# A Relativistic Electron Can't Extract Net Energy from a 'Long' Laser Pulse

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In this note I write out the argument of Sprangle *et al.*, [Phys. Rev. E**52**, 5443 (1995), Opt. Comm. **124**, 69 (1995)] at some length – and add to it a result, eq. (9), that I have not seen published elsewhere. The specific point of this argument is that the longitudinal component of the laser field near a focus, neglected in the analysis of Malka *et al.* [Phys. Rev. Lett. **78**, 3314 (1997)], cancels any energy transferred to a relativistic electron by the transverse component.

The Gaussian approximation to a laser beam that propagates along the $+z$ axis and is polarized in the $\mathbf{x}$ direction is to leading order:

$$\mathbf{E} = \frac{E_0 \hat{\mathbf{x}}}{\sqrt{1+\varsigma^2}} e^{-i\tan^{-1}\varsigma} e^{i\rho^2\varsigma/(1+\varsigma^2)} e^{-\rho^2/(1+\varsigma^2)} e^{i\varphi}, \qquad (1)$$

where $\rho^2 = (x^2+y^2)/w_0^2$, the radius of the waist is $w_0$, $\varsigma = z/z_0$, the Rayleigh range is

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{k w_0^2}{2}, \qquad (2)$$

the phase is

$$\varphi = kz - \omega t, \qquad (3)$$

the frequency of the wave is $\omega$, the wave number is $k = \omega/c$, the wavelength is $\lambda$ and $c$ is the speed of light.

Expression (1) describes the main features of the focus of the laser beam, but it does not satisfy the Maxwell equation $\nabla \cdot \mathbf{E} = 0$. To see this, note that a divergence-free field that points only in the $x$ direction cannot vary with $x$.

We find below that an electric field which satisfies Maxwell's equations in the next approximation has a small longitudinal component, and that this apparently small component is sufficient to cancel completely any net energy transfer to a relativistic electron that appears possible if only eq. (1) is used.

Equation (1) describes a continuous-wave laser beam, rather than a pulse.

Equation (1) of Malka *et al.* is the same as my eq. (1) with the addition of a factor $g(\varphi)$ that describes the laser pulse envelope in time. The new part of what follows is to find a condition on the form of $g$ so that (1) satisfies Maxwell's equations to a good approximation. I will find that the assumption by Malka *et al.* that $g = \sin^2(\varphi/\varphi_0)$ for $0 < \varphi/\varphi_0 < \pi$ and zero elsewhere is not satisfactory. This oversight is in addition to their erroneous claim that the longitudinal part of field $\mathbf{E}$ can be neglected.

Before I deal with the issue of acceleration of electrons, I review the derivation of a better approximation to a Gaussian laser pulse, following L.W. Davis [Phys. Rev. A**19**, 1177 (1979)]. The knowledgeable reader might want to skip ahead to eqs. (26) and (27).

Davis' insight is that the form of eq. (1) can more properly be used for the vector potential than for the electric field. In general, the divergence of the vector potential need not be zero and there are solutions to Maxwell's equations with nonuniform vector potential that point only along the $x$-axis.

A second useful insight is that when the wave equation for the vector potential is written in terms of the dimensionless variables

$$\xi = \frac{x}{w_0}, \quad \upsilon = \frac{y}{w_0}, \quad \rho^2 = \xi^2 + \upsilon^2, \quad \text{and} \quad \varsigma = \frac{z}{z_0}, \qquad (4)$$

then a series expansion suggests itself. Namely, the focal region has transverse and longitudinal extent in the ratio

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{k w_0}. \qquad (5)$$

The aspect ratio $\theta_0$ (also the diffraction angle) is typically much less than one, and so can serve as the expansion parameter.

We seek fields that propagate in the $+z$ direction, have limited transverse extent, and for which the vector potential has only an $x$ component. We try

$$\mathbf{A}(\mathbf{r}, t) = \hat{\mathbf{x}} \psi(\mathbf{r}) g(\varphi) e^{i\varphi}, \qquad (6)$$

where $\psi$ and $g$ vary 'slowly'. The vector potential must satisfy the free-space wave equation:

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}. \qquad (7)$$

Inserting trial solution (6) into (7) we find that

$$\nabla^2 \psi + 2ik \frac{\partial \psi}{\partial z} \left(1 - \frac{ig'}{g}\right) = 0, \qquad (8)$$

where $g' = dg/d\varphi$. Since $\psi$ is a function of $\mathbf{r}$ while $g$ and $g'$ are functions of the phase $\varphi$, eq. (8) cannot be satisfied in general. Often the discussion is restricted to the case where $g' = 0$, *i.e.*, to continuous waves. However, we see that to proceed with our description of pulsed beams we must accept the condition that

$$g' \ll g. \qquad (9)$$

First, consider the proposal of Malka *et al.* that $g = \sin^2(\varphi/\varphi_0)$. Then $g'/g = (2/\varphi_0)\cot(\varphi/\varphi_0)$. Even for the plausible restriction that $\varphi_0 \gg 2$, $g'/g$ blows up at the beginning and end of the pulse (which is defined on the interval $0 \le \varphi \le \pi\varphi_0$).

Another popular form for a laser pulse is a Gaussian: $g = \exp[-(\varphi/\varphi_0)^2]$. Then $g'/g = -2\varphi/\varphi_0^2$ which does not satisfy (9) for $|\varphi| \gtrsim \varphi_0$.

A more appropriate form for a pulsed beam is a hyperbolic secant (as arises in studies of solitons):

$$g(\varphi) = \text{sech}\left(\frac{\varphi}{\varphi_0}\right). \qquad (10)$$

Then $g'/g = -(1/\varphi_0)\tanh(\varphi/\varphi_0)$ which is much less than one everywhere for $\varphi_0 \gg 1$.

One might hope that a poor approximation would suffice in regions where the field is weak. But when the field is probed by a moving electron, the latter spends a long time in the weak-field region, so time integrals such as energy transfer have significant contributions from that region.

Hence, I strongly recommend that future numerical (and analytic) calculations involving laser pulses use form (10).

We now suppose that condition (9) is satisfied, so that (8) can be approximated as

$$\nabla_\perp^2 \psi + 4i\frac{\partial \psi}{\partial \varsigma} + \theta_0^2 \frac{\partial^2 \psi}{\partial \varsigma^2} = 0, \qquad (11)$$

where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \upsilon^2}, \qquad (12)$$

in terms of the dimensionless variables introduced in eqs. (4-5). This form suggests the series expansion

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + ... \qquad (13)$$

Inserting this into eq. (11) and collecting terms of order $\theta_0^0$ and $\theta_0^2$, we find

$$\nabla_\perp^2 \psi_0 + 4i\frac{\partial \psi_0}{\partial \varsigma} = 0, \qquad (14)$$

and

$$\nabla_\perp^2 \psi_2 + 4i\frac{\partial \psi_2}{\partial \varsigma} = -\frac{\partial^2 \psi_0}{\partial \varsigma^2}, \qquad (15)$$

respectively. Equation (14) can be recognized as the paraxial wave equation whose Gaussian solution was given in eq. (1). That is,

$$\psi_0 = f e^{-f\rho^2}, \qquad (16)$$

where

$$f = \frac{-i}{\varsigma - i} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1+\varsigma^2}}. \qquad (17)$$

Davis cleverly guessed the solution to eq. (15):

$$\psi_2 = \left(\frac{f}{2} - \frac{f^3\rho^4}{4}\right)\psi_0, \qquad (18)$$

although we will not need this here.

We work in the Lorentz gauge (and Gaussian units), so the scalar potential $\phi$ obeys

$$\frac{\partial \phi}{\partial t} = -c\nabla \cdot \mathbf{A}. \qquad (19)$$

Similarly to eq. (6), we suppose that $\phi$ can be written

$$\phi(\mathbf{r}, t) = \Phi(\mathbf{r}) g(\varphi) e^{i\varphi}. \qquad (20)$$

Then,

$$\frac{\partial \phi}{\partial t} = -\omega\phi\left(1 - \frac{ig'}{g}\right) \approx -\omega\phi, \qquad (21)$$

when condition (9) is satisfied. In this case,

$$\phi = -\frac{i}{k}\nabla \cdot \mathbf{A}. \qquad (22)$$

The electric and magnetic fields can now be deduced from the approximate vector potential

$$\mathbf{A} = \hat{\mathbf{x}}\psi_0 g(\varphi) e^{i\varphi} \qquad (23)$$

via

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{k}\nabla(\nabla \cdot \mathbf{A}) + ik\mathbf{A}, \qquad (24)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad (25)$$

again approximating as one the factors $1 - ig'/g$ that arise. The results accurate to order $\theta_0$ are, after dividing out a factor of $ik$:

$$\begin{aligned} E_x &= \psi_0 g e^{i\varphi}, \\ E_y &= 0, \\ E_z &= \frac{i\theta_0}{2}\frac{\partial \psi_0}{\partial \xi} g e^{i\varphi} = -i\theta_0 f \xi E_x, \end{aligned} \qquad (26)$$

$$\begin{aligned} B_x &= 0, \\ B_y &= E_x, \\ B_z &= \frac{i\theta_0}{2}\frac{\partial \psi_0}{\partial \upsilon} g e^{i\varphi} = -i\theta_0 f \upsilon E_x. \end{aligned} \qquad (27)$$

These expression satisfy $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$ plus terms of order $\theta_0^2$.

After these lengthy preliminaries we are ready to consider vacuum laser acceleration of electrons.

We consider an electron moving at velocity $c$ along the line $x = z\theta$, where $\theta$ is a small angle. The electron passes through a laser field given by eqs. (26-27), with $y = 0$ always, and we suppose that it passes the origin at $t = 0$. In terms of the dimensionless variables $\xi$, $\varsigma$ and $\rho$, the trajectory is

$$\rho = \xi = \frac{\theta\varsigma}{\theta_0}. \tag{28}$$

The electric field component along the electron's trajectory is

$$E_\parallel = E_x \sin\theta + E_z \cos\theta \approx E_x\theta(1 - if\varsigma) = E_x\theta f, \tag{29}$$

for small $\theta$, noting that eqs. (26) and (28) lead to

$$E_z = -i\theta_0 f\xi E_x = -i\theta f\varsigma E_x, \tag{30}$$

and that eq. (17) leads to the identity

$$1 - if\varsigma = f. \tag{31}$$

The electron has coordinate $z$ at time $t = z/(c\cos\theta)$ so

$$\varphi = kz - \omega t \approx -\frac{kz\theta^2}{2} = -\frac{\theta^2}{\theta_0^2}\varsigma. \tag{32}$$

Then,

$$E_x = E_0 fg e^{-f\rho^2} e^{i\varphi}$$
$$\approx E_0 fg e^{-f\varsigma^2\theta^2/\theta_0^2} e^{-i\varsigma\theta^2/\theta_0^2} = E_0 fg e^{-if\varsigma\theta^2/\theta_0^2}, \tag{33}$$

using eqs. (26), (31) and (32). Inserting this into eq. (29) we have

$$E_\parallel = \theta E_0 f^2 g e^{-if\varsigma\theta^2/\theta_0^2}$$
$$= \frac{d}{d\varsigma}\left(\frac{i\theta_0^2}{\theta} E_0 g e^{-if\varsigma\theta^2/\theta_0^2}\right), \tag{34}$$

again neglecting a term in $g'/g$. That is, the force on the electron along its trajectory can be derived from a potential. The change in energy along the trajectory is then just the change in the potential. However, the potential

$$U = -\frac{i\theta_0^2}{\theta} E_0 g e^{-if\varsigma\theta^2/\theta_0^2} \tag{35}$$

has the same value at $\varsigma = -\infty$ and $+\infty$, so the (relativistic) electron gains no net energy as it crosses the laser beam.

This is the argument of Sprangle *et al.*, with the addition that it holds for laser pulses as well as for continuous waves, so long as condition (9) is satisfied – which condition is required if (26) and (27) are to be (approximate) solutions to Maxwell's equations.

Everyone agrees that if one uses $E_x$ from (26) but ignores $E_z$, a net energy gain will be calculated for a free electron crossing a laser beam.

See also L. Feng and Y.-K. Ho, Phys. Lett. **A184**, 440 (1994), J. Phys. B**27**, 2417 (1994).

The interesting case of extremely short pulses ($g'/g \approx 1$) has been considered by H.M. Lai, Phys. Fluids **23**, 2373 (1980). The above analysis does not hold in this limit, and significant energy can in principle be transferred from a single-cycle pulse to an electron. However, for a pulse of even a few cycles, the energy transfer is greatly suppressed.