# A Sliding-Block Problem

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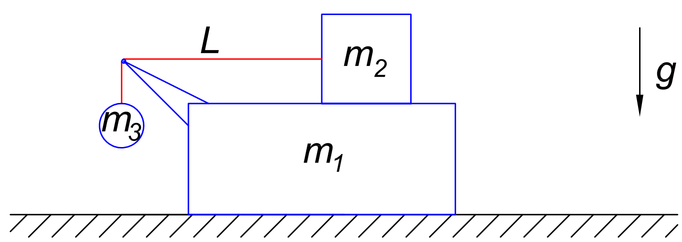
## 1 Problem

Discuss the motion of a the system sketched below in which a block of mass $m_1$ slides without friction on a horizonal surface, a block of mass $m_2$ slides without friction on top of block 1, and mass $m_3$ is attached to block 2 by a string of length $L$ that passes over a tiny, frictionless pulley supported by block 1.

You may ignore the possibility that block 1 tips, and limit your discussion to motion without oscillation of mass 3, such that the accelerations of the three masses are all constant.

*This problem was inspired by [1].*

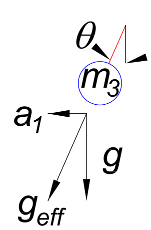



## 2 Solution

### 2.1 Uniformly Accelerated Motion

As mass 3 falls, mass 2 accelerates to the left and masses 1 and 3 accelerate to the right, such that the horizontal ($x$) coordinate of the center of mass of the system remains constant. During this motion, the portion of the string between the pulley and mass 3 takes on a nonzero angle $\theta$ to the vertical, and in general this angle is not constant, with mass 3 both translating and rotating.

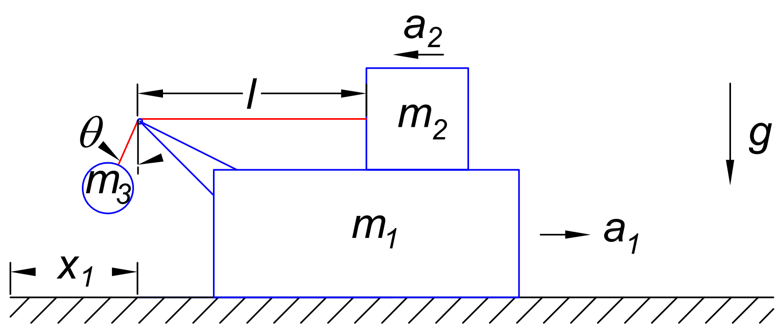
Here, we suppose that the system is launched in such a way that angle $\theta$ remains constant (and nonzero), and mass 3 does not rotate as the three masses accelerate.




To determine angle $\theta$, we go to the accelerated frame in which mass 1 is at rest. Then, there exists a "coordinate force" $-m_3\,\mathbf{a}_1$ on mass 3, where $\mathbf{a}_1$ is the acceleration of mass 1 in the inertial lab frame, such that the effective gravity $\mathbf{g}_{\text{eff}}$ on mass 3 is as sketched in the figure above. If angle $\theta$ is constant and mass 3 does not rotate, then mass 3 moves only along the direction of the string (in the rest frame of mass 1), such that,

$$\tan\theta = \frac{a_1}{g}, \qquad a_1 = g\tan\theta. \tag{1}$$

The rest of the analysis is performed in the lab frame, with notation in the sketch below.




The horizontal coordinate of the (tiny) pulley is $x_1$, such that

$$\ddot{x}_1 = a_1 = g\tan\theta \tag{2}$$

is the acceleration of mass 1. We denote the length of the string between the pulley and mass 2 as $l$, such that the horizontal coordinate of mass 2 is (to within a constant) $x_2 = x_1 + l$, and the acceleration of mass 2 is

$$\ddot{x}_2 = a_2 = a_1 + \ddot{l} = g\tan\theta + \ddot{l}. \tag{3}$$

The horizontal coordinate of mass 3 is (to within a constant) $x_3 = x_1 - (L - l)\sin\theta$, and

$$\ddot{x}_3 = a_1 + \ddot{l}\sin\theta = g\tan\theta + \ddot{l}\sin\theta. \tag{4}$$

Since the horizontal position of the center of mass of the system is constant (in the lab frame), the horizontal acceleration of the center of mass is zero,

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_3\ddot{x}_3 = (m_1 + m_2 + m_3)g\tan\theta + (m_2 + m_3\sin\theta)\ddot{l} = 0. \tag{5}$$

To obtain additional relations, we consider the tension $T > 0$ in the string. The acceleration $a_2$ of block 2 can then be related as

$$T = -m_2 a_2 = -m_2(g\tan\theta + \ddot{l}). \tag{6}$$

The string acts on the pulley such that the horizontal acceleration of block 1 is related by

$$m_1 a_1 = m_1 g\tan\theta = T(1 - \sin\theta) = -m_2(g\tan\theta + \ddot{l})(1 - \sin\theta). \tag{7}$$

This determines $\ddot{l}$ to be

$$\ddot{l} = -g\tan\theta\left(1 + \frac{m_1}{m_2(1 - \sin\theta)}\right). \tag{8}$$

Using this in eq. (5), we obtain an equation for angle $\theta$,

$$m_1 + m_2 + m_3 = (m_2 + m_3 \sin\theta)\left(1 + \frac{m_1}{m_2(1-\sin\theta)}\right), \tag{9}$$

$$\sin^2\theta - \left(2 + \frac{m_1(m_2+m_3)}{m_2 m_3}\right)\sin\theta + 1 = 0. \tag{10}$$

For example, suppose that $m_1 = 2m_2 = 4m_3$. Then, eq. (10) becomes

$$\sin^2\theta - 8\sin\theta + 1 = 0, \qquad \sin\theta = \frac{8 \pm \sqrt{60}}{2} = 0.127,\ 7.87, \tag{11}$$

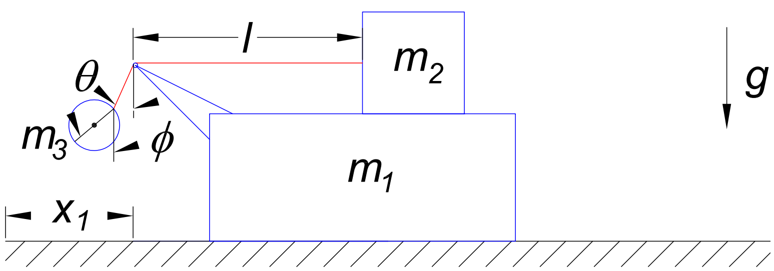
for which the physical solution is $\theta = 7.3^\circ$. The accelerations are

$$a_1 = 0.128g, \qquad \ddot{l} = -0.421g, \qquad a_2 = -0.293g, \tag{12}$$

$$\ddot{x}_3 = 0.075g, \qquad \ddot{y}_3 = \ddot{l}\cos\theta = -0.417g, \qquad |\mathbf{a_3}| = 0.424g. \tag{13}$$

## 2.2 Small Oscillations of $m_3$

We now consider more general motion of the system, in which mass 3 oscillates as well as translates. For this, we suppose the mass 3 is a solid sphere of radius $r$, and moment of inertia $2m_3 r^2/5$ about its center.




As before, $x_2 = x_1 + l$ to within a constant, so

$$v_2^2 = \dot{x}_2^2 = \dot{x}_1^2 + \dot{l}^2 + 2\dot{x}_1\dot{l}, \tag{14}$$

while now the coordinates of mass 3 are

$$\begin{aligned}
x_3 &= x_1 - (L-l)\sin\theta - r\sin\phi, && (15)\\
y_3 &= -(L-l)\cos\theta - r\cos\phi, && (16)\\
\dot{x}_3 &= \dot{x}_1 + \dot{l}\sin\theta - (L-l)\,\dot{\theta}\cos\theta - r\,\dot{\phi}\cos\phi, && (17)\\
\dot{y}_3 &= \dot{l}\cos\theta + (L-l)\,\dot{\theta}\sin\theta + r\,\dot{\phi}\sin\phi, && (18)\\
v_3^2 &= \dot{x}_1^2 + \dot{l}^2 + (L-l)^2\,\dot{\theta}^2 + r^2\,\dot{\phi}^2 + 2\dot{x}_1\,\dot{l}\sin\theta - 2\dot{x}_1(L-l)\,\dot{\theta}\cos\theta - 2\dot{x}_1\,r\,\dot{\phi}\cos\phi \\
&\quad + 2r\,\dot{l}\,\dot{\phi}\sin(\phi-\theta) - 2r(L-l)\,\dot{\theta}\,\dot{\phi}\cos(\phi-\theta). && (19)
\end{aligned}$$

The potential energy is, to within a constant, $V = -m_3\,g[(L-l)\cos\theta + r\cos\phi]$.

The Lagrangian of the system is

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}(m_1+m_2+m_3)\,\dot{x}_1^2 + \frac{1}{2}(m_2+m_3)\,\dot{l}^2 + (m_2+m_3\sin\theta)\,\dot{x}_1\,\dot{l} + \frac{m_3}{2}(L-l)^2\,\dot{\theta}^2 \\
& + \frac{m_3}{2} r^2\,\dot{\phi}^2 - m_3\,\dot{x}_1(L-l)\,\dot{\theta}\cos\theta - m_3\,r\,\dot{x}_1\,\dot{\phi}\cos\phi + m_3\,r\,\dot{l}\,\dot{\phi}\sin(\phi-\theta) \\
& + m_3\,r(L-l)\dot{\theta}\,\dot{\phi}\cos(\phi-\theta) + \frac{2m_3\,r^2}{5}\frac{\dot{\phi}^2}{2} + m_3\,g[(L-l)\cos\theta + r\cos\phi]. \qquad (20)
\end{aligned}
$$

The equations of motion for the four coordinates $x_1$, $l$, $\theta$ and $\phi$ are

$$
\begin{aligned}
& (m_1+m_2+m_3)\,\ddot{x}_1 + (m_2+m_3\sin\theta)\,\ddot{l} \\
& + m_3[(L-l)\,\ddot{\theta}\cos\theta - (L-l)\,\dot{\phi}^2\sin\theta - \dot{l}\,\dot{\phi}\cos\theta + r\,\ddot{\phi}\cos\phi - r\,\dot{\phi}^2\sin\phi] = 0, \qquad (21) \\
& (m_2+m_3\sin\theta)\,\ddot{x}_1 + (m_2+m_3)\,\ddot{l} + m_3\,r\,\ddot{\phi}\sin(\phi-\theta) + m_3(L-l)\,\dot{\theta}^2 \\
& + m_3\,r\,\dot{\phi}^2\cos(\phi-\theta) = -m_3\,g\cos\theta, \qquad (22) \\
& -m_3\,\ddot{x}_1(L-l)\cos\theta + m_2(L-l)^2\,\ddot{\theta} + m_3\,r(L-l)\,\ddot{\phi}\cos(\phi-\theta) \\
& -2m_2(L-l)\,\dot{l}\,\dot{\theta} - m_3\,r(L-l)\,\dot{\phi}^2\sin(\phi-\theta) = -m_3\,g(L-l)\sin\theta, \qquad (23) \\
& -m_3\,r\,\ddot{x}_1\cos\phi + m_3\,r\,\ddot{l}\sin(\phi-\theta) + m_3\,r(L-l)\,\ddot{\theta}\cos(\phi-\theta) + 7m_3\,r^2\,\ddot{\phi}/5 \\
& -m_3\,r\,\dot{l}\,\dot{\theta}\cos(\phi-\theta) + m_3\,r(L-l)\,\dot{\theta}^2\sin(\phi-\theta) = -m_3\,g\,r\sin\phi. \qquad (24)
\end{aligned}
$$

For the case analyzed in sec. 2 above, with $\theta = \phi =$ constant, the equations of motion (21)-(24) simplify to

$$
\begin{aligned}
& (m_1+m_2+m_3)\,\ddot{x}_1 + (m_2+m_3\sin\theta)\,\ddot{l} = 0, \qquad (25) \\
& (m_2+m_3\sin\theta)\,\ddot{x}_1 + (m_2+m_3)\,\ddot{l} = -m_3\,g\cos\theta, \qquad (26) \\
& \ddot{x}_1 = g\tan\theta, \qquad (27) \\
& \ddot{x}_1 = g\tan\phi = g\tan\theta. \qquad (28)
\end{aligned}
$$

Equation (25) is the same as eq. (5), which expresses that the $x$-coordinate of the center of mass is constant. Equations (27)-(28) are the same as eq. (2), which relates to the effective gravity in the accelerated frame of mass 1. Using these relations in eq. (26), we recover eq. (10) for $\sin\theta$ after some algebra.

For a system that starts from rest at time $t = 0$, we write the solution found in sec. 2 as

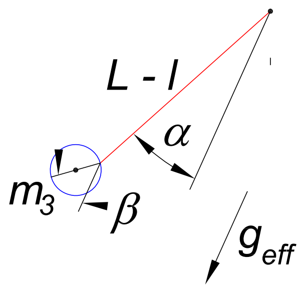
$$
x_{1,0}(t), \qquad l_0(t), \qquad \theta = \phi = \theta_0, \qquad g_{\mathrm{eff},0} = g/\cos\theta_0. \qquad (29)
$$

Returning to the general case, in which mass 3 oscillates and rotates as it falls, we are reminded of Poe's *The Pit and the Pendulum* [2]. The physics of a lengthening pendulum was considered by Lecornu in 1895 [3] and Lord Rayleigh (1902) [4], and that of a (slowly) shortening pendulum was the topic of a famous brief exchange between Lorentz and Einstein at the 1911 Solvay Conference [5], where Einstein's remark anticipated the notion of adiabatic invariance.[1,2]

[1]Einstein's comment, that the ratio $E/\nu$ of the energy $E$ of an oscillator to its frequency $\nu$ should be constant, may have been motivated by Planck's quantum condition that $E = h\nu$ for an oscillator.

[2]The formal development of the concept of adiabatic invariance is attributed to Ehrenfest [6].

The oscillator motion of mass 3 is approximately that of a compound pendulum subject to a time-dependent effective gravitational acceleration $g_{\rm eff}(t)$, such that the angular frequency $\omega$ of oscillation is of order $\sqrt{g_{\rm eff}(t)/[L-l(t)]}$. Here, we make only a kind of adiabatic approximation that the motion of mass 3 at time $t$ is as if $g_{\rm eff}$ and $L-l$ have constant values, namely those at time $t$. This case is represented in the figure below.




The Lagrangian for this subsystem, with angular coordinates $\alpha$ and $\beta$, is

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{m_3}{2}(L-l)^2\,\dot\alpha^2 + \frac{m_3}{2}\frac{7r^2}{5}\,\dot\beta^2 + m_3\,r(L-l)\,\dot\alpha\,\dot\beta\cos(\alpha-\beta) \\
& + m_3\,g_{\rm eff}[(L-l)\cos\alpha + r\cos\beta],
\end{aligned} \tag{30}
$$

for which the equation of motion are

$$
(L-l)^2\,\ddot\alpha + r(L-l)\,\ddot\beta\cos(\alpha-\beta) - r(L-l)\,\dot\beta^2\sin(\alpha-\beta) \;=\; -g_{\rm eff}(L-l)\sin\alpha, \tag{31}
$$

$$
\frac{7r^2}{5}\,\ddot\beta + r(L-l)\,\ddot\alpha\cos(\alpha-\beta) + r(L-l)\,\dot\alpha^2\sin(\alpha-\beta) \;=\; -g_{\rm eff}\,r\sin\beta. \tag{32}
$$

We next consider small oscillations, $\alpha = \alpha_0\,e^{i\omega t}$, $\beta = \beta_0\,e^{i\omega t}$, for which the equations of motion simply further to,

$$
[(L-l)\,\omega^2 - g_{\rm eff}]\,\alpha_0 + r\,\omega^2\,\beta_0 \;=\; 0, \tag{33}
$$

$$
(L-l)\,\omega^2\,\alpha_0 + \left(\frac{7r}{5}\,\omega^2 - g_{\rm eff}\right)\beta_0 \;=\; 0. \tag{34}
$$

For a solution to exist, $\omega$ must satisfy,

$$
\frac{2r(L-l)}{5}\,\omega^4 - g_{\rm eff}\left(L-l+\frac{7r}{5}\right)\omega^2 + g_{\rm eff}^2 = 0, \tag{35}
$$

which implies that there are two modes of oscillation, with angular frequencies,

$$
\omega = \sqrt{\frac{5g_{\rm eff}}{4\,r(L-l)}\left(L-l+\frac{7r}{5} \pm \sqrt{(L-l)^2 + \frac{6\,r(L-l)}{5} + \frac{49r^2}{25}}\right)}. \tag{36}
$$

As usual for a compound pendulum, the small angles $\alpha$ and $\beta$ have the same sign for the lower frequency of oscillation, and opposite signs for the higher frequency.

Our adiabatic approximation is that the time-dependent frequencies $\omega(t)$ are obtained by using $l(t) = l_0(t)$, and $g_{\rm eff}(t) = g_{{\rm eff},0}$ from eq. (29) in eq. (36).

# References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[6] P. Ehrenfest, *Adiabatic invariants and the Theory of Quanta*, Phil. Mag. **33**, 500 (1917),
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