(Non)decoherence in the Daya Bay Reactor Antineutrino Experiment

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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The topic of decoherence in neutrino oscillations is intellectually interesting, but it is not relevant to a successful neutrino-oscillation experiment such as Daya Bay, in that if oscillations were observed, then there was essentially no decoherence in the experiment. As such, little knowledge of decoherence can be extracted from the data of a successful neutrino experiment, and a discussion of decoherence based only on such data will be of limited significance.[1]

A better picture of the (non)decoherence effects in a successful neutrino-oscillation experiment should be based on more than just the nominal data of the experiment.

1 The Coherence Length in the Daya Bay Experiment

As reviewed in the Appendix, standard analysis of neutrino oscillations[2] leads to the introduction of the concepts of the oscillation length $L_{\rm osc}$ and the coherence length $L_{\rm coh}$ in context of two neutrinos with mass eigenstates m_1 and m_2 that are created in a flavor state of definite momentum $p \gg m_i c$, where c is the speed of light in vacuum,

$$L_{\rm osc} = \frac{4E\hbar}{\Delta m_{12}^2 c^3} \approx \frac{4E\hbar c}{|m_1^2 - m_2^2|\, c^4}\,, \qquad \Delta m_{12}^2 = \left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right) E \approx m_1^2 - m_2^2\,, \tag{1}$$

where $E = (E_1 + E_2)/2$ is the average neutrino energy. The period of the oscillation for propagation of the neutrinos in the x-direction is $\lambda_x = \pi L_{\rm osc}$ for given energy E. Similarly, the period of the oscillation in the neutrino-energy spectrum at fixed distance $x \gg \lambda_x$ is approximately

$$\lambda_E \approx \frac{\pi L_{\rm osc} E}{x} = \frac{E}{N_{\rm osc}} \qquad \left(N_{\rm osc} \equiv \frac{x}{\lambda_x} \gg 1\right) \tag{2}$$

where $N_{\rm osc} \gg 1$ is the number of the oscillations observable in the energy spectrum.

The coherence length is the distance after which the wavepackets for neutrinos of types 1 and 2 of the same momentum, but different energies, cease to overlap,

$$L_{\rm coh}(E) = \frac{L_{\rm osc}(E)}{\sqrt{2}\pi\sigma_{\rm rel}(E)}\,, \tag{3}$$

[1]These comments were inspired in part by the Daya Bay internal note [1], which discussed decoherence using only the data from the neutrino detectors.

[2]There are two versions of the "standard" neutrino-oscillation analyses based on the approximation of plane-wave states, which violate energy-momentum conservation when neutrinos oscillate. Some people assume that the oscillating neutrino has a definite energy, but not a definite momentum (perhaps starting with [2]; in this approach energy, but not momentum, is conserved in neutrino oscillations. Other people assume that a neutrino has a definite momentum but not energy (perhaps starting with [3]; in this approach momentum, but not energy, is conserved in neutrino oscillations. The analysis reviewed here follows the latter approach.

where $\sigma_{\rm rel} = \sigma_E/E$, and σ_E is the relevant rms energy spread.

While it might seem natural that $\sigma_{\rm rel}$ is determined by "intrinsic" effects related to the source, or to the source-detector distance D, in experiments on reactor neutrinos $\sigma_{\rm rel}$ is almost entirely determined by the detector energy resolution,[3]

$$\sigma_{\rm rel} \approx \frac{\sigma_{E_{\rm det}}}{E} \qquad \text{(reactor-neutrino experiment)}. \tag{4}$$

A consequence of this is that if the detector resolution is sufficient to resolve oscillations in the neutrino energy spectrum, then the coherence length is automatically longer than the source-detector distance, and there will be little/no decoherence in the data.

If we approximate $\sigma_{E_{\rm det}}/E$ by

$$\sigma_{\rm rel} \approx \frac{\sigma_{E_{\rm det}}}{E} \approx \frac{\sigma_{E_{\rm prompt}}}{E_{\rm prompt}} \approx \frac{0.08}{\sqrt{E_{\rm prompt}}}\,, \tag{5}$$

where the value 0.08 holds for the Daya Bay experiment, then we arrive at a prediction of the coherence length. For example, at $E = 4$ MeV, the peak energy of the reactor antineutrino spectrum, for which $E_{\rm prompt} \approx 3$ MeV, the detector energy resolution is

$$\sigma_{\rm rel} = \frac{\sigma_{E_{\rm prompt}}}{E_{\rm prompt}} \approx \frac{0.08}{\sqrt{3}} = 0.046 \qquad \text{(Daya Bay detector resolution}, E = 4 \text{ MeV)}. \tag{6}$$

At neutrino energy $E = 4$ MeV, $L_{\rm osc} \approx 2$ km $\approx D$ for oscillations related to neutrino-mixing angle θ_{13}, where D is the distance from the reactors to the Daya Bay Far Detector. Then, eq. (3) leads to the prediction that for a spectral analysis of the neutrino oscillations,

$$L_{\rm coh} \approx \frac{L_{\rm osc}}{0.046\sqrt{2}\pi} \approx 5L_{\rm osc} \approx 5D \approx 10 \text{ km} \qquad \text{(Daya Bay}, E = 4 \text{ MeV)}. \tag{7}$$

That is, decoherence is unimportant in the spectral analysis [7] of the Daya Bay experiment.[4]

1.1 Decoherence When the Neutrino Energy is Not Used in the Analysis

To illustrate further the notion of "decoherence", we consider the relative rate of electron antineutrinos that could be detected as a function of distance from a nuclear reactor, if the neutrino energy were not measured (or knowledge of the neutrino energy not used in the analysis).[5]

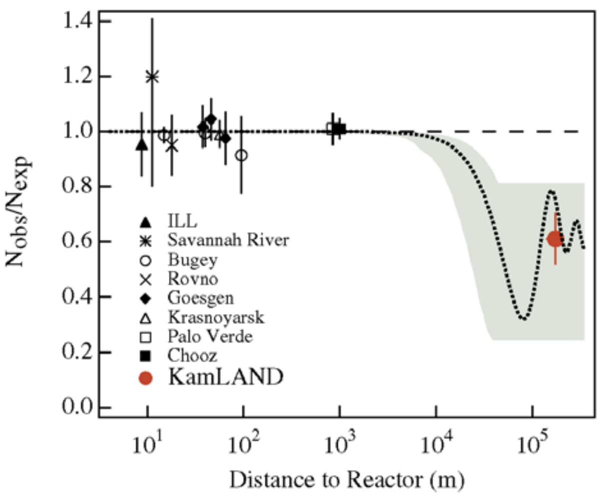
[3]Strictly, $1/\sigma_E^2 = 1/\sigma_{E_{\rm source}}^2 + 1/\sigma_{E_{\rm det}}^2$, as perhaps first discussed in eq. (30) of [4]. See also eq. (53) of [5] and eq. (15) of [6]. Further details are given in Appendix 2.1.1

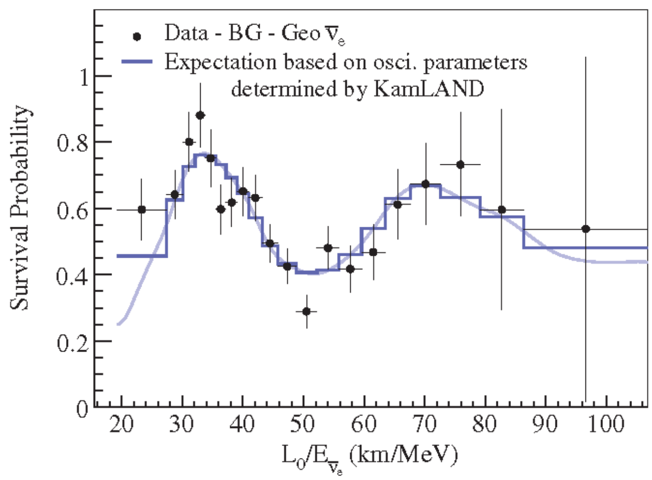
[4]If the Far Detector of the Daya Bay experiment were moved to a larger distance $D = NL_{\rm osc}$ from the nuclear reactors, the neutrino energy spectrum would show N oscillations, but for $N \gtrsim 5$ these oscillations would not be resolved due to insufficient detector energy resolution. In the latter case, we would say that the neutrino oscillations have decohered.

See Fig. 3 of [7] for evidence of roughly one oscillation in the energy spectrum of the Daya Bay Far Detector.

[5]The Daya Bay analyses reported in [8, 9] are not of this type, but use the observed neutrino energy in a fit of the data to a model of the oscillating-neutrino interaction rate *vs.* distance.

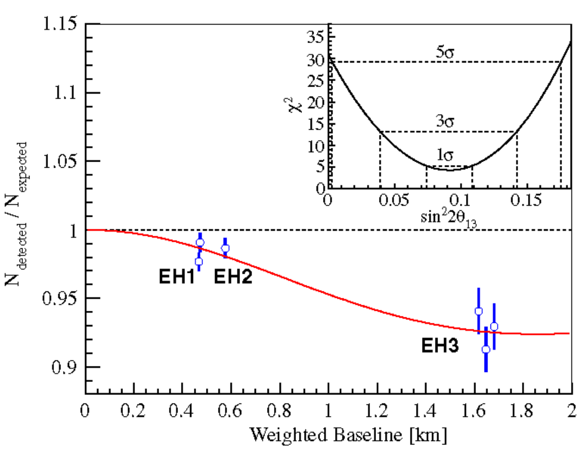
Then, as the energy of the detected neutrinos, roughly $2 < E < 8$ MeV, varies by a factor of ≈ 4, the oscillation length of these neutrinos varies by a factor of 4, and the oscillations become "smeared out" with distance from the reactor. At large distances, oscillations cannot be observed *vs.* distance, and the survival probability is constant at $P(L) \approx 1 - \sin^2(2\theta_{12}) \left\langle \sin^2(\Delta m^2_{12} L/4E) \right\rangle \approx 1 - 0.5 \sin^2(2\theta_{12}) \approx 0.6$ for oscillations where $\sin^2(2\theta_{12}) \approx 0.8$, as in [10] (KamLAND), from which the left figure below is taken.[6]

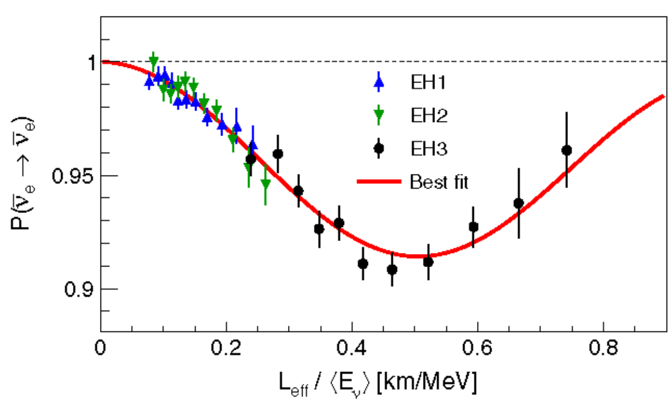




On the other hand, the reconstructed neutrino energy E can be used to plot the data *vs.* L/E, as in the right figure above (from [16]), in which can evidence for neutrino oscillations is more clearly seen.

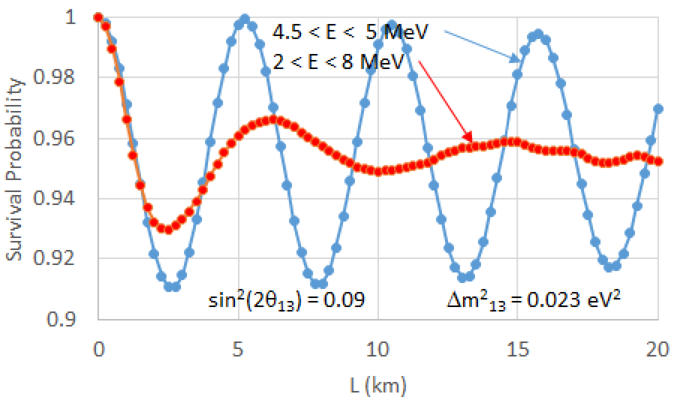
To illustrate this effect for the Daya Bay experiment, where the relevant neutrino-mixing angle is θ_{13}, with $\sin^2(2\theta_{13}) \approx 0.09$, we recall the left figure below (from [8]), in which the neutrino energy is not used in making the plot, and the right figure below (from [9]), in which the energy is used. Again, better evidence for oscillations is obtained when the measured neutrino energy (with its uncertainty due to the detector energy resolution) is used

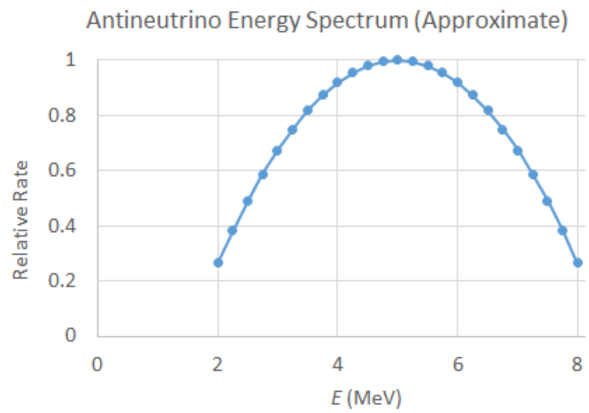




[6]Discussion of decoherence in the KamLAND data is given in [11].
See also [12], where the damping of the oscillations to $1 - 0.5 \sin^2(2\theta_{12})$ is called an effect of quantum decoherence.
Discussion of decoherence in data from atmospheric and astrophysical neutrinos is given, for example, in [13, 14, 15].

We illustrate this point further with a calculation based on parameters for $1-3$ neutrino oscillations, assuming two different energy bands in the analysis shown in the left figure below.





The red curve is for an analysis that ignores the neutrino energy, such that the neutrino oscillation is damped/distorted beyond $\approx L_{\rm osc}(\langle E\rangle)/[\Delta E/E]$ ($\approx$ 2 km for this Daya Bay example)).[7] The blue curve in the left figure above is for an analysis that restricts the neutrino energy to $4.5 < E < 5$ MeV. The coherence length in this case is ≈ 15 km, about 6 times longer than for the analysis with $2 < E < 8$ MeV, with only slight degradation of the amplitude of the oscillation at 15 km $= L_{\rm coh}(4.5 < E < 5)$.

In these examples, neutrino oscillations occur, but the effect is not observable as an oscillation at large distances, which loss of information we call "decoherence".[8]

The energy range ΔE used in the data analysis can be changed/varied after the data are collected. This "delayed choice" affects the amount of "decoherence" in the analysis. However, even if the range of reconstructed energy E is made very narrow in the analysis, the restricted data sample corresponds to neutrinos of energy range $\approx \sqrt{2}\pi\sigma_E$, where σ_E is the detector energy resolution. Hence, the coherence length in a data analysis cannot be larger than $EL_{\rm osc}/\sqrt{2}\pi\sigma_E$, which could be called the "quantum coherence length," but it could be shorter if a choice is made after the data were collected to use $\Delta E > \sqrt{2}\pi\sigma_E$. In the latter case, we could speak of the "classical coherence length" $EL_{\rm osc}/\Delta_E$.

The amount of decoherence depends on the range ΔE of energies sampled in the detector/data analysis, as well as on the source-detector distance. Decoherence is often stated as an effect of the "environment" on a quantum system, and in the present examples, the "environment" includes the "empty space" between the source and the detector, as well as the detector itself.

These examples reinforce that the quantity $L_{\rm osc}(\langle E\rangle)/[\Delta E/E]$ should be regarded as

[7] If there were no "smearing"/decoherence, the first minimum in the red curve in the above left plot would have value 1 - 0.09 = 0.91 rather than 0.93 (at $L \approx 2.5$ km $\approx L_{\rm osc}(\langle E\rangle) \approx L_{\rm coh}$). Hence, some effect of "decoherence" is already observable in the figure at $L_{\rm coh} \approx 2.5$ km.

[8] Some people (for example, [17]) consider that the "smearing" of the oscillations due to limited energy resolution in a neutrino detector is not an effect of "decoherence", although the "smearing" precludes observation of the oscillations at large distances. In this view, the "coherence length" is not the length over which an oscillatory signal can be well observed, but a more abstract concept of less relevance to experimental measurements.

the coherence length L_{coh} in an experiment where neutrinos within energy range ΔE are observed.

In sum, **the coherence length depends on the detector/data analysis, as well as the neutrino-production process**.

2 Limits on the "Intrinsic" Value of σ_{rel}

While the detector resolution largely determines the value of σ_{rel} in a reactor neutrino experiment, it may be interesting to discuss what can be said about the "intrinsic" contribution to this quantity.

2.1 Limits Based on Knowledge of the Source

Lower limits on the "intrinsic" value of σ_{rel} can be deduced from properties of the nuclear reactor.

For example, the typical lifetime of the beta decay that produced a reactor antineutrino is (I think) $\tau \approx 10$ s. Then $c\tau \approx 3 \times 10^4$ km, such that by the uncertainty principle,

$$\sigma_E \gtrsim \frac{\hbar}{\tau} = \frac{\hbar c}{c\tau} \approx 7 \times 10^{-21} \text{ MeV}, \tag{8}$$

and for neutrino energy of 4 MeV,

$$\sigma_{\mathrm{rel}} \gtrsim \frac{\hbar c}{c\tau E} \approx 2 \times 10^{-21} \qquad (\text{beta-decay lifetime}, E = 4 \text{ MeV}). \tag{9}$$

This is, of course, a very weak limit.

A much stronger limit is based on the knowledge that the nucleus whose decay produced the antineutrino was localized roughly by the size of an atom, say $\sigma_x \approx 2 \times 10^{-10}$ m $= 2 \times 10^{-13}$ km. Then,

$$\sigma_E \approx c\,\sigma_p \gtrsim \frac{\hbar c}{\sigma_x} \approx 10^{-3} \text{ MeV}, \tag{10}$$

and for neutrino energy of 4 MeV,[9]

$$\sigma_{\mathrm{rel}} \gtrsim \frac{\hbar c}{\sigma_x E} \approx 2.5 \times 10^{-4} \qquad (\text{source-atom size}, E = 4 \text{ MeV}). \tag{11}$$

2.2 Limits Based on the Source-Detector Distance

Limits on the "intrinsic" value of σ_{rel} can easily be calculated from the source-detector distance D, whose maximum value is ≈ 2 km in the Daya Bay experiment.

[9] The limit (11) is deduced in sec. 2.1.6 of [17], but is not found in [1] since this limit is not based on the Daya Bay neutrino data.

A lower limit comes from the fact that the neutrino exists only for time $\Delta t \approx D/c$, noting that since the neutrino energy is much larger than its mass, the neutrino velocity is essentially the speed of light c. Then, by the uncertainty principle,

$$\sigma_E \gtrsim \frac{\hbar}{\Delta t} \approx \frac{\hbar c}{c\,\Delta t} = \frac{\hbar c}{D} \approx \frac{2\times 10^{-16}\ \text{MeV-km}}{2\ \text{km}} = 10^{-16}\ \text{MeV}. \tag{12}$$

Hence, for the typical reactor-neutrino energy $E = 4$ MeV, we have that

$$\sigma_{\text{rel}} = \frac{\sigma_E}{E} > \frac{10^{-16}\ \text{MeV}}{4\ \text{MeV}} = 2.5\times 10^{-17}. \tag{13}$$

On the other hand, the fact that oscillations are observed in the Daya Bay experiment implies that $L_{\text{coh}} > D \approx L_{\text{osc}}$. Hence, from eq. (6) above we infer that

$$\sigma_{\text{rel}} = \frac{\sigma_E}{E} = \frac{L_{\text{osc}}}{L_{\text{coh}}\sqrt{2\pi}} \gtrsim \frac{1}{\sqrt{2\pi}} = 0.23. \tag{14}$$

The paper [1] does not mention these simple calculations, but describes a lengthy procedure whose result is

$$2.38\times 10^{-17} < \sigma_{\text{rel}} < 0.232. \tag{15}$$

Thus, use only of the largest Daya Bay source-detector distance D reproduces the main result of [1].

Hence, it appears that while the lengthy analysis presented in [1] is technically correct, it is readily anticipated in a few lines, and in any case it does not find the easily predicted value that $L_{\text{coh}} \approx 5L_{\text{osc}} \approx 10$ km in a spectral analysis of the Daya Bay experiment for $E = 4$ MeV, and that $L_{\text{coh}} \approx L_{\text{osc}} \approx 2$ km for an analysis in which the neutrino energy is not used.

A Appendix: Two-Neutrino Oscillations

A.1 Standard Analysis

We review the standard concepts and notation of neutrino oscillations supposing that there are only two types of neutrinos, both with mass.[10] Production of these neutrinos in a weak interaction via a W-boson emphasizes the so-called flavor states, ν_a and ν_b, while the neutrino states with definite mass are ν_1 and ν_2. These two pairs of states are related by 2×2 unitary matrix with a single parameter, the mixing angle θ_{12} [19],[11]

$$\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} \\ -\sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \tag{16}$$

[10]This Appendix was extracted from [18].

[11]The two-neutrino mixing angle θ_{12} was introduced prior to its relative, the Cabibbo angle [20], that describes the weak-interaction coupling of the u-quark to the d-s quark system, where u, d and s are flavor states of the strong interaction, which differ from the flavor states of the weak interaction. The formalism for the strong-weak three-quark mixing was introduced in [21], and the present notation in terms of quark-mixing angles first appeared in [22]. The latter notation is also commonly used for three-neutrino mixing.

which implies the states a and b can transform back and forth between each other.[12] The neutrino flavor states a and b do not have a well defined mass, according to eq. (16), if the neutrino states 1 and 2 have different masses (as occurs in Nature).[13]

When a neutrino is produced in a nuclear decay, or in the decay of a meson, it is produced in a flavor state rather than a mass state, and it typically accompanied by the associated flavor antilepton. Energy and momentum are conserved in this decay process, but the energy and momentum of the neutrino are different depending for the different neutrino mass state components of the neutrino flavor state.

We review the standard formalism for neutrino oscillations (perhaps first given in [26], and in somewhat more detail in [27]).

The usual procedure is to consider plane-wave states of neutrinos 1 and 2 that have well defined energies E_i and momenta P_i large compared to their (rest) masses m_i, which wave/particles propagate essentially at the speed c of light in vacuum.[14] Then, for propagation along the x-axis, the momenta P_i are

$$c^2 P_i^2 = E_i^2 - m_i^2 c^4, \qquad P_i \approx \frac{E_i}{c}\left(1 - \frac{m_i^2 c^4}{2E_i^2}\right), \tag{17}$$

and

$$\psi_i(x,t) = \psi_{i,0}\, e^{i(P_i x - E_i t)/\hbar} \approx \psi_{i,0}\, e^{iE_i(x/c - t)/\hbar} e^{-i m_i^2 c^3 x/2E_i \hbar}. \tag{18}$$

A neutrino created in a decay at, say, time $t = 0$ is not really in a plane-wave state (18), but rather has a wave packet with a spread of energies ΔE, which implies the time spread of the wave packet is $\Delta t \approx \hbar/\Delta E$ and a spatial width $\Delta x \approx \hbar c/\Delta E$. If wave packets of neutrino mass types 1 and 2 are created together (at the origin and at time $t = 0$), then these packets continue to overlap significantly, and interfere, until their centroids are separated by roughly the pulse width Δx. This occurs at the so-called coherence time $t_{\rm coh}$ related by

$$\Delta x \approx \frac{\hbar c}{\Delta E} = |v_1 - v_2|\, t_{\rm coh} = \left|\frac{c^2 P_1}{E_1} - \frac{c^2 P_2}{E_2}\right| t_{\rm coh} = \left|\frac{m_1^2 c^4}{2E_1^2} - \frac{m_2^2 c^4}{2E_2^2}\right| c t_{\rm coh}, \tag{19}$$

[12] The possibility of such transitions in a two-state system of elementary particles was first noted by Gell-Mann and Pais in 1955 [23] for the K^0-$\bar{K}^0$ system, and first considered for neutrinos by Pontecorvo in 1957 [24]. In meson-antimeson systems such as K^0-$\bar{K}^0$, the meson and antimeson have the same mass (assuming CPT invariance is valid), and can decay to the same final states, such that transitions $K^0 \leftrightarrow \bar{K}^0$ are possible. The neutrino oscillations considered here are not between neutrinos and antineutrinos, but between different flavor states of neutrinos (or of antineutrinos). For discussion of possible $\nu \leftrightarrow \bar{\nu}$ oscillations, see [2].

[13] If a neutrino could be produced in either of the mass states 1 or 2, it would remain in that state until observed (provided it propagates in vacuum; propagation through matter involves interactions that depend on neutrino flavor which lead to oscillations between neutrino mass states [3, 25]). If there were a method of observation of mass states, the neutrino would always be observed in the same mass state in which it was created.

[14] We work in the lab frame. In contrast, discussion of K^0-$\bar{K}^0$ oscillations are typically given in the "rest frame" of the K, which is not strictly well defined since the eigenstates K_L^0 and K_S^0 has different masses. However, the neutral-Kaon mass difference is very small, $\Delta m_K/\overline{m}_K \approx 10^{-14}$, so little error is incurred by this procedure. However, for neutrinos it could be that $\Delta m_\nu/\overline{m}_\nu > 1$, so the notion of a single rest frame for oscillating neutrinos is doubtful.

where $E = (E_1 + E_2)/2$ is the average energy of the two neutrinos. We introduce the coherence length $L_{\rm coh}$ according to (see, for example, [28, 29, 30]),[15,16]

$$L_{\rm coh} = ct_{\rm coh} = \frac{\hbar c}{\Delta E \left| \frac{m_1^2 c^4}{2E_1^2} - \frac{m_2^2 c^4}{2E_2^2} \right|} \approx \frac{2E^2 \hbar c}{\Delta E \, |m_1^2 - m_2^2| \, c^4} \,. \tag{20}$$

The usual analysis continues with the approximation (often not stated explicitly) that the first phase factor in eq. (18) can be ignored, and we write[17]

$$\psi_i(x,t) \approx \psi_{i,0} \, e^{-im_i^2 c^3 x/2E_i\hbar}, \qquad \text{where} \qquad x \approx ct. \tag{21}$$

The exponential phase factor $e^{-im_i^2 c^3 x/2E_i\hbar}$ is slightly different for mass states 1 and 2, which leads to an oscillatory interference term in the spatial/time dependence of an initial single-flavor state.

For a neutrino created at the origin at time $t = 0$ in a pure flavor state a with $\psi_{a,0} = 1$, $\psi_{b,0} = 0$, the initial mass states are

$$\psi_{1,0} = \cos\theta_{12}, \qquad \psi_{2,0} = \sin\theta_{12}, \tag{22}$$

according to eq. (16), so the evolution of the flavor states is, using eq. (21),

$$\psi_a(x) = \cos\theta_{12}\psi_1 + \sin\theta_{12}\psi_2 = \cos^2\theta_{12} \, e^{-im_1^2 c^3 x/2E_1\hbar} + \sin^2\theta_{12} \, e^{-im_2^2 c^3 x/2E_2\hbar}, \tag{23}$$

$$\psi_b(x) = -\sin\theta_{12}\psi_1 + \cos\theta_{12}\psi_2 = -\cos\theta_{12}\sin\theta_{12} \left(e^{-im_1^2 c^3 x/2E\hbar} - e^{-im_2^2 c^3 x/2E\hbar} \right). \tag{24}$$

The probability that the initial flavor state a is still a after the neutrino has traveled distance x is

$$\begin{aligned} P_{a\to a}(x,E) &= |\psi_a(x)|^2 = \cos^4\theta_{12} + \sin^4\theta_{12} + 2\cos^2\theta_{12}\sin^2\theta_{12}\cos\left[\left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right)\frac{c^3 x}{2\hbar}\right] \\ &= \cos^4\theta_{12} + \sin^4\theta_{12} + 2\cos^2\theta_{12}\sin^2\theta_{12}\left(1 - 2\sin^2\frac{\Delta m_{12}^2 c^3 x}{4E\hbar}\right) \\ &= 1 - \sin^2 2\theta_{12}\sin^2\frac{\Delta m_{12}^2 c^3 x}{4E\hbar} = 1 - \sin^2 2\theta_{12}\sin^2\frac{x}{L_{\rm osc}}\,, \end{aligned} \tag{25}$$

where the squared mass difference Δm_{12}^2 and oscillation length L are given by[18]

$$\Delta m_{12}^2 = \left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right) E \approx m_1^2 - m_2^2, \qquad L_{\rm osc} = \frac{4E\hbar}{\Delta m_{12}^2 c^3} \approx \frac{4E\hbar c}{|m_1^2 - m_2^2| \, c^4}\,. \tag{26}$$

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[15]The neutrino coherence length was perhaps first discussed in [31].

[16]Calling the length defined by eq. (20) the coherence length is perhaps unfortunate in that the meaning here is significantly different from the usage in optics, where the optical coherence length is usually taken to be the spatial width $\hbar c/\Delta E$ of a wave packet (in vacuum) with energy spread ΔE.

[17]It is actually more common to write $\psi_i(x,t) \approx \psi_{i,0} \, e^{-im_i^2 c^4 t/2E_i\hbar}$ where $t = x/c$. Writing $\psi_i(x,t)$ as a function of x is closer to experimental practice, as emphasized in [29, 30].

[18]The oscillations of a neutral meson-antimeson system are usually expressed in the (nominal) rest frame of the meson as $\cos(|m_1 - m_2| \, c^2 t^\star/\hbar)$ (see, for example, [32]). In the lab frame the oscillation has, for $E \gg mc^2$, the approximate form $\cos[(|m_1 - m_2| \, c^2 x/\hbar c)(mc^2/E)] = \cos(|m_1^2 - m_2^2| \, c^4 x/2E\hbar c) = 1 - 2\sin^2(|m_1^2 - m_2^2| \, c^4 x/4E\hbar c)$, where $m = (m_1 + m_2)/2$ is the average mass of the states 1 and 2 of definite lifetime (sometimes called the "long" and "short" states as in K_L^0 and K_S^0). Again, $L_{\rm osc} = 4E\hbar c/\left|m_1^2 - m_2^2\right| c^4$.

and again $E = (E_1 + E_2)/2$ is the average neutrino energy. The period of the oscillation in x is $\lambda_x = \pi L_{\rm osc}$ for fixed energy E, and the period of the oscillation in E is approximately

$$\lambda_E \approx \frac{\pi L_{\rm osc} E}{x} = \frac{E}{N_{\rm osc}} \qquad \left(N_{\rm osc} \equiv \frac{x}{\lambda_x} \gg 1 \right) \tag{27}$$

when the fixed distance $x = N_{\rm osc}\lambda_x \gg \lambda_x$, where $N_{\rm osc} \gg 1$ is the number of the oscillation being observed.

The probability that the initial flavor state a is has become flavor b after the neutrino has traveled distance x is

$$\begin{aligned} P_{a\to b}(x, E) &= |\psi_b(x)|^2 = 2\cos^2\theta_{12}\sin^2\theta_{12}\left\{1 - \cos\left[\left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right)\frac{c^3 x}{2\hbar}\right]\right\} \\ &= \sin^2 2\theta_{12}\sin^2\frac{\Delta m_{12}^2 c^3 x}{4E\hbar} = \sin^2 2\theta_{12}\sin^2\frac{x}{L_{\rm osc}} = 1 - P_{a\to a}(x). \end{aligned} \tag{28}$$

Equations (25) and (28) are the standard representation of two-neutrino oscillations for neutrinos produce in a flavor state.

The coherence length (20) is related to the oscillation length (26) by

$$L_{\rm coh} \approx \frac{E}{\Delta E} L_{\rm osc}. \tag{29}$$

Under the assumption that the relevant energy spectrum is approximately Gaussian with variance σ_E, it has become conventional to write[19]

$$L_{\rm coh} = \frac{E}{\sqrt{2}\pi\sigma_E} L_{\rm osc}. \tag{30}$$

A.2 Effect of Detector Resolution

Equation (29) indicates that in the extreme case that the neutrino wave packet is maximally broad, with energy spread $\Delta E \approx E$ as for (anti)neutrinos from the decay of heavy nuclei, only a few oscillations might be observable. However, this assumes that the neutrino is detected without any determination of its energy or momentum.

If the neutrino is detected in a manner that determines its energy (or momentum) to some accuracy $\sigma_{E_{\rm det}}$ which is smaller than the energy spread $\sigma_{E_{\rm source}}$ associated with the source, the energy spread that appears in eqs. (29)-(30) should be $\sigma_{E_{\rm det}}$ rather than the source-related energy spread as considered above.[20]

In practice, the relative energy resolution of neutrino detectors is a few percent, which has no effect on the coherence of oscillations of neutrinos from two-body decays, but will be the determining factor for the coherence length of neutrinos from three-body decays. That is, a detector with sensitivity to neutrino energy makes a selection among the full spectrum

[19]The definition (30) may have first been given in eq. (24) of [6].

[20]Strictly, $1/\sigma_E^2 = 1/\sigma_{E_{\rm source}}^2 + 1/\sigma_{E_{\rm det}}^2$, as perhaps first discussed in eq. (30) of [4]. See also eq. (53) of [5] and eq. (15) of [6]. Earlier discussion of the role of the detector, as in [33], emphasized its size rather than its energy resolution.

of energy of neutrinos incident upon it, which permits observation of oscillations of neutrinos from a three-body decay at greater distance from the source than would be possible if the detector merely identified the presence of a neutrino but with no information as to its energy (or momentum).

In particular, if oscillations are to be observed in the energy spectrum of neutrinos from a three-body decay at a single distance x from the source (rather than, say, as oscillations as a function of distance for neutrinos of any energy), the energy resolution of the detector must be better than 1/4 of a period λ_E of the oscillation in energy. Then, eq. (27) implies that $E/\Delta E_{\rm det} \gtrsim 4N_{\rm osc}$, and the coherence length is $L_{\rm coh} \gtrsim 4N_{\rm osc}L_{\rm osc} = 4x/\pi$ for observation of oscillation number $N_{\rm osc}$ at distance $x = N_{\rm osc}\lambda_x = N_{\rm osc}\pi L_{\rm osc}$. That is, the requirement that the detector resolution be good enough to resolve the energy oscillations insures that the coherence length for the oscillations is long enough that they can be observed.[21]

For example, in the context of a three-neutrino scenario, where $L_{12} \approx 30L_{13}$, it is possible to resolve the so-called mass hierarchy by observation of rapid 1-3 oscillations with $N_{\rm osc} \approx 30$ at the peak of the first (slower) 1-2 oscillation [34]. The relative detector energy resolution for neutrinos needs to be better than 1/120 to resolve the oscillations, whereas to avoid any effects of decoherence, the energy resolution should be somewhat better than this.

While only a single oscillation has been observed in neutrino experiments (and in the K^0*-*$\bar{K}^0$ *system) to date, oscillations over nine periods have recently been observed in the* B_s^0*-*$\bar{B}_s^0$ *system [35], which indicates that* $E_{B\bar{B}}/\Delta E \gtrsim 10$ *for* B_s^0 *production at a hadron collider. Since heavy quark states such as the* B_s^0 *are produced in pp collisions via "fusion" of gluons whose initial energies are not well defined, but the relative detector energy resolution for the* B_s^0 *is less that 10%, the experimental results [35] are evidence that* $\Delta E \approx \Delta E_{\rm det}$ *in this case, where good detector resolution has extended the coherence length of the meson-antimeson oscillations.*

A.2.1 Further Details

When a neutrino (or antineutrino) of flavor a is produced in the decay

$$A \to B + \nu_a, \tag{31}$$

in the rest frame of particle A of mass m_A, and the neutrino flavor state ν_a is related to neutrino mass eigenstates ν_1 and ν_2 of masses m_1 and m_2 by

$$|\nu_a\rangle = \cos\theta_{12}|\nu_1\rangle + \sin\theta_{12}|\nu_2\rangle, \tag{32}$$

the final state wavefunction in entangled, and can be written as

$$|B,\nu_a\rangle = \cos\theta_{12}|B_1\rangle|\nu_1\rangle + \sin\theta_{12}|B_2\rangle|\nu_2\rangle. \tag{33}$$

Energy and momentum conservation are that

$$m_A = E_{B_1} + E_{\nu_1} = E_{B_2} + E_{\nu_2}, \qquad 0 = \mathbf{P}_{B_1} + \mathbf{P}_{\nu_1} = \mathbf{P}_{B_2} + \mathbf{P}_{\nu_2}, \tag{34}$$

[21] If the energy resolution is barely sufficient to resolve the oscillations, the coherence length is only slightly larger than the source-detector distance, and there may be some loss of amplitude of the oscillations.

where, of course, the energy and momentum for a state of mass m are related by $E^2 = m^2c^4 + P^2c^2$.

We are particularly interested in the case of a three-body β-decay, where B is a two-particle system, and the neutrino energies form a continuum over a range of several MeV.

In general, the neutrino is detected with an energy resolution smaller than the width of its β-decay spectrum, so that for detected neutrinos, we can speak of σ_E ($\approx 0.08\sqrt{E}$ for the Daya Bay experiment) as the detector energy resolution rather than as the width of the β-decay spectrum.[22]

The usual argument[23] is that the process of detection of a neutrino leaves it with a definite energy E, even if this value is not well known due to the uncertainty in the measurement of that energy, which is reported as $\bar{E}$. Then, the observed behavior of detected neutrinos is to be obtained by weighting their survival probability by an approximately Gaussian detector-resolution function.

From eq. (25), the probability that a neutrino of energy E and flavor a is still of that flavor after traveling distance x is

$$P_{a\to a}(x, E) = 1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m_{12}^2 c^3 x}{4E\hbar}\,. \tag{35}$$

The probability that a neutrino is detected as having energy $\bar{E}$ by a detector with rms energy resolution $\sigma_E(\bar{E})$ is,

$$P_{a\to a}(x, \bar{E}) \propto \int dE\, e^{-(E-\bar{E})/2\sigma_E^2} P_{a\to a}(x, E). \tag{36}$$

The Gaussian factor in eq. (35) can be rewritten as[24]

$$\begin{aligned} e^{-(E-\bar{E})^2/2\sigma_E^2} &= e^{-x^2E^2\bar{E}^2(E-\bar{E})^2/2\sigma_E^2E^2\bar{E}^2x^2} = e^{-E^2\bar{E}^2(x/E-x/\bar{E})^2/2\sigma_E^2x^2} \approx e^{-\bar{E}^4(x/E-x/\bar{E})^2/2\sigma_E^2x^2} \\ &\equiv e^{-\bar{E}^2(w-\bar{w})^2/2\sigma_E^2\bar{w}^2}, \end{aligned} \tag{37}$$

where $w = x/E$ and $\bar{w} = x/\bar{E}$. The probability (36) can now be represented in the (normalized) form,

$$\begin{aligned} P_{a\to a}(\bar{w}) &= \int dw\, \frac{e^{-\bar{E}^2(w-\bar{w})^2/2\sigma_E^2\bar{w}^2}}{\sqrt{2\pi}\sigma_E\bar{w}/\bar{E}} \left(1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m_{12}^2 c^3 w}{4\hbar}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \int dw\, \frac{e^{-\bar{E}^2(w-\bar{w})^2/2\sigma_E^2\bar{w}^2}}{\sqrt{2\pi}\sigma_E\bar{w}/\bar{E}} \left(1 - \cos \frac{\Delta m_{12}^2 c^3 w}{2\hbar}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \left(1 - \int dw'\, \frac{e^{-\bar{E}^2w'^2/2\sigma_E^2\bar{w}^2}}{\sqrt{2\pi}\sigma_E\bar{w}/\bar{E}} \cos \frac{\Delta m_{12}^2 c^3 (w' + \bar{w})}{2\hbar}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \left(1 - \cos \frac{\Delta m_{12}^2 c^3 \bar{w}}{2\hbar}\, e^{-\sigma_E^2\Delta m_{12}^4 c^6 \bar{w}^2/8\bar{E}^2\hbar^2}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \left(1 - \cos \frac{2x}{L_{\rm osc}(\bar{E})}\, e^{-2(\sigma_E^2/\bar{E})^2x^2/L_{\rm osc}^2(\bar{E})}\right), \end{aligned} \tag{38}$$

[22] In the unrealistic case of extremely fine detector resolution σ_E would not go to zero, but to a small value governed by other considerations, such as the size of the atom that contained the state A. That is, $\sigma_E^2 = \sigma_{E_{\rm det}}^2 + \sigma_{E_{\rm other}}^2$.

[23] See, for example, [36].

[24] This type of transformation was used in [5], on which this section is based.

using Gradshteyn and Ryzhik 3.896.4. For very fine detector energy resolution, $\sigma_E/\bar{E} \ll 1$, we recover the form (25). For coarse energy resolution the survival probability does not oscillate, but simply has the value $1 - (1/2)\sin^2 2\theta_{12}$ independent of x, and we say that the oscillations have decohered, as illustrated in the top left figures on pp. 3-4. The damping/coherence length in the last form of eq. (38) is

$$L_{\rm coh}(\bar{E}) = \frac{L_{\rm osc}(\bar{E})}{\sqrt{2}\,\sigma_E/\bar{E}}\,. \tag{39}$$

A.3 Effect of Source Size

If the neutrino source is large compared to an oscillation length the evidence for neutrino oscillations in a detector will be "washed out". This is not strictly an effect of decoherence, in that neutrinos produced in different primary interactions do not interfere with one another.[25,26]

This effect is important in studies of oscillations of reactor neutrinos, where the distances between the detector and multiple reactors must be not too different. Also, since supernova neutrinos have energies and oscillations lengths similar to those for reactor neutrinos, but the size of supernovas is large compared to the kilometer scale of the oscillation lengths, oscillations of supernova neutrinos cannot be observed.[27]

References

[1] F.P. An *et al.*, *Probing decoherence effects in wave packet treatment of neutrino oscillations at Daya Bay* (June 5, 2016),
http://physics.princeton.edu/~mcdonald/dayabay/Naumov/DYB_Decoh_prl_v4.pdf

[2] J.N. Bahcall and H. Primakoff, *Neutrino-antineutrino oscillations*, Phys. Rev. D **18**, 3463 (1978), http://physics.princeton.edu/~mcdonald/examples/neutrinos/bahcall_prd_18_3463_78.pdf

[3] L. Wolfenstein, *Neutrino oscillations in matter*, Phys. Rev. D **17**, 2369 (1978),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/wolfenstein_prd_17_2369_78.pdf

[4] C. Giunti, C.W. Kim and U.W. Lee, *When do neutrinos cease to oscillate?* Phys. Lett. B **421**, 237 (1998), http://physics.princeton.edu/~mcdonald/examples/neutrinos/giunti_pl_b421_237_98.pdf

[5] T. Ohlsson, *Equivalence between Gaussian averaged neutrino oscillations and neutrino decoherence*, Phys. Lett. B **502**, 159 (2001),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/ohlsson_pl_b502_159_01.pdf

[25] Dirac has written [37] "Each photon then interferes only with itself. Interference between two different photons never occurs."

[26] The source of the neutrino could be determined by detection of its partner B in the production reaction $A \to B + \nu$ (without precise determination of the energy or momentum of B), so in the case of multiple source points the observed probability distributions (25) and (28) are the sum of those for the various possible production points.

[27] A separate issue is that for oscillations to be observed at distances from the source that are very large compared to an oscillation length, the period of the oscillations in the energy spectrum is very short, and extremely good detector energy resolution would be required to resolve the oscillations from a "point" source.

[6] M. Beuthe, *Towards a unique formula for neutrino oscillations in vacuum*, Phys. Rev. D **66**, 013003 (2002),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/beuthe_prd_66_013003_02.pdf

[7] F.P. An *et al.*, *Spectral Measurement of Electron Antineutrino Oscillation Amplitude and Frequency at Daya Bay*, Phys. Rev. Lett. **112**, 061801 (2014),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/an_prl_112_061801_14.pdf

[8] F.P. An *et al.*, *Observation of Electron-Antineutrino Disappearance at Daya Bay*, Phys. Rev. Lett. **108**, 171803 (2012),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/an_prl_108_171803_12.pdf

[9] F.P. An *et al.*, *New Measurement of Antineutrino Oscillation with the Full Detector Configuration at Daya Bay*, Phys. Rev. Lett. **115**, 111802 (2015),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/an_prl_115_111802_15.pdf

[10] K. Eguchi *et al.*, *First Results from KamLAND: Evidence for Reactor Antineutrino Disappearance*, Phys. Rev. Lett. **90**, 021802 (2003),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/eguchi_prl_90_021802_03.pdf

[11] G. Barenboim *et al.*, *Quantum decoherence and neutrino data*, Nucl. Phys. B **758**, 90 (2006), http://physics.princeton.edu/~mcdonald/examples/neutrinos/barenboim_np_b758_90_06.pdf

[12] D. Morgan *et al.*, *Probing quantum decoherence in atmospheric neutrino oscillations with a neutrino telescope*, Astrop. Phys. **25**, 311 (2006),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/morgan_ap_25_311_06.pdf

[13] E. Lisi, A. Marrone, and D. Montanino, *Probing Possible Decoherence Effects in Atmospheric Neutrino Oscillations*, Phys. Rev. Lett. **85**, 61166 (2000),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/lisi_prl_85_1166_00.pdf

[14] G. Fogli *et al.*, *Status of atmospheric neutrino $\nu_\mu \to \nu_\tau$ oscillations and decoherence after the first K2K spectral data*, Phys. Rev. D **67**, 093006 (2003),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/fogli_prd_67_093006_03.pdf

[15] D. Hooper, D. Morgan, and E. Winstanley, *Probing quantum decoherence with high-energy neutrinos*, Phys. Lett. B **609**, 206 (2005),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/hooper_pl_b609_206_05.pdf

[16] S. Abe *et al.*, *Precision Measurement of Neutrino Oscillation Parameters with KamLAND*, Phys. Rev. Lett. **100**, 221803 (2008),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/abe_prl_100_221803_08.pdf

[17] M. Gonchar *et al.*, *Quantum decoherence in neutrino oscillations* (May 5, 2016),
http://dayabay.ihep.ac.cn/DocDB/0104/010403/016/dubna_decoherence_technote_v5.pdf

[18] K.T. McDonald, *Do Neutrino Oscillations Conserve Energy?* (July 21, 2013),
http://physics.princeton.edu/~mcdonald/examples/neutrino_osc.pdf

[19] Z. Maki, M. Nakagawa and S. Sakata, *Remarks on the Unified Model of Elementary Particles*, Prog. Theor. Phys. **28**, 870 (1962),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/maki_ptp_28_870_62.pdf

[20] N. Cabibbo, *Unitary Symmetry and Leptonic Decays*, Phys. Rev. Lett. **10**, 531 (1963),
http://physics.princeton.edu/~mcdonald/examples/QED/cabibbo_prl_10_531_63.pdf

[21] M. Kobayashi and T. Maskawa, *CP Violation in the Renormalizable Theory of Weak Interaction*, Prog. Theor. Phys. **49**, 652 (1973),
http://physics.princeton.edu/~mcdonald/examples/QED/kobayashi_ptp_49_652_73.pdf

[22] L.-L. Chau and W.-Y. Keung, *Comments on the Parametrization of the Kobayashi-Maskawa Matrix*, Phys. Rev. Lett. **53**, 1802 (1984),
http://physics.princeton.edu/~mcdonald/examples/QED/chau_prl_53_1802_84.pdf

[23] M. Gell-Mann and A. Pais, *Behavior of Neutral Particles under Charge Conjugation*, Phys. Rev. **97**, 1387 (1955),
http://physics.princeton.edu/~mcdonald/examples/QED/gellmann_pr_97_1387_55.pdf

[24] B. Pontecorvo, *Mesonium and Antimesonium*, Sov. Phys. JETP **6**, 429 (1957),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/pontecorvo_spjetp_6_429_57.pdf

[25] S.P. Mikheyev and A.Y. Smirnov. *Resonant Amplification of ν Oscillations in Matter and Solar-Neutrino Spectroscopy*, Nuovo Cim. **9C**, 17 (1986),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/mikheyev_nc_9c_17_86.pdf

[26] V. Gribov and B. Pontecorvo, *Neutrino Astronomy and Lepton Charge*, Phys. Lett. **28B**, 493 (1969), http://physics.princeton.edu/~mcdonald/examples/neutrinos/gribov_pl_28b_493_69.pdf

[27] S.M. Bilenky and B. Pontecorvo, *Again on Neutrino Oscillations*, Lett. Nuovo Cim. **17**, 569 (1976), http://physics.princeton.edu/~mcdonald/examples/neutrinos/bilenky_lnc_17_569_76.pdf

[28] L. Stodolsky, *When the wavepacket is unnecessary*, Phys. Rev. D **58**, 036006 (1998),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/stodolsky_prd_58_036006_98.pdf

[29] H.J. Lipkin, *What is coherent in neutrino oscillations*, Phys. Lett. **B579**, 355 (2004),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/lipkin_pl_b579_355_04.pdf

[30] H.J. Lipkin, *Quantum theory of neutrino oscillations for pedestrians: simple answers to confusing questions*, Phys. Lett. **B642**, 366 (2006),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/lipkin_pl_b642_366_06.pdf

[31] S. Nussinov, *Solar Neutrinos and Neutrino Mixing*, Phys. Lett. **63**B, 201 (1976),
http://physics.princeton.edu/~mcdonald/examples/neutrinos/nussinov_pl_63b_201_76.pdf

[32] T.D. Lee and C.S. Wu, *Decays of Neutral K Mesons*, Ann. Rev. Nucl. Sci. **16**, 511 (1966), http://physics.princeton.edu/~mcdonald/examples/EP/lee_arns_16_511_66.pdf

[33] B. Kayser, *On the quantum mechanics of neutrino oscillation*, Phys. Rev. D **24**, 110 (1981), http://physics.princeton.edu/~mcdonald/examples/neutrinos/kayser_prd_24_110_81.pdf

[34] S.T. Petcov and M. Piai, *The LMA MSW solution of the solar neutrino problem, inverted neutrino mass hierarchy and reactor neutrino experiment*, Phys. Lett. **B533**, 94 (2002), http://physics.princeton.edu/~mcdonald/examples/neutrinos/petcov_pl_b533_94_02.pdf

[35] R. Aaij *et al.*, *Precision measurement of the B_s^0-$\bar{B}_s^0$ oscillation frequency with the decay $B_s^0 \to D_s^- \pi^+$*, New J. Phys. **15**, 053021 (2013), http://physics.princeton.edu/~mcdonald/examples/EP/aaij_njp_15_053021_13.pdf

[36] H.J. Lipkin, *Stodolsky's Theorem and Neutrino Oscillation Phases – for pedestrians*, https://arxiv.org/abs/hep-ph/0212093

[37] P.A.M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Clarendon Press, London, 1958), p. 9, http://physics.princeton.edu/~mcdonald/examples/QM/dirac_PQM_p9.pdf