsilon^{i_1 \ldots i_n}.
\]
9.5.8 Vector Product

Vector product of $v_1 \ldots v_{n-1}$:

$$w^i = \sum_{i_1 \ldots i_n = 1}^{n} \varepsilon^{i_1 \ldots i_{n-1} i} v_1^{i_1} \ldots v_{n-1}^{i_{n-1}} v_n^{i},$$

$$w_i = \sum_{i_1 \ldots i_n = 1}^{n} \varepsilon_{i_1 \ldots i_{n-1}} v_1^{i_1} v_2^{i_2} \ldots v_{n-1}^{i_{n-1}} v_n^{i}.$$ 

$$||p_{ik}|| = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ v_1^1 & v_1^2 & \cdots & v_1^n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^1 & v_n^2 & \cdots & v_n^n \end{vmatrix},$$

$$||q_{ik}|| = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ v_1^1 & v_1^2 & \cdots & v_1^n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^1 & v_n^2 & \cdots & v_n^n \end{vmatrix}.$$ 

$$W^i = \frac{1}{\sqrt{a}} Q_{i} \quad (Q_{i} \text{ is the algebraic complement of } q_{i}),$$

$$W_i = \sqrt{a} P_{i} \quad (P_{i} \text{ is the algebraic complement of } p_{i}).$$

$$\sum_i W^i v_{r|i} = 0, \quad \sum_1 W_i v_r^i = 0 \quad (r = 1, 2, \ldots, n - 1).$$

9.5.9 Extension Of A Field

$$dV = \sqrt{\alpha} dx_1 dx_2 \ldots dx_n.$$ 

$$\int_C \sqrt{\alpha} dx_1 \ldots dx_n = \int_C \sqrt{\alpha} D d\mathbf{x}_1 \ldots d\mathbf{x}_n.$$ 

$$\bar{\alpha} = D^2 \alpha, \quad \sqrt{\alpha} = D \sqrt{\alpha}.$$ 

$$\int_C \sqrt{\alpha} dx_1 \ldots dx_n = \int_C \sqrt{\alpha} d\mathbf{x}_1 \ldots d\mathbf{x}_n.$$
9.5.10 Curl Of A Vector In Three Dimensions

In general, in $n$ dimensions the curl of a vector is the two indices antisymmetric system

$$p_{i\ell} = X_{i|\ell} - X_{\ell|i}.$$  

$$X_{i|\ell} = \frac{\partial X_i}{\partial x_\ell} - \sum_p X_p \begin{bmatrix} i & \ell \\ p & \end{bmatrix},$$

$$X_{\ell|i} = \frac{\partial X_\ell}{\partial x_i} - \sum_p X_p \begin{bmatrix} \ell & i \\ p & \end{bmatrix}.$$  

$$p_{i\ell} = \frac{\partial X_i}{\partial x_\ell} - \frac{\partial X_\ell}{\partial x_i}.$$  

In 3 dimensions:

$$R_h^i = \sum_{i,\ell=1}^3 \varepsilon^{hi\ell} X_{\ell|i},$$

that is:

$$R^1 = \frac{1}{\sqrt{a}}(X_{3|2} - X_{2|3}) = \frac{1}{\sqrt{a}}p_{32},$$

and analogous relations for $R^2$ and $R^3$. Summing up:

$$\begin{align*}
R^1 &= \frac{1}{\sqrt{a}} p_{32} = \frac{1}{\sqrt{a}} \left( \frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3} \right), \\
R^2 &= \frac{1}{\sqrt{a}} p_{13} = \frac{1}{\sqrt{a}} \left( \frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1} \right), \\
R^3 &= \frac{1}{\sqrt{a}} p_{21} = \frac{1}{\sqrt{a}} \left( \frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2} \right).
\end{align*}$$

9.5.11 Sections Of A Manifold. Geodesic Manifolds

Let us consider $m$ directions $\lambda_\alpha (\alpha = 1, 2, \ldots, m)$. The directions $\xi$ with parameters

$$\xi^i = \sum_{\alpha=1}^m \rho_\alpha \lambda^i_\alpha$$

and the moments

\[ \xi_i = \sum_{\alpha=1}^{m} \rho_\alpha \lambda_\alpha |_i \]

are defined for arbitrary \( \rho \) provided that:

\[ \sum_{i=1}^{m} \xi_i^2 = 1 \]

that is:

\[ \sum_{\alpha,\beta=1}^{m} \sum_{i=1}^{m} \rho_\alpha \rho_\beta \lambda^i_\alpha \lambda^i_\beta |_i = \sum_{\alpha,\beta=1}^{m} \rho_\alpha \rho_\beta \sum_{i=1}^{m} \lambda^i_\alpha \lambda^i_\beta |_i = \sum_{\alpha,\beta=1}^{m} \rho_\alpha \rho_\beta \cos(\hat{\alpha_\beta}) = 1. \]

The section\(^2\) \( G \) is defined by means of \( m \) directions (it is a set of \( \infty^{m-1} \) directions).

The geodesic surface of pole \( P \) is made of the geodesic curves outgoing from \( P \) along the section \( \lambda, \mu \).

The geodesic manifold \( V^m \) with \( m \) dimensions and with pole \( P \) is made of the \( \infty^{m-1} \) geodesic lines outgoing from \( P \) along a section \( G_m \); it contains \( \infty^m \) points. Geodesic surfaces correspond to \( m = 2 \), while geodesic hypersurfaces to \( m = n - 1 \).

9.5.12 Geodesic Coordinates Along A Given Line

\[ \mathbf{\vec{x}}_i = \mathbf{\vec{x}}_i(x_1, x_2, \ldots, x_n), \]

\[ \mathbf{\vec{x}}_i = f_1(s). \]

\[ d\mathbf{\vec{y}}_i = dx_i + \sum_{k,j=i}^{m} \left\{ \begin{array}{c} k \ j \\ i \end{array} \right\} dx_k dx_j. \]

\[ d\mathbf{\vec{x}}_i = \sum_{\ell=1}^{n} S_{i\ell} d\mathbf{\vec{y}}_\ell = \sum_{\ell=1}^{n} S_{i\ell} dx_\ell + \sum_{k,i,\ell=1}^{n} S_{i\ell} \left\{ \begin{array}{c} k \ j \\ l \end{array} \right\} dx_k dx_j. \]

\(^2\) The symbol \( G \) is introduced by the author in reference to the initial of the Italian word “giacitura”, which means “section”.

\[ S_{i\ell} = S_{i\ell}(s). \]

\[
\frac{\partial x_i}{\partial x_{\ell}} = S_{i\ell}, \quad \frac{\partial^2 x_i}{\partial x_k \partial x_j} = \sum_{\ell=1}^{n} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix},
\]

\[
\frac{\partial S_{i\ell}}{\partial x_m} = \frac{\partial S_{im}}{\partial x_{\ell}}, \quad \frac{\partial^2 x_i}{\partial x_k \partial x_j} = \frac{\partial S_{ik}}{\partial x_j},
\]

\[
\frac{\partial S_{i\ell}}{\partial x_j} = \sum_{\ell=1}^{m} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix},
\]

\[
\sum_{i=1}^{n} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix} = \sum_{i=1}^{n} S_{i\ell} \begin{pmatrix} j & k \\ \ell \end{pmatrix}.
\]

\[
\frac{\partial S_{ik}}{ds} = \sum_{j=i}^{n} \frac{\partial S_{ik}}{\partial x_j} \dot{x}_j = \sum_{i,j=1}^{n} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix} \dot{x}_j.
\]

\((k = 1, 2, \ldots, n; i = 1, 2, \ldots, n).\)

\[
\frac{d\bar{x}_i}{ds} = \sum_{k=1}^{n} \frac{\partial \bar{x}_i}{\partial x_k} \dot{x}_k = \sum_{k=1}^{n} S_{ik} \dot{x}_k.
\]

\[
\bar{x}_i = \int \sum_{k=1}^{n} S_{ik} \dot{x}_k ds + \sum_{\ell=1}^{n} S_{i\ell} \delta x_{\ell} + \sum_{k,j,\ell=1}^{n} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix} \delta x_k \delta x_j
\]

\[
= \int \sum_{k=1}^{n} S_{ik} \dot{x}_k ds + \sum_{\ell=1}^{n} S_{i\ell} \left( \delta x_{\ell} + \sum_{k,j=1}^{n} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix} \delta x_k \delta x_j \right)
\]

Second proof:

\[
\bar{x}_i = p_i(s) + \sum_{\ell=1}^{m} S_{i\ell}(s) \left( \delta x_{\ell} + \frac{1}{2} \sum_{k,j=1}^{n} S_{i\ell} \begin{pmatrix} k & j \\ \ell \end{pmatrix} \delta x_k \delta x_j \right)
\]

+ first-order infinitesimals.
\[ \delta x_i = dx_i + \delta' x_i, \]
\[ dx_i = \dot{x}_i ds + \frac{1}{2} \ddot{x}_i ds^2, \]
\[ \delta x_i = \dot{x}_i ds + \frac{1}{2} \ddot{x}_i ds^2 + \delta' x_i. \]

(a) \[ \bar{x}_i = p_i(s) + \sum_{\ell=1}^{n} S_{i\ell}(s) \left( \dot{x}_\ell ds + \frac{1}{2} \ddot{x}_\ell ds^2 \right) \]
\[ + \frac{1}{2} \sum_{j=1}^{n} \left\{ \begin{array}{c} k \ j \\ l \end{array} \right\} \dddot{x}_k \dddot{x}_j ds^2 \]
\[ + \sum_{\ell=1}^{n} S_{i\ell}(s) \left( \delta' x_\ell + \frac{1}{2} \sum_{k,j=1}^{n} \left\{ \begin{array}{c} k \ j \\ \ell \end{array} \right\} \delta' x_k \delta' x_\ell \right) \]
\[ + \sum_{\ell,k,j=1}^{n} S_{i\ell}(s) \left\{ \begin{array}{c} k \ j \\ \ell \end{array} \right\} \ddot{x}_k \delta x_\ell ds. \]
\[ p_i(s + ds) = p_i(s) + \dot{p}_i(s) ds + \frac{1}{2} \ddot{p}_i(s) ds^2, \]
\[ S_{i\ell}(s + ds) = S_{i\ell}(s) + \dot{S}_{i\ell}(s) ds + \ldots, \]
\[ \left\{ \begin{array}{c} k \ j \\ \ell \end{array} \right\}_{s+ds} = \left\{ \begin{array}{c} k \ j \\ \ell \end{array} \right\}_s + \ldots. \]

(b) \[ \bar{x}_i = p_i(s + ds) + \sum_{\ell=1}^{n} S_{i\ell}(s + ds) (\delta' x_\ell) \]
\[ + \frac{1}{2} \sum_{k,j=1}^{n} \left\{ \begin{array}{c} k \ j \\ l \end{array} \right\} \delta' x_k \delta' x_j \]
\[ = p_i(s) + \sum_{i=1}^{n} S_{i\ell}(s) \left( \delta' x_\ell + \frac{1}{2} \sum_{k,j=1}^{n} \left\{ \begin{array}{c} k \ j \\ \ell \end{array} \right\} \delta' x_k \delta' x_j \right) \]
\[ + \dot{p}_i(s) ds + \frac{1}{2} \ddot{p}_i(s) ds^2 + \sum_{\ell=1}^{n} \dot{S}_{i\ell}(s) ds \delta' x_\ell. \]
\[
\begin{align*}
(a-b) \quad & \dot{p}_i(s) \, ds + \frac{1}{2} \ddot{p}_i(s) \, ds^2 + \sum_{\ell=1}^{n} \dot{S}_{i\ell}(s) \, ds \, \delta' x_{\ell} \\
& = \sum_{\ell=1}^{n} S_{i\ell}(s) \left( \dot{x}_{\ell} ds + \frac{1}{2} \ddot{x}_{\ell} ds^2 + \frac{1}{2} \sum_{k,j=1}^{n} \left\{ k \begin{array}{c} j \\ \ell \end{array} \right\} \dot{x}_k \dot{x}_j ds^2 \right) \\
& + \sum_{\ell,k,j=1}^{n} S_{i\ell}(s) \left\{ k \begin{array}{c} j \\ \ell \end{array} \right\} \dot{x}_k \delta' x_{j} \, ds.
\end{align*}
\]

\[1^\text{st order}: \quad \dot{p}_i(s) \, ds = \sum_{\ell=1}^{n} S_{i\ell}(s) \dot{x}_\ell \, ds,
\]

\[2^\text{nd order}: \quad \frac{1}{2} \ddot{p}_i(s) \, ds^2 + \sum_{\ell=1}^{n} \dot{S}_{i\ell}(s) \, ds \, \delta' x_{\ell} \\
= \frac{1}{2} \sum_{\ell=1}^{n} S_{i\ell} \ddot{x}_{\ell} ds^2 + \frac{1}{2} \sum_{\ell,k,j=1}^{n} S_{i\ell}(s) \left\{ k \begin{array}{c} j \\ \ell \end{array} \right\} \dot{x}_k \dot{x}_j ds^2 \\
+ \sum_{\ell,k,j=1}^{n} S_{i\ell}(s) \left\{ k \begin{array}{c} j \\ \ell \end{array} \right\} \dot{x}_k \delta' x_{j} \, ds.
\]  

For arbitrary \( \delta' x_{j} \):

\[
\begin{align*}
\dot{p}_i(s) &= \sum_{\ell=1}^{n} S_{i\ell}(s) \dot{x}_\ell = \sum_{j=1}^{n} S_{ij}(s) \dot{x}_j \quad (1) \\
\ddot{p}_i(s) &= \sum_{\ell=1}^{n} S_{i\ell} \ddot{x}_{\ell} + \sum_{\ell,k,j=1}^{n} S_{i\ell}(s) \left\{ k \begin{array}{c} j \\ \ell \end{array} \right\} \dot{x}_k \dot{x}_j \quad (2) \\
\dot{S}_{ij}(s) &= \sum_{\ell,k=1}^{n} S_{i\ell}(s) \left\{ k \begin{array}{c} j \\ \ell \end{array} \right\} \dot{x}_k \quad (3)
\end{align*}
\]

From (3), by summing over every value of \( i \), we obtain the quantities \( S_{ij} \) (with \( n^2 \) arbitrary constants; for example they are given by the initial
values). By taking the derivative of (1) with respect to \( s \) and replacing \( S_{ij}(s) \) with their expression in (3), we get (2) identically. Then, it is enough to satisfy only (1). We find:

\[
p_i(s) = \int \sum_{\ell=1}^{n} S_{i\ell}(s) \dot{x}_\ell(s) \, ds.
\]

Since the integrals are defined up to a constant, we thus have \( 2n^2 \) arbitrary constants, \( n^2 \) of which are trivial (additive constants). The final formula coincides with the one already obtained above:

\[
\bar{x}_i = \int \sum_{\ell=1}^{n} S_{i\ell} \dot{x}_\ell \, ds + \sum_{\ell=1}^{n} S_{i\ell} \left( \delta x_\ell + \sum_{k,j=1}^{n} \binom{k}{j} \delta x_k \delta x_j \right)
\]

\( S_{i\ell} \) being the solutions of the \( n \) differential systems (3).

9.6. RIEMANN’S SYMBOLS AND PROPERTIES RELATING TO CURVATURE

9.6.1 Cyclic Displacement Round An Elementary Parallelogram

\[
x_i \to x_i + \delta x_i \to x_i + \delta x_i + \delta' x_i \to x_i + \delta' x_i \to x_i,
\]

\[
u^i \to u_i^i \to u_2^i \to u_3^i \to u_4^i.
\]

\[
du^i = - \sum_{k,j=1}^{n} \binom{k}{j} u^k dx_j = \sum X_j^i dx_j,
\]

\[
X_j^i = - \sum_{k=1}^{n} \binom{k}{i} u^k
\]

\( (X_j^i \) is not a tensor). Up to 2nd order infinitesimals:
\[ \delta u^i = - \sum_{k,j=1}^{n} \{ k \, j \} \, (u^k)_0 \delta x_j, \]

\[ X^i_j = - \sum_{k=1}^{n} \{ k \, j \} \, (u^k)_0 - \sum_{k,\ell=1}^{n} \left( \frac{\partial}{\partial x_\ell} \left\{ \begin{array}{c} k \\ i \end{array} \right\} \right)_0 (u^x)_0 \delta x_\ell \]

\[ + \sum_{k,\ell,m=1}^{n} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \left\{ \begin{array}{c} m \\ \ell \end{array} \right\} (u^m)_0 \delta x_\ell. \]

\[ \Delta u^i = \oint \sum_{j=1}^{n} X^i_j \, dx_j = \int_{P_1} \sum_{j=1}^{n} X^i_j \, dx_j + \int_{P_2} \ldots + \int_{P_3} \ldots + \int_{P_{n=1}} \]

\[ = \left( \int_{P_1} \ldots + \int_{P_2} \ldots \right) + \left( \int_{P_2} \ldots + \int_{P_{n=1}} \ldots \right) \]

\[ = \sum_{k,j,\ell=1}^{n} \frac{\partial}{\partial x_\ell} \left\{ \begin{array}{c} k \\ j \end{array} \right\} u^k \delta x_\ell \, dx_j - \sum_{k,\ell,m,j=1}^{n} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \left\{ \begin{array}{c} m \\ \ell \end{array} \right\} u^m \delta x_\ell \, dx_j \]

\[ + \sum_{k,\ell,m,j=1}^{n} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \left\{ \begin{array}{c} m \\ \ell \end{array} \right\} u^m \, dx_\ell \delta x_j. \]

\[ \Delta u^r = - \sum_{i,h,k=1}^{n} u^i \, dx_k \, \delta x_k \left[ \sum_{p=1}^{n} \left( \left\{ \begin{array}{c} i \\ k \end{array} \right\} \left\{ \begin{array}{c} p \\ h \end{array} \right\} \right) \right. \]

\[ - \left( \left\{ \begin{array}{c} i \\ h \end{array} \right\} \left\{ \begin{array}{c} p \\ k \end{array} \right\} \right) \right) \]

\[ + \frac{\partial}{\partial x_k} \left\{ \begin{array}{c} i \\ r \end{array} \right\} - \frac{\partial}{\partial x_k} \left\{ \begin{array}{c} i \\ r \end{array} \right\} \right]. \]
\[
\Delta u^r = + \sum_{i,h,k=1}^n \{ir, hk\} u^i \, dx_h \, \delta x_k
\]

which is the Riemann curvature.

### 9.6.2 Fundamental Properties Of Riemann’s Symbols Of The Second Kind

\[
\{ir, hk\} = - \frac{\partial}{\partial x_h} \left\{ \begin{array}{c} i \\ k \\ r \end{array} \right\} + \frac{\partial}{\partial x_k} \left\{ \begin{array}{c} i \\ h \\ r \end{array} \right\}
\]

\[
- \sum_{p=1}^n \left[ \left\{ \begin{array}{c} p \\ h \\ r \end{array} \right\} \left\{ \begin{array}{c} i \\ k \\ p \end{array} \right\} - \left\{ \begin{array}{c} p \\ k \\ r \end{array} \right\} \left\{ \begin{array}{c} i \\ h \\ p \end{array} \right\} \right]
\]

Properties of Riemann’s symbols of the second kind:

1) \( \{ir, hk\} = a^r_{ikh} \) (covariance with respect to the indices \( i, h, k \), contravariance with respect to the index \( r \))

2) \( \{ir, hk\} = -\{ir, kh\} \),

3) \( \{ir, hk\} + \{hr, ki\} + \{kr, ih\} = 0 \).

Up to 2\(^{nd}\) order infinitesimals:

\[
\{11, 12\} = \frac{\partial}{\partial x_1} \left\{ \begin{array}{cc} 1 & 2 \\ 1 & \end{array} \right\} - \frac{\partial}{\partial x_2} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} = 0,
\]

\[
\{21, 12\} = \frac{\partial}{\partial x_1} \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} - \frac{\partial}{\partial x_2} \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \frac{2}{3} + \frac{1}{3} = 1,
\]

\[
\{12, 12\} = \frac{\partial}{\partial x_1} \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} - \frac{\partial}{\partial x_2} \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = -\frac{1}{3} - \frac{2}{3} = -1,
\]

\[
\{22, 12\} = \frac{\partial}{\partial x_1} \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} - \frac{\partial}{\partial x_2} \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = 0.
\]
\[ \Delta u^r = - \sum_{i,h,k=1}^{n} \{ir, hk\} u^i \delta x_h \delta' x_k. \]

\[ r = 1 : \quad \Delta u^1 = -u^2(\delta x_1 \delta' x_2 - \delta x_2 \delta' x_1), \]
\[ r = 2 : \quad \Delta u^2 = u^1(\delta x_1 \delta' x_2 - \delta x_2 \delta' x_1). \]

9.6.3 Fundamental Properties And Number Of Riemann’s Symbols Of The First Kind

\[(ij, hk) = \sum_{r=1}^{n} a_{jr}\{ir, hk\} \]
\[= -\sum_{r=1}^{n} a_{jr} \frac{\partial}{\partial x_h} \{i \ k \} + \sum_{r=1}^{n} a_{jr} \frac{\partial}{\partial x_k} \{i \ h \} \]
\[ - \sum\limits_{p,r=1}^{n} a_{jr} \left[ \{p \ h \} \{i \ k \} \right] \]
\[- \left[ \{p \ k \} \{i \ h \} \right], \]
\[= -\frac{\partial}{\partial x_h} \left[ i \ k \right] + \sum_{r=1}^{n} \frac{\partial a_{jr}}{\partial x_h} \{i \ k \} \]
\[+ \frac{\partial}{\partial x_k} \left[ i \ h \right] - \sum_{r=1}^{n} \frac{\partial a_{jr}}{\partial x_k} \{i \ h \} \]
\[- \sum_{rp=1}^{n} \left( \left[ p \ h \right] \{i \ k \} - \left[ p \ k \right] \{i \ h \} \right). \]

\(^3\) In the original manuscript, the following note appears: Change the sign of Riemann’s symbols. Also, the following is pointed out, referring to equations reported in Levi-Civita I: Notes on the Tallis formulae: Eq. (3), p.201 is correct; Eq. (4), p.201, change the sign; Eq. (26), p.219 is correct.
Since:
\[
\sum_{r=1}^{n} \frac{\partial a_{jr}}{\partial x_h} \left\{ \begin{array}{c} i \\ k \end{array} \right\} = \sum_{p=1}^{n} \left[ \begin{array}{c} p \\ h \end{array} \right] \left\{ \begin{array}{c} i \\ k \end{array} \right\} + \sum_{p=1}^{n} \left[ \begin{array}{c} j \\ h \end{array} \right] \left\{ \begin{array}{c} i \\ k \end{array} \right\}
\]
eq 0,
\]
etc., we finally have:

\[
(ij, hk) = - \frac{\partial}{\partial x_h} \left[ \begin{array}{c} i \\ k \end{array} \right] + \frac{\partial}{\partial x_k} \left[ \begin{array}{c} i \\ h \end{array} \right] + \sum_{p=1}^{n} \left( \left[ \begin{array}{c} j \\ h \end{array} \right] \left\{ \begin{array}{c} i \\ k \end{array} \right\} \right) - \left[ \begin{array}{c} j \\ p \end{array} \right] \left\{ \begin{array}{c} i \\ p \end{array} \right\} + \left\{ \begin{array}{c} i \\ h \end{array} \right\}.
\]

Properties of Riemann’s symbols of the first kind:

1) covariance with respect to every index,
2) \((ij, hk) = -(ij, kh),\)
3) \((ij, hk) = -(ji, hk).\)

In fact:

\[
\frac{\partial}{\partial x_h} \left[ \begin{array}{c} i \\ k \end{array} \right] - \frac{\partial}{\partial x_k} \left[ \begin{array}{c} i \\ h \end{array} \right] = \frac{1}{2} \left( \frac{\partial^2 a_{jk}}{\partial x_i \partial x_h} - \frac{\partial^2 a_{ih}}{\partial x_i \partial x_k} - \frac{pr^2 a_{ik}}{\partial x_j \partial x_h} + \frac{\partial^2 a_{ih}}{\partial x_j \partial x_k} \right)
\]

\[
= - \left( \frac{\partial}{\partial x_h} \left[ \begin{array}{c} j \\ k \end{array} \right] - \frac{\partial}{\partial x_k} \left[ \begin{array}{c} j \\ h \end{array} \right] \right),
\]
etc.;

\[
\sum_{p=1}^{n} \left[ \begin{array}{c} j \\ h \end{array} \right] \left\{ \begin{array}{c} i \\ k \end{array} \right\} = \sum_{p,q=1}^{n} \alpha^{pq} \left[ \begin{array}{c} j \\ h \end{array} \right] \left[ \begin{array}{c} i \\ k \end{array} \right] \left[ \begin{array}{c} j \\ q \end{array} \right] = \sum_{p=1}^{n} \left[ \begin{array}{c} i \\ k \end{array} \right] \left\{ \begin{array}{c} j \\ h \end{array} \right\}.
(ij, hk) + (hj, ki) + (kj, ih) = 0,

(ij, hk) + (ih, kj) + (ik,jh) = 0,

(ij, hk) = (hk, ij).

In fact:

\[
\frac{\partial}{\partial x_h} \begin{bmatrix} i & k \\ j \\ \end{bmatrix} - \frac{\partial}{\partial x_k} \begin{bmatrix} i & h \\ j \end{bmatrix} = \frac{1}{2} \left( \frac{\partial^2 a_{jk}}{\partial x_i \partial x_h} - \frac{\partial^2 a_{ik}}{\partial x_j \partial x_h} + \frac{\partial^2 a_{jh}}{\partial x_i \partial x_k} + \frac{\partial^2 a_{ih}}{\partial x_j \partial x_k} \right)
\]

etc.; for the remaining proof, see property 3).

7) **Number of the independent symbols of first kind.**
Given the indices \(i, j, h, k\), irrespectively of their ordering, we have two independent symbols if all the indices are different from each other; one independent symbol if three indices are different and the fourth is equal to one of them; one independent symbol if we have two pairs of different symbols; no non-vanishing symbol in the other cases. Thus the total number of independent symbols results to be:

\[
2 \frac{n(n-1)(n-2)(n-3)}{24} + 3 \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2} = \frac{n(n-1)}{12} [(n-2)(n-3) + 6(n-2) + 6] = \frac{n^2(n^2 - 1)}{12}.
\]

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9.6.4 Bianchi Identity And Ricci Lemma

The Bianchi identity for the covariant derivatives of the Riemann’s symbols is:

$$\{ir, hk\}_\ell + \{ir, k\ell\}_h + \{ir, \ell k\}_k = A_{ihk\ell}^r = 0.$$  

It can be easily verified by performing the covariant derivatives in locally cartesian coordinates.

The same holds for the Ricci lemma:

$$(ij, hk)_\ell + (ij, k\ell)_h + (ij, \ell h)_k = 0.$$  

9.6.5 Tangent Geodesic Coordinates Around The Point $P_0$

$$\bar{x}_i = (\lambda^i)_0 s, \quad x_i = \int_0^s \lambda^i ds$$

$$(\lambda^i$$ are evaluated in the point $P_0$; in order to have geodesic coordinates in $P$ it is enough that the formula holds up to $s^2$ terms, as we certainly assume).

$$\ddot{x}_i = \sum_{k,j=1}^n \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \dot{x}_k \dot{x}_j = 0.$$  

$$\ddot{x}_i = -\sum_{k,j=1}^n \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \dot{x}_k \dot{x}_j,$$

$$\dddot{x}_i = -\sum_{k,j=1}^n \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \frac{d}{dx_r} \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \ddot{x}_k \ddot{x}_j - 2 \sum_{k,j=1}^n \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \ddot{x}_k \ddot{x}_j$$

$$-\sum_{k,j,r=1}^n \frac{d}{dx_r} \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \ddot{x}_k \ddot{x}_j$$

$$+ 2 \sum_{k,j,r,s=1}^n \left\{ \begin{array}{c} k \\ j \\ i \end{array} \right\} \left\{ \begin{array}{c} r \\ s \\ j \end{array} \right\} \dddot{x}_k \dddot{x}_r \dddot{x}_s.$$
At a point $P_0$:

\[
\dot{x}_r = \lambda^r,
\]

\[
\ddot{x}_r = - \sum_{h,k=1}^{n} \left\{ \begin{array}{c} h \\ k \\ r \end{array} \right\} \lambda^h \lambda^k,
\]

\[
\dddot{x}_r = \sum_{i,h,k=1}^{n} \dot{x}_i \dot{x}_h \dot{x}_k \left[ - \frac{\partial}{\partial x_i} \left\{ \begin{array}{c} h \\ k \\ r \end{array} \right\} + 2 \sum_{p=1}^{n} \left\{ \begin{array}{c} k \\ r \\ p \end{array} \right\} \left\{ \begin{array}{c} h \\ k \end{array} \right\} \right].
\]
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